HOPF AUTOMORPHISMS AND TWISTED EXTENSIONS

SUSAN MONTGOMERY, MARIA D. VEGA, AND SARAH WITHERSPOON

Abstract. We give some applications of a Hopf algebra constructed from a group acting on another Hopf algebra \( A \) as Hopf automorphisms, namely Molnar’s smash coproduct Hopf algebra. We find connections between the exponent and Frobenius-Schur indicators of a smash coproduct and the twisted exponents and twisted Frobenius-Schur indicators of the original Hopf algebra \( A \). We study the category of modules of the smash coproduct.

1. Introduction

Molnar [Mi] defined smash coproducts of Hopf algebras, putting them on equal footing with the better-known smash products by viewing both as generalizations of semidirect products of groups. Recently smash coproducts have made an appearance as examples of new phenomena in representation theory [BW, DE]. In this paper we propose several applications of smash coproducts. In particular, the smash coproduct construction will allow us to “untwist” some invariants defined via the action of a Hopf algebra automorphism, such as the twisted exponents and the twisted Frobenius-Schur indicators.

We note that considering Hopf automorphisms is a timely topic, since there has been recent progress in determining the automorphism groups of some Hopf algebras [AD, Ke, R3, SV, Y]. There has also been much recent work on indicators; their importance lies in the fact that they are invariants of the category of representations of the Hopf algebra, and may be defined for more abstract categories [NSc]. Moreover the notion of twisted indicators can be extended to pivotal categories [SV3].

We start by defining the smash coproduct \( A \boxtimes k^G \), for any Hopf algebra \( A \) with an action of a group \( G \) by Hopf automorphisms, in the next section. In Section 3 we recall the notions of exponent and twisted exponent [SV2] of a Hopf algebra, and find connections between the exponent of \( A \boxtimes k^G \) and twisted exponents of \( A \) itself. In Section 4 we assume the Hopf algebra \( A \) is semisimple. We recall definitions of Frobenius-Schur indicators [KSZ] and twisted Frobenius-Schur indicators [SV] for simple modules over the Hopf algebra, and give relationships between the indicators of the smash coproduct \( A \boxtimes k^G \) and twisted indicators of \( A \) itself.

In Section 5 we do not assume the Hopf algebra is semisimple. We introduce the twisted Frobenius-Schur indicators of the regular representation of such a Hopf algebra, simultaneously generalizing indicators for not necessarily semisimple Hopf algebras [KMN] and twisted indicators for semisimple Hopf algebras [SV]. Again we find a connection with the Frobenius-Schur indicator of a smash coproduct. We compute an example for which the Hopf algebra

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2. THE SMASH COPRODUCT

Our Hopf algebra was defined by Molnar [Ml, Theorem 2.14], who called it the smash coproduct, although our definition seems different at first glance. See also [R2, p. 357].

Let $A$ be a Hopf algebra over a field $k$ and let a group $G$ act as Hopf algebra automorphisms of $A$. Let $k^G$ be the algebra of set functions from $G$ to $k$ under pointwise multiplication; that is, if $\{p_x \mid x \in G\}$ denotes the basis of $k^G$ dual to $G$, then $p_x p_y = \delta_{x,y} p_x$ for all $x, y \in G$. Recall that $k^G$ is a Hopf algebra with comultiplication given by $\Delta(p_x) = \sum_{y \in G} p_y \otimes p_y^{-1}x$. counit $\varepsilon(p_x) = \delta_{1,x}$ and antipode $S(p_x) = p_{x^{-1}}$ for all $x \in G$.

Then we may form the smash coproduct Hopf algebra $K = A \bowtie k^G$ with algebra structure the usual tensor product of algebras. Denote by $a \bowtie p_x$ the element $a \otimes p_x$ in $K$, for each $a \in A$ and $x \in G$. Comultiplication is given by

$$\Delta(a \bowtie p_x) = \sum_{y \in G} (a_1 \bowtie p_y) \otimes ((y^{-1} \cdot a_2) \bowtie p_{y^{-1}x})$$

for all $x \in G$, $a \in A$. The counit and antipode are determined by

$$\varepsilon(a \bowtie p_x) = \delta_{1,x} \varepsilon(a) 1 \quad \text{and} \quad S(a \bowtie p_x) = (x^{-1} \cdot S(a)) \bowtie p_{x^{-1}}.$$  

For simplicity, we will frequently omit the $\bowtie$ symbol in our notation. The normalized integral of $K$ is $\Lambda_K = \Lambda_A \bowtie p_1 = \Lambda_A p_1$.

Note that Molnar defines the smash coproduct for the right coaction of any commutative Hopf algebra $H$. We show that our construction is actually his smash coproduct with $H = k^G$, by dualizing our $G$-action to a $k^G$-coaction.

**Lemma 2.1.** (1) $K$ as above is isomorphic to the smash coproduct as in [Ml, Theorem 2.14], and thus is a Hopf algebra.

(2) If $A$ is finite-dimensional, then $K^* \cong A^* \# k^G$, the smash product Hopf algebra as in [Ml, Theorem 2.13].

**Proof.** (1) Given the left action of $G$ on $A$, we define $\rho : A \to A \otimes k^G$ by $a \mapsto \sum_{x \in G} (x \cdot a) \otimes p_x$. Then $\rho$ is a right comodule map, using the fact that the $G$-action on $A$ satisfies $x \cdot (y \cdot a) = (xy \cdot a)$ and $1 \cdot a = a$ for all $x, y \in G$ and $a \in A$.

Next we note that $A$ is a right comodule algebra under $\rho$ since the $G$-action is multiplicative, that is $(x \cdot a)(x \cdot b) = x \cdot (ab)$. Also $A$ is a right comodule coalgebra, as the $G$-action preserves the coalgebra structure of $A$, that is, $x \cdot (a_1 \otimes a_2) = \sum_a (x \cdot a)_1 \otimes (x \cdot a)_2$. Thus $A$ is a right $k^G$-comodule bialgebra.

Finally the antipode also dualizes to the antipode given by Molnar, and thus Molnar’s theorem [Ml, Theorem 2.14] applies.

(2) This is a special case of Molnar’s result [Ml, Theorem 5.4]. □
3. Hopf powers and exponents

In any Hopf algebra $H$, we denote the $n$th Hopf power of an element $x \in H$ by $x^{[n]} = \sum x_1x_2x_3\ldots x_n$; that is, first apply $\Delta_H$ $n - 1$ times to $x$ and then multiply. Note that $x \mapsto x^{[n]}$ is a linear map.

For $H$ semisimple, recall that the exponent of $H$, $\exp(H)$, is the smallest positive integer $n$, if it exists, such that $x^{[n]} = \varepsilon(x)1$ for all $x \in H$.

Recently [SV2] introduced the twisted exponent, where $\exp$ is twisted by an automorphism of $H$ of finite order. Assume that $\tau \in \text{Aut}(H)$ and that $n$ is a multiple of the order of $\tau$. Define the $n$th $\tau$-twisted Hopf power of $x$ to be

$$x^{[n,\tau]} := \sum x_1(\tau \cdot x_2)(\tau^2 \cdot x_3)\ldots(\tau^{n-1} \cdot x_n).$$

Then $\exp_\tau(H)$ is the smallest positive integer $n$, if it exists, such that $n$ is a multiple of the order of $\tau$ and $x^{[n,\tau]} = \varepsilon(x)1$ for all $x \in H$. Since $\tau$ is a Hopf automorphism, $\varepsilon(\tau \cdot x) = \varepsilon(x)$ for any $x \in H$, and thus $\varepsilon(x^{[n,\tau]}) = \varepsilon(x^{[n]}) = \varepsilon(x)$. If $H$ is not semisimple and $S^2 \neq 1$ yet $S$ is still bijective, there is a more general definition of the twisted exponent in [SV2].

We will need the following proposition which is a special case of [SV2, Proposition 3.4].

**Proposition 3.1.** Suppose that the Hopf automorphism $\tau$ of the semisimple Hopf algebra $H$ has order $r$, $\exp_\tau(H)$ is finite, and $m$ is a positive integer. Then $x^{[mr,\tau]} = \varepsilon(x)1$ for all $x \in H$ if and only if $\exp_\tau(H)$ divides $m$.

Next we give some formulas for our Hopf algebras $K = A \sharp k^G$.

**Lemma 3.2.** Let $w = a \sharp p_x \in A \sharp k^G$, the smash coproduct as above. Then

$$(a \sharp p_x)^{[n]} = \sum_{z \in G, \ v^n=x} a^{[n,v^{-1}]} p_z.$$ 

In particular for $w = \Lambda_K = \Lambda_A p_1$, replace $z$ by $z^{-1}$. Then

$$\Lambda_{A}^{[n]} = \sum_{z \in G, \ v^n=1} \Lambda_A^{[n,v]} p_z.$$ 

**Proof.** A calculation shows that

$$(a \sharp p_x)^{[n]} = \sum_{z \in G, \ v^n=x} a_1(z^{-1} \cdot a_2)(z^{-2} \cdot a_3)\ldots(z^{-(n-1)} \cdot a_n) p_z,$$

which gives the first equation in the lemma. The second follows from the first. $\Box$

We now find a relation among the (twisted) exponents of $A$, $G$, and $K = A \sharp k^G$.

**Theorem 3.3.** The exponent of $K$ is the least common multiple of $\exp(G)$ and $\exp_z(A)$ for all $z \in G$.

**Proof.** Let $n = \exp(K)$, so that

$$(a \sharp p_x)^{[n]} = \varepsilon(a \sharp p_x)1 = \varepsilon(a)\delta_{x,1}1 = \varepsilon(a)\delta_{x,1} \sum p_z$$

for all $a \in A$ and $x \in G$. When $a = 1$, then $(p_x)^{[n]} = \delta_{x,1}1$ implies that $\exp(G) = \exp(k^G)$ divides $n$. Thus $z^n = 1$ for all $z \in G$. By the above calculation, $(a \sharp p_1)^{[n]} = \varepsilon(a)1$, and so by
Lemma 3.2, \( a^{[n, z^{-1}]} = \varepsilon(a) \) for all \( z \in G \) and \( a \in A \). Therefore by Proposition 3.1, \( \exp(K) \) is a common multiple of \( \exp(G) \) and \( \exp_z(A) \) for all \( z \in G \).

Now let \( m \) be any common multiple of \( \exp(G) \) and \( \exp_z(A) \) for all \( z \in G \). By Lemma 3.2 and Proposition 3.1,

\[
(a \zeta p_x)^[m] = \sum_{z \in G, z^m = x} a^{[m, z^{-1}]} p_z = \delta_1(x) \sum_{z \in G} a^{[m, z^{-1}]} p_z = \delta_1(x) \varepsilon(a) \sum_{z \in G} p_z = \varepsilon(a \zeta p_x) 1_K.
\]

Again by Proposition 3.1, \( \exp(K) \) divides \( m \). \( \square \)

We will use the following lemma in calculations.

**Lemma 3.4.** Let \( H \) be a Hopf algebra for which \( S^2 = 1 \) and let \( \tau \) be a Hopf automorphism of \( H \) whose order divides \( n \). Then \( S(x^{[n, \tau]}(a)) = \tau^{-1}(S(x^{[n, \tau^{-1}]})(a)) \) for all \( x \in H \).

**Proof.** Since \( S \) is an anti-algebra and anti-coalgebra map and \( \tau^n = 1 \) by hypothesis,

\[
S(x^{[n, \tau]}) = S \left( \sum_x x_1(\tau \cdot x_2)(\tau^2 \cdot x_3) \cdots (\tau^{n-1} \cdot x_n) \right) = \sum_x \tau^{n-1}(S(x_n)) \tau^{n-2}(S(x_{n-1})) \cdots \tau^2(S(x_3)) \tau(S(x_2)) S(x_1) = \sum_x \tau^{-1}(S(x_n)) \tau^{-2}(S(x_{n-1})) \cdots \tau^{-1(n-2)}(S(x_3)) \tau^{-1(n-1)}(S(x_2)) S(x_1) = \tau^{-1} \left( \sum_x S(x_n) \tau^{-1}(S(x_{n-1})) \cdots \tau^{-1(n-3)}(S(x_3)) \tau^{-1(n-2)}(S(x_2)) \tau^{-1(n-1)}(S(x_1)) \right) = \tau^{-1}(S(x^{[n, \tau^{-1}]})).
\]

We raise the question as to how the twisted exponent is affected if \( \tau \) is replaced by a suitable power of \( \tau \). If the order of \( \tau \) is \( n \), and \( m \) is relatively prime to \( n \), then a guess is that \( \exp_{\tau^m}(H) = \exp_{\tau}(H) \). At least it is true when \( m = n - 1 \):

**Lemma 3.5.** Let \( H \) be a Hopf algebra for which \( S^2 = 1 \) and let \( \tau \) be a Hopf automorphism of \( H \). Then \( \exp_{\tau^{-1}}(H) = \exp_{\tau}(H) \).

**Proof.** Assume that \( \exp_{\tau}(H) = n \). Then for all \( x \in H \),

\[
\sum_x x_1 \tau(x_2) \cdots \tau^{n-1}(x_n) = \varepsilon(x) 1.
\]

Apply the antipode \( S \) to this equation. Since \( S \) commutes with \( \tau \),

\[
\sum_x \tau^{n-1}(S(x_n)) \cdots \tau(S(x_2)) S(x_1) = \varepsilon(x) 1 = \sum_x \tau^{n-1}((Sx)_1) \cdots \tau((Sx)_{n-1}) Sx_n.
\]
Now replace $Sx$ by $y$ to see
\[ \sum_y \tau^{n-1}(y_1) \cdots \tau(y_{n-1}) y_n = \varepsilon(y) 1. \]

Finally apply $\tau$, using that $\tau^{-i} = \tau^{n-i}$:
\[ \sum_y y_1 \tau^{-1}(y_2) \cdots (\tau^{-1})^{n-2}(y_{n-1}) (\tau^{-1})^{n-1}(y_n) = \varepsilon(y) 1. \]

This proves the lemma. \(\square\)

4. Modules and Frobenius-Schur indicators

In this section, we take $A$ to be a semisimple Hopf algebra with action of a finite group $G$ by Hopf automorphisms, and let $K = A \natural k^G$, the smash coproduct as above. Then for any (left) $K$-module $M$, we may write
\[ M = \bigoplus_{x \in G} M_x \]
where $M_x = p_x \cdot M$ is a $K$-submodule of $M$ for each $x \in G$. Note that each $M_x$ is also an $A$-module, by restricting the action to $A$.

Let $\nu^K_m$ denote the $m$th Frobenius-Schur indicator for $K$-modules as in [KSZ], and let $\nu^A_{m,x}$ denote the $m$th twisted Frobenius-Schur indicator for $A$-modules, twisted by $x$, as in [SV]. That is, if $V$ is a $K$-module with character (or trace function) $\chi_V$, then
\[ \nu^K_m(V) = \chi_V(\Lambda^K_m) \]
where $\Lambda_K$ is the normalized integral of $K$. If $W$ is an $A$-module with character $\chi_W$ and $x$ is an automorphism of $A$ whose order divides $m$, then
\[ \nu^A_{m,x}(W) = \chi_W(\Lambda^A_{m,x}), \]
where $\Lambda_A$ is the normalized integral of $A$. See [SV] for general results on twisted indicators and for computations of $\nu^A_{m,x}$ when $A = H_8$, the smallest semisimple noncommutative, noncocommutative Hopf algebra.

Our next theorem gives a relationship between the Frobenius-Schur indicators of $K$ and the twisted Frobenius-Schur indicators of $A$.

**Theorem 4.1.** For every $K$-module $M$,
\[ \nu^K_m(M) = \sum_{x \in G, \ x^m = 1} \nu^A_{m,x^{-1}}(M_x). \]

**Proof.** Write $M = \bigoplus_{x \in G} M_x$ as before. Then $\nu^K_m(M) = \sum_{x \in G} \nu^K_m(M_x)$, and we will now compute $\nu^K_m(M_x)$ for an element $x$ of $G$, writing $\Lambda = \Lambda_A$ for ease of notation: By Lemma 3.2,
\[ \nu^K_m(M_x) = \chi_{M_x}(\Lambda^K_m) = \chi_{M_x}(\sum_{z \in G, \ z^m = 1} \Lambda^{[m,z]} p_z^{-1}) = \delta_{x^m,1} \chi_{M_x}(\Lambda^{[m,x^{-1}]}) = \delta_{x^m,1} \nu^A_{m,x^{-1}}(M_x). \]

Summing over all elements of $G$, we obtain the stated formula. \(\square\)
As a consequence, for example, if $x$ is an element of $G$ of order $n$ and $M$ is a $K$-module for which $M = M_x$ (i.e. $M_y = 0$ for all $y \neq x$), then $\nu^k_m(M) = 0$ for all $m < n$.

In our next result, we show that a twisted Frobenius-Schur indicator may always be realized as a Frobenius-Schur indicator for a smash coproduct. Let $\tau$ be any Hopf automorphism of $A$ of finite order $n$, and let $G = \langle \tau \rangle$ be the cyclic subgroup of the automorphism group generated by $\tau$. Set $K = A \otimes k^G$.

**Theorem 4.2.** For any $A$-module $N$, extend $N$ to be a $K$-module $M$ by letting $M_{\tau^{-1}} = N$ and $M_x = 0$ for all $x \in G, x \neq \tau^{-1}$. Then for every positive integer multiple $m$ of $n$,

$$\nu^A_m(N) = \nu^K_m(M).$$

Thus every value of a twisted indicator for $A$ is the value of an ordinary indicator for a smash coproduct over $A$.

**Proof.** By Theorem 4.1,

$$\nu^K_m(M) = \sum_{x \in G, x^m=1} \nu^A_{m,x^{-1}}(M_x) = \nu^A_{m,\tau}(M_{\tau^{-1}}) = \nu^A_{m,\tau}(N).$$

$\square$

**Example 4.3.** We illustrate the theorem using a non-trivial automorphism of $A = H_8$, the Kac-Palyutkin algebra of dimension 8 which is neither commutative nor cocommutative. The Hopf automorphism group was found in [SV], Section 4.2. Let $A$ be generated by $x, y, z$ with the usual relations $x^2 = y^2 = 1, z^2 = \frac{1}{2}(1 + x + y - xy), xy = yx, xz = zy$, where $x, y$ are group-like and $\Delta(z) = \frac{1}{2}(1 \otimes 1 + 1 \otimes x + y \otimes 1 - y \otimes x)(z \otimes z)$.

Let $\tau = \tau_1$ be the automorphism of $A$ of order 2 that interchanges $x$ and $y$ and sends $z$ to $\frac{1}{2}(-z + xz + yz + xy)$, and let $\chi$ be the character of the unique two-dimensional simple module $N$ of $A$. Then from [SV], $\nu^A_{2,\tau}(N) = -1$.

Letting $G = \langle \tau \rangle$ and $K = A \otimes k^G$, $N$ becomes a $K$-module $M$ by setting $M_{\tau} = N$ and $M_1 = 0$. Then $\nu^K_1(M) = -1$.

**5. Frobenius-Schur indicators for nonsemisimple Hopf algebras**

Let $A$ be a finite-dimensional Hopf algebra that is not necessarily semisimple. When $A$ is not semisimple, there does not exist a normalized integral, and so we cannot use the definition of indicator from the previous section. Instead we extend the work in [KMN] and define twisted Frobenius-Schur indicators for $A$ itself and obtain connections to Frobenius-Schur indicators of smash coproducts. Fix $\tau$, a Hopf automorphism of $A$ whose order divides the positive integer $m$. We define a variant of the $m$th twisted Hopf power map of $A$ to be $P_{m-1,\tau} : A \to A$, given by

$$P_{m-1,\tau}(a) = \sum \tau^{m-1}(a_1)\tau^{m-2}(a_2)\cdots \tau^2(a_{m-2})\tau(a_{m-1})$$

for all $a \in A$. We will use this map to define twisted Frobenius-Schur indicators, and then we will show how it relates to the twisted Hopf power maps defined in Section 3, by giving equivalent definitions of twisted Frobenius-Schur indicators in Theorem 5.1 and Corollary 5.2.

The $m$th twisted Frobenius-Schur indicator of $A$ is

$$\nu_{m,\tau}(A) := \text{Tr}(S \circ P_{m-1,\tau}).$$
the trace of the map $S \circ P_{m-1,\tau}$ from $A$ to $A$, where $S$ is the antipode of $A$.

We choose this definition as it specializes to the definition of the Frobenius-Schur indicator of the regular representation $A$ for an arbitrary finite-dimensional Hopf algebra in [KMN] when $\tau$ is the identity, and also to the definition of twisted Frobenius-Schur indicators in the semisimple case given in [SV, Theorem 5.1]. The indicator of the regular representation has also been considered in [Sh].

The following theorem generalizes part of [KMN, Theorem 2.2].

**Theorem 5.1.** Let $\Lambda$ be a left integral of $A$ and $\lambda$ a right integral of $A^*$ for which $\lambda(\Lambda) = 1$. Then

$$\nu_{m,\tau}(A) = \lambda(S(\Lambda)\tau^{[m,\tau]}).$$

**Proof.** By [R, Theorem 1],

$$\text{Tr}(S \circ P_{m-1,\tau}) = \sum \lambda(S(\Lambda_2)S \circ P_{m-1,\tau}(\Lambda_1))$$

$$= \sum \lambda(S(\Lambda_m)S(\tau^{m-2}(\Lambda_1)\tau^{m-3}(\Lambda_2)\cdots \tau(\Lambda_{m-1})))$$

$$= \sum \lambda(S(\Lambda_m)\tau(S(\Lambda_{m-1}))\cdots \tau^{m-1}(S(\Lambda_1)))$$

$$= \sum \lambda(S(\Lambda)\tau((S(\Lambda_2)\cdots \tau^{m-1}(S(\Lambda_m)))) = \lambda(S(\Lambda)\tau^{[m,\tau]}).$$

□

A similar proof to that of [KMN, Corollary 2.6] yields the following result that will be useful for computations.

**Corollary 5.2.** Let $\Lambda_r$ be a right integral of $A$ and $\lambda_r$ be a right integral of $A^*$ for which $\lambda_r(\Lambda_r) = 1$. Then

$$\nu_{m,\tau}(A) = \lambda_r(\Lambda_r\tau^{[m,\tau]}).$$

Similarly let $\Lambda_l$ be a left integral of $A$ and $\lambda_l$ be a left integral of $A^*$ for which $\lambda_l(\Lambda_l) = 1$. Then

$$\nu_{m,\tau}(A) = \lambda_l(\tau^{-1}(\Lambda_l\tau^{[m,\tau]})).$$

**Proof.** The first statement follows immediately from Theorem 5.1 and the fact that if $\Lambda_l$ is a left integral, then $\lambda_r := S(\Lambda_l)$ is a right integral, and the value of $\lambda_r$ on each is the same.

For the second statement, if $\lambda_r$ is a right integral, let $\lambda_l := \lambda_r \circ S$, a left integral of $A^*$. Then again by Theorem 5.1 and also Lemma 3.4,

$$\lambda_l(\tau^{-1}(\Lambda_l\tau^{[m,\tau]})) = \lambda_r(S(\tau^{-1}(\Lambda_l\tau^{[m,\tau]})))$$

$$= \lambda_r(\tau^{-1}(S(\Lambda_l\tau^{[m,\tau]})))$$

$$= \lambda_r(S(\Lambda_l)\tau^{[m,\tau]}) = \lambda_l(\Lambda_l\tau^{[m,\tau]}).$$

□

Now let $G$ be a group of Hopf algebra automorphisms of $A$, as in Section 2. The next result is a connection between twisted indicators of $A$ and indicators of the smash coproduct $K = A \triangleright \leftarrow k^G$.

**Theorem 5.3.** \[ \nu_m(K) = \sum_{g \in G, \ g^m = 1} \nu_{m,g}(A). \]
Proof. Note that $\Lambda_K = \Lambda \otimes p_1$ and $\lambda_K^* = \lambda \otimes (\sum_{z \in G} z)$ (since e.g. $\varepsilon(z \cdot a) = \varepsilon(a)$). By [KMN, Theorem 2.2] and our Lemma 3.4,

$$\nu_m(K) = \lambda_K^*(S_K(\Lambda_K^m)) = \left( \lambda \otimes \left( \sum_{z \in G} z \right) \right) \left( S_K \left( \sum_{g \in G, g^n = 1} \Lambda^m \otimes p_{g^{-1}} \right) \right)$$

$$= \sum_{g \in G, g^n = 1} \lambda(g(S(A_1g(A_2) \cdots g^{m-1}(A_m))))$$

$$= \sum_{g \in G, g^n = 1} \lambda(S(A))^{m,g^{-1}}$$

$$= \sum_{g \in G, g^n = 1} \nu_{m,g^{-1}}(A) = \sum_{g \in G, g^n = 1} \nu_{m,g}(A).$$

□

In the next section, we compute an example, a nonsemisimple Hopf algebra of dimension 8 and its Hopf automorphism group.

6. NONSEMISIMPLE EXAMPLE

Let $A$ be the Hopf algebra defined as

$$A = k\langle g, x, y \mid gx = -xg, gy = -yg, xy = -yx, g^2 = 1, x^2 = y^2 = 0 \rangle$$

with coalgebra structure given by:

$$\Delta(g) = g \otimes g, \varepsilon(g) = 1, S(g) = g,$$

$$\Delta(x) = x \otimes g + 1 \otimes x, \varepsilon(x) = 0, S(x) = gx,$$

$$\Delta(y) = y \otimes g + 1 \otimes y, \varepsilon(y) = 0, S(y) = gy.$$ 

The element $\Lambda = xy + xyg$ is both a right and left integral for $A$, and $\lambda = (xy)^*$ is both a right and left integral for $A^*$ such that $\lambda(\Lambda) = 1$.

Lemma 6.1. Let $V$ be the $k$-span of $x$ and $y$. Then $\text{Aut}(A) \cong GL_2(V)$.

Proof. This is close to the examples considered in [AD], as $A$ is pointed and generated by its group-like and skew-primitive elements. However we provide an elementary proof for completeness.

The coradical of $A$ is given by $A_0 = k\langle g \rangle$. Any automorphism $\tau$ of $A$ stabilizes $A_0$ and so fixes $g$. The next term of the coradical filtration is

$$A_1 = A_0 \oplus V \oplus gV,$$

since $V$ is the set of $(g,1)$-primitives and $gV$ is the set of $(1,g)$-primitives. Consequently $V$ and $gV$ are each stable under the action of $\tau$. But the $\tau$-action on $V$ determines the $\tau$-action on $gV$, and also on $A = A_1 \oplus W$, where $W$ is the span of $xy$ and $gxy$.

Conversely it is easy to check that any invertible linear action on $V$ preserves all of the relations of $A$, and thus gives an automorphism.

□
For an automorphism $\tau$ of order 2 or 3, we are able to compute some values of the indicators, using Corollary 5.2. We identify $\tau$ with a matrix
\[
\begin{pmatrix}
x \\
y
\end{pmatrix} \xrightarrow{\tau} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\
y \end{pmatrix},
\]
where $a, b, c, d \in k$, such that $\text{Det}(\tau) = ad - bc \neq 0$.

**Proposition 6.2.** Case (1). If $\tau^2 = 1$ and $m$ is even, then $\nu_{m, \tau}(A) = \frac{m^2}{2} (1 + \text{Det}(\tau))$. Consequently,
\[
\nu_{m, \tau}(A) = \begin{cases} m^2, & \text{if } \text{Det}(\tau) = 1 \\ 0, & \text{if } \text{Det}(\tau) = -1. \end{cases}
\]

Case (2). If $\tau^3 = 1$, then $\nu_{3, \tau}(A) = (\text{Tr}(\tau) + \text{Det}(\tau))^2 + (\text{Tr}(\tau) + 1)(1 - \text{Det}(\tau))$. Consequently,
\[
\nu_{3, \tau}(A) = \begin{cases} 9, & \text{if } \tau = \text{id} \\ 0, & \text{if } \tau \neq \text{id}. \end{cases}
\]

**Proof.** We verify the formulas by using the first part of Corollary 5.2.

Case (1): Recall that $\Lambda = xy + xyy$ and $\lambda = (xy)^*$ are right integrals. We must find $\lambda(\Lambda^{[m, \tau]})$. First we will show that $\lambda((xy)^{[m, \tau]}) = \frac{m^2}{2} (1 + \text{Det}(\tau))$, and then we will argue that $\lambda((xyg)^{[m, \tau]}) = \lambda((xy)^{[m, \tau]})$. In order to find $(xy)^{[m, \tau]}$, first note that
\[
\Delta^{m-1}(x) = x \otimes g^{\otimes m-1} + 1 \otimes x \otimes g^{\otimes m-2} + \cdots + 1^{\otimes i-1} \otimes x \otimes g^{\otimes i-i} + \cdots + 1^{\otimes m-1} \otimes x,
\]
\[
\Delta^{m-1}(y) = y \otimes g^{\otimes m-1} + 1 \otimes y \otimes g^{\otimes m-2} + \cdots + 1^{\otimes i-1} \otimes y \otimes g^{\otimes i-i} + \cdots + 1^{\otimes m-1} \otimes y,
\]
each sum consisting of $m$ terms. Set
\[
x_1 = x \otimes g^{\otimes m-1}, \ldots, x_i = 1^{\otimes i-1} \otimes x \otimes g^{\otimes i-i}, \ldots, x_m = 1^{\otimes m-1} \otimes x,
\]
the index indicating the position of $x$ in the tensor product, and similarly define $y_1, y_2, \ldots, y_m$.

Letting $\mu$ denote the multiplication map, by definition we have
\[
(xy)^{[m, \tau]} = \mu((1 \otimes \tau)^{\otimes \frac{m^2}{2}} \left( \sum_{i,j=1}^{m} x_i y_j \right)).
\]

Since $\tau(g) = g$ and $\lambda = (xy)^*$, in computing $\lambda((xy)^{[m, \tau]})$, the only terms in the above expansion of $(xy)^{[m, \tau]}$ yielding a nonzero value of $\lambda$ are those with an even number of factors of $g$. These are precisely the terms $x_i y_j$ for which $i, j$ have the same parity, of which there are $\frac{m^2}{2}$ terms. If $i, j$ are both odd (of which there are $\frac{m^2}{4}$ pairs), then in $(xy)^{[m, \tau]}$, the $(i, j)$ term is simply $xy$ by the following observations: (1) $\tau$ is applied only to factors of $g$ or 1, which are fixed by $\tau$, (2) if $i < j$, there are an even number of factors of $g$ between $x$ and $y$ after applying $\mu$, and (3) if $i > j$, there are an odd number of factors of $g$ between $x$ and $y$ after applying $\mu$ (since $x_i$ is to the left of $y_j$), so moving factors of $g$ to the right, past $x$, results in a factor of $(-1)$, and then applying the relation $yx = -xy$ results in another factor of $(-1)$, so that the end result is a term $xy$. If $i, j$ are both even (of which there are $\frac{m^2}{4}$ pairs), then in $(xy)^{[m, \tau]}$, the $(i, j)$ term is $\tau(xy) = \text{Det}(\tau)xy$, by similar reasoning. Therefore
\[
\lambda((xy)^{[m, \tau]}) = \lambda \left( \frac{m^2}{4} \text{Det}(\tau)xy + \frac{m^2}{4} \text{Det}(\tau)xy \right) = \frac{m^2}{4} (1 + \text{Det}(\tau)).
\]
Finally, in order to compute \( \lambda((xyg)^{[m,\tau]}) \), note that we need only include an extra factor of \( g^\otimes m \) on the right:

\[
(xyg)^{[m,\tau]} = \mu\left( (1 \otimes \tau)^\otimes m \right) \left( \sum_{i,j=1}^{m} x_i y_j \right) (g^\otimes m).
\]

Since \( m \) is even, the number of new factors of \( g \) to be included, in comparison to our previous calculation, is even, and so a similar analysis applies. One checks that the extra factors of \( g \) do not affect the result, and so

\[
\lambda((xyg)^{[m,\tau]}) = \lambda((xy)^{[m,\tau]}) = \frac{m^2}{4} (1 + \det(\tau)).
\]

Consequently, \( \nu_{m,\tau}(A) = \lambda(A^{[m,\tau]}) = \frac{m^2}{2} (1 + \det(\tau)) \).

To see the conclusion of Case (1), note that since \( \tau^2 = 1 \), the determinant of \( \tau \) is either 1 or \(-1\).

Case (2): A similar analysis applies. Note that \( \lambda((xy)^{[3,\tau]}) = \mu(1 \otimes \tau \otimes \tau^2)(\sum_{i,j=1}^{3} x_i y_j) \) and that \( \tau^2(x) = (a^2 + bc)x + b(a + d)y \), \( \tau^2(y) = c(a + d)x + (d^2 + bc)y \). In evaluating \( \lambda((xy)^{[3,\tau]}) \), we again need only consider \((i, j)\) terms for which \( i, j \) have the same parity. By contrast, in evaluating \( \lambda((xyg)^{[3,\tau]}) \), we need only consider \((i, j)\) terms for which \( i, j \) have different parity. Thus we find

\[
\lambda((xyg)^{[3,\tau]}) = \lambda(xy + x\tau^2(y) + y\tau^2(x)g + \tau^2(xy) + \tau(xy)) \\
= 1 + (d^2 + bc) + (a^2 + bc) + (a^2 + bc)(d^2 + bc) - bc(a + d)^2 + (ad - bc),
\]

\[
\lambda((xyg)^{[3,\tau]}) = \lambda(xyg^2\tau(y)g^4 + y\tau(x)g^5 + g\tau(x)g^2\tau^2(y)g + g\tau(y)g\tau^2(x)g^2) \\
= d + a + a(d^2 + bc) - bc(a + d) - bc(a + d) + d(a^2 + bc).
\]

Adding these together, we have

\[
\lambda(A^{[3,\tau]}) = 1 + a + d + a^2 + ad + d^2 + a^2d + ad^2 + a^2d^2 - abc - 2abcd - bcd + bc + b^2c^2 \\
= (\Tr(\tau) + \det(\tau))^2 + (1 + \Tr(\tau))(1 - \det(\tau)).
\]

To see the conclusion in Case (2), one can check the possible Jordan forms of the matrix for \( \tau \). \hfill \Box

7. TENSOR PRODUCTS AND CATEGORY OF MODULES

The following theorem generalizes [BW, Theorem 2.1] from the case that \( A \) is a group algebra, to the case that \( A \) is a Hopf algebra. Let \( K = A \bowtie k^G \) as before, and recall that for \( M \) a \( K \)-module and \( x \in G \), \( M_x \) denotes \( p_x \cdot M \), a \( K \)-submodule of \( M \), and \( M = \bigoplus_{x \in G} M_x \). If \( y \in G \), define \( yM_x \) to be \( M_x \) as a vector space, with \( A \)-module structure given by \( a \cdot yM_x = (y^{-1} \cdot a) \cdot M_x \) for all \( a \in A \), \( m \in M \).

**Theorem 7.1.** Let \( M, N \) be \( K \)-modules. Then

(i) \( (M \otimes N)_x \cong \bigoplus_{y \in G, \ yz = x} M_y \otimes yN_z \), and

(ii) \( (M^*)_x = x^{-1}(M_{x^{-1}}^*) \).
Proof. The proof is a straightforward generalization of that of [BW, Theorem 2.1]. We include details for completeness. We will prove the statement for modules of the form $M = M_y$, $N = N_z$. Let $\phi : M_y \otimes N_z \to M_y \otimes ^g N_z$, where the target module is a $K$-module on which $p_{yz}$ acts as the identity and $p_w$ acts as 0 for $w \neq yz$, be defined by $\phi(m \otimes n) = m \otimes n$ for all $m \in M_y$, $n \in N_z$. We check that $\phi$ is a $K$-module homomorphism: Let $x \in G$, $a \in A$. Apply $\Delta$ to $a \bowtie p_x$ to obtain

$$\phi((a \bowtie p_x)(m \otimes n)) = \sum \delta_{x,yz} \phi(a_1 m \otimes (y^{-1} \cdot a_2)n).$$

On the other hand,

$$\phi(a \bowtie p_x)(m \otimes n) = \sum \delta_{x,yz} a_1 m \otimes (y^{-1} \cdot a_2)n.$$

As $\phi$ is a bijection by its definition, it is an isomorphism of $K$-modules.

For each $y \in G$, let $\psi : y^{-1}(M_y)^* \to (M^*)_{y^{-1}}$ be the $K$-linear map defined by $\psi(f) = f$. Let $x \in G$, $a \in A$. Then

$$\psi((a \bowtie p_x)(f)) = f \circ ((x^{-1} \cdot S(a)) \bowtie p_{x^{-1}})$$

and this function is nonzero only if $x = y^{-1}$, in which case $\psi((a \bowtie p_x)(f))(m) = f((y \cdot S(a))m).

On the other hand,

$$(a \bowtie p_x)\psi(f)(m) = \psi(f)((S(a) \bowtie p_{x^{-1}} m) = \delta_{x,y^{-1}} f((y \cdot S(a))m).$$

Therefore $\psi$ is a $K$-module homomorphism, and since it is a bijection by its definition, it is an isomorphism of $K$-modules.

As a consequence of the theorem, the category of $K$-modules is equivalent to the semidirect product tensor category $\mathcal{C} \rtimes G$ where $\mathcal{C}$ is the category of $A$-modules. By definition, $\mathcal{C} \rtimes G$ is the category $\oplus_{g \in G} \mathcal{C}$, with objects $\oplus_{g \in G}(M_g, g)$ where each $M_g$ is an object of $\mathcal{C}$, and tensor product $(M, g) \otimes (N, h) = (M \otimes ^g N, gh)$. See [T], where the notation $\mathcal{C}[G]$ is used instead for this semidirect product category. For other occurrences of $\mathcal{C} \rtimes G$ in the literature, see, for example, [GNaNi, Ni].

References


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Department of Mathematics, University of Southern California, Los Angeles, CA
E-mail address: smontgom@usc.edu

Department of Mathematics, North Carolina State University, Raleigh, NC
E-mail address: mdvega@ncsu.edu

Department of Mathematics, Texas A&M University, College Station, TX
E-mail address: sjw@math.tamu.edu