A Module-Theoretic Approach to Clifford Theory for Blocks

S. J. Witherspoon*
Department of Mathematics, University of Toronto
Toronto, Ontario M5S 3G3, Canada
sjw@math.toronto.edu

Abstract

This work concerns a generalization of Clifford theory to blocks of group-graded algebras. A module-theoretic approach is taken to prove a one-to-one correspondence between the blocks of a fully group-graded algebra lying over a given block of its identity component, and conjugacy classes of blocks of a twisted group algebra. In particular, this applies to blocks of a finite group lying over blocks of a normal subgroup.

Introduction

In [8], Dade developed the block theory of group-graded algebras, in particular extending the Clifford correspondence to blocks of group-graded algebras. The Clifford correspondence for representations is a one-to-one correspondence between the irreducible representations of a group G having a given constituent when restricted to a normal subgroup, and the irreducible projective representations (with respect to a particular 2-cocycle) of a certain subgroup of the corresponding quotient group [3]. Dade's generalization is achieved by an analysis of centralizer algebras, and he has recently used it to make progress towards solving conjectures of Alperin and Dade [11, 12]. In this work we reinterpret these ideas, treating blocks as bimodules to provide a new proof of the Clifford correspondence for blocks of group-graded algebras. This approach clarifies the connection between the Clifford correspondence for blocks and that for representations or modules.

Specifically, let G be a finite group and A a *fully G-graded algebra* over an algebraically closed field k. That is, $A = \sum_{g\in G} A_g$ is a direct sum of subspaces A_g with $A_g A_h = A_{gh}$ for all $g, h \in G$. Here $A_g A_h$ denotes the set of all finite sums of products xy, where $x \in A_g$ and $y \in A_h$. The first example of a fully G-graded algebra is $A = k\Gamma$ for a group Γ with normal subgroup N and $G = \Gamma/N$. In this case a subspace A_g is spanned

*Research supported in part by NSERC grant # OGP0170281.

by the elements of the coset g. As the identity component is $A_1 = kN$, comparing blocks of A_1 to blocks of A amounts to comparing blocks of N to blocks of Γ.

We consider A to be a right $A^{op} \otimes A$-module, where A^{op} is the opposite algebra to A, via left and right multiplication by elements of A. A block of A is an indecomposable direct summand of the $A^{op} \otimes A$-module A. A block $\widetilde{B}$ of A *lies over* a block B of A_1 if B is a direct summand of the module $\widetilde{B}$ restricted to $A_1^{op} \otimes A_1$. That this definition is equivalent to that of Dade [8, p. 218] follows from the ideas contained in the proof of Lemma 1.1. The group G acts by conjugation on the blocks of A_1, an element $g \in G$ sending B to $A_{g^{-1}} B A_g$. In Section 3 we assume B is G-invariant and derive a one-to-one correspondence between the blocks of A lying over a given block B of A_1 and G-conjugacy classes of blocks of a twisted group algebra for a subgroup of G, as stated in the theorem below. A reduction to this invariant case is given in Section 1.

A starting point for the Clifford correspondence for blocks is provided by a more general correspondence for indecomposable modules discussed at the start of Section 2. In this context, we let $B\uparrow^{A_1^{op}\otimes A}$ denote the induced $A_1^{op} \otimes A$-module $B \otimes_{A_1^{op}\otimes A_1} (A_1^{op} \otimes A)$ and $\mathcal{E} = \mathrm{End}_{A_1^{op}\otimes A}(B\uparrow^{A_1^{op}\otimes A})$. Then $\mathcal{E}$ is a *G-graded algebra*; that is $\mathcal{E} = \sum_{g\in G} \mathcal{E}_g$ is a direct sum of subspaces $\mathcal{E}_g$ with $\mathcal{E}_g\mathcal{E}_h \subseteq \mathcal{E}_{gh}$ for all $g, h \in G$. In Section 2 we describe the Miyashita action of G on $\mathcal{E}$ which permutes the blocks. Let $J_G(\mathcal{E})$ denote the *graded Jacobson radical* of $\mathcal{E}$, the intersection of all maximal graded right ideals. We show in Section 3 that $\mathcal{E}/J_G(\mathcal{E})$ is a twisted group algebra for a subgroup of G.

Theorem (Clifford correspondence) *Let B be a G-invariant block of A_1. The blocks of A lying over B correspond one-to-one with G-conjugacy classes of blocks of the twisted group algebra $\mathcal{E}/J_G(\mathcal{E})$.*

In order to prove the theorem, in Section 2 we relate blocks of A lying over B to blocks of the endomorphism algebra $\mathcal{E}$ in the following way. Let Δ be the diagonal subalgebra $\sum_{g\in G}(A_{g^{-1}})^{op} \otimes A_g$ of $A^{op} \otimes A$. The identity component A_1 of A is naturally a Δ-module; we denote this Δ-module by $(A_1)_\Delta$. The $A^{op} \otimes A$-module $(A_1)_\Delta\uparrow^{A^{op}\otimes A}$ induced from Δ is isomorphic to A. If the block B of A_1 is G-invariant, then B is naturally a Δ-module, and so we obtain an ideal direct summand $B_\Delta\uparrow^{A^{op}\otimes A} \cong ABA$ of A. We show in Section 1 that the blocks of A lying over B are precisely the indecomposable direct summands of the $A^{op} \otimes A$-module $B_\Delta\uparrow^{A^{op}\otimes A}$. In turn, these direct summands are in one-to-one correspondence with the blocks of another endomorphism algebra, $\mathrm{End}_{A^{op}\otimes A}(B_\Delta\uparrow^{A^{op}\otimes A})$. The algebra $\mathcal{E}^G$ of fixed points of $\mathcal{E} = \mathrm{End}_{A_1^{op}\otimes A}(B\uparrow^{A_1^{op}\otimes A})$ is isomorphic to $\mathrm{End}_{A^{op}\otimes A}(B_\Delta\uparrow^{A^{op}\otimes A})$, and blocks of $\mathcal{E}^G$ correspond to G-conjugacy classes of blocks of $\mathcal{E}$, as we show in Section 2.

Our results also hold in the more general situation where k is replaced by a p-modular system (K, R, k) for a prime p, where k is algebraically closed. Here R is a complete discrete valuation ring with maximal ideal $\mathfrak{p}$, quotient field K, and residue class field $k = R/\mathfrak{p}$ of characteristic p. The basic theory needed includes the Krull-Schmidt-Azumaya Theorem [6, Proposition 56.4] and existence of projective covers of modules which follows from [5, Theorem 6.23] and [6, Propositions 56.2 and 56.4]. In this situation, A is a fully G-graded R-algebra (free and finitely generated as an R-module), and the twisted group algebra $\mathcal{E}/J_G(\mathcal{E})$ of the Clifford correspondence is a k-algebra as $J_G(\mathcal{E}) \supseteq \mathfrak{p}\mathcal{E}$. For simplicity however, we state our results in the case of a single base field k. All algebras and modules will be finite dimensional over k, and tensor products will be over k unless otherwise indicated.

The author thanks J. L. Alperin for suggesting this work and P. R. Boisen, E. C. Dade, and H. Ellers for helpful conversations.

1 An endomorphism algebra

In this section we give a correspondence between blocks lying over a given G-invariant block B of A_1 and blocks of the endomorphism algebra $\mathrm{End}_{A^{op}\otimes A}(B_\Delta \uparrow^{A^{op}\otimes A})$, at the same time giving a reduction to this invariant case. Let G_B be the subgroup of G fixing an arbitrary block B of A_1, that is

$$G_B = \{g \in G \mid A_{g^{-1}} B A_g = B\}.$$

Let $A_{G_B} = \sum_{g\in G_B} A_g$ and $\Delta_{G_B} = \sum_{g\in G_B} (A^{op})_g \otimes A_g$, where $(A^{op})_g = (A_{g^{-1}})^{op}$, a fully $\delta(G_B)$-graded algebra where $\delta(G_B) = \{(g,g) \mid g \in G_B\} \subseteq G \times G$. Note that B is naturally a Δ_{G_B}-module, denoted $B_{\Delta_{G_B}}$. We observe that if $\widetilde{B}$ is a block of A lying over B, so that B is a direct summand of $\widetilde{B}\downarrow_{A_1^{op}\otimes A_1}$, then $\widetilde{B}$ lies over all G-conjugates of B: The restricted module $\widetilde{B}\downarrow_{A_1^{op}\otimes A_1}$ is fixed by the G-action, as $\widetilde{B}$ is. So for any $g \in G$, $B^g = A_{g^{-1}} B A_g$ is also a direct summand of $\widetilde{B}\downarrow_{A_1^{op}\otimes A_1}$. In addition $\widetilde{B}$ lies over no other blocks of A_1, a consequence of the proof of the following lemma. We refer the reader to [2], [9], or [16] for facts about conjugate modules and induced modules for a fully group-graded ring.

Lemma 1.1 *The blocks of A lying over B correspond one-to-one with the blocks of* $\mathrm{End}_{A^{op}\otimes A}(B_{\Delta_{G_B}} \uparrow^{A^{op}\otimes A})$.

Proof: First we prove that the blocks of A lying over B correspond one-to-one with the indecomposable summands of the $A^{op}\otimes A$-module $B_{\Delta_{G_B}} \uparrow^{A^{op}\otimes A} \cong ABA$. Write $A_1 \cong B_1 \oplus \cdots \oplus B_k \oplus M$ as $A_1^{op}\otimes A_1$-modules, where $B = B_1, \ldots, B_k$ are the distinct conjugates of B, and M is the sum of the remaining blocks of A_1. Let $\Delta = \sum_{g\in G} (A^{op})_g \otimes A_g$. The induced $A^{op}\otimes A$-module $(A_1)_\Delta \uparrow^{A^{op}\otimes A}$ is isomorphic to A [2, Lemma 3.3]; this isomorphism is given simply by sending an element $\sum_i a_i \otimes (b_i \otimes c_i)$ of $(A_1)_\Delta\uparrow^{A^{op}\otimes A}$ to $\sum_i b_i a_i c_i$. As $B_1 \oplus \cdots \oplus B_k$ and M are naturally Δ-submodules of $(A_1)_\Delta$, we have

$$A \cong (B_1 \oplus \cdots \oplus B_k)_\Delta\uparrow^{A^{op}\otimes A} \oplus M_\Delta\uparrow^{A^{op}\otimes A}$$

as $A^{op}\otimes A$-modules. The Δ-module $(B_1\oplus\cdots\oplus B_k)_\Delta$ is isomorphic to $B_{\Delta_{G_B}}\uparrow^\Delta$ by an argument similar to that used in [2, Lemma 3.3]. Therefore $B_{\Delta_{G_B}}\uparrow^{A^{op}\otimes A}\cong B_{\Delta_{G_B}}\uparrow^\Delta\uparrow^{A^{op}\otimes A}\cong (B_1 \oplus \cdots \oplus B_k)_\Delta\uparrow^{A^{op}\otimes A}$, and so $A \cong B_{\Delta_{G_B}}\uparrow^{A^{op}\otimes A} \oplus M_\Delta\uparrow^{A^{op}\otimes A}$.

Let $\widetilde{B}$ be a block of A contained in $B_{\Delta_{G_B}}\uparrow^{A^{op}\otimes A}$. By Mackey's Theorem for group-graded algebras [16],

$$B_{\Delta_{G_B}}\uparrow^{A^{op}\otimes A}\downarrow_{A_1^{op}\otimes A_1}\cong \sum_{(s,t)\in\delta(G_B)\backslash G\times G} B\otimes_{A_1^{op}\otimes A_1}((A^{op})_s\otimes A_t).$$

A similar argument to [2, Lemma 3.3] shows that the $A_1^{op}\otimes A_1$-module $A_{s^{-1}}BA_t$ is isomorphic to the conjugate module $B\otimes_{A_1^{op}\otimes A_1}((A^{op})_s\otimes A_t)$. As B is indecomposable and A is fully graded, these conjugate modules $A_{s^{-1}}BA_t$ are indecomposable as well. Since $\widetilde{B}$ divides $B_{\Delta_{G_B}}\uparrow^{A^{op}\otimes A}$, the Krull-Schmidt Theorem now implies that some $A_{s^{-1}}BA_t$ divides $\widetilde{B}\downarrow_{A_1^{op}\otimes A_1}$. Therefore B is a summand of $A_s(\widetilde{B}\downarrow_{A_1^{op}\otimes A_1})A_{t^{-1}} \cong \widetilde{B}\downarrow_{A_1^{op}\otimes A_1}$; that is, $\widetilde{B}$ lies over B.

Now assume $\widetilde{B}$ is a block of A lying over B, but does not divide $B_{\Delta_{G_B}}\uparrow^{A^{op}\otimes A}$. Then $\widetilde{B}$ divides $M_\Delta\uparrow^{A^{op}\otimes A}$, and so B divides $M_\Delta\uparrow^{A^{op}\otimes A}\downarrow_{A_1^{op}\otimes A_1}$. As before, we apply Mackey's Theorem to $M_\Delta\uparrow^{A^{op}\otimes A}\downarrow_{A_1^{op}\otimes A_1}$, and conclude that B divides $A_{s^{-1}}MA_t$ for some $s,t\in G$. This implies that the conjugate $A_1^{op}\otimes A_1$-module $B^{s^{-1}} = A_sBA_{s^{-1}}$ divides $A_sA_{s^{-1}}MA_tA_{s^{-1}} = MA_{ts^{-1}}$. Letting e be the primitive central idempotent of A_1 corresponding to the block $B^{s^{-1}}$, we derive a contradiction, as $eB^{s^{-1}}\neq 0$ and $eM = 0$. Therefore the blocks of A lying over B are exactly the indecomposable summands of $B_{\Delta_{G_B}}\uparrow^{A^{op}\otimes A}$.

As a summand of A, $B_{\Delta_{G_B}}\uparrow^{A^{op}\otimes A}$ is an algebra, and $\mathrm{End}_{A^{op}\otimes A}(B_{\Delta_{G_B}}\uparrow^{A^{op}\otimes A})\cong Z(B_{\Delta_{G_B}}\uparrow^{A^{op}\otimes A})$, the center of $B_{\Delta_{G_B}}\uparrow^{A^{op}\otimes A}$. Thus blocks of $B_{\Delta_{G_B}}\uparrow^{A^{op}\otimes A}$ correspond one-to-one with blocks of $\mathrm{End}_{A^{op}\otimes A}(B_{\Delta_{G_B}}\uparrow^{A^{op}\otimes A})$. □

We next show that the blocks of A lying over B correspond one-to-one with the blocks of A_{G_B} lying over B, a reduction allowing us to consider only the G-invariant case from now on. This is a version of the Fong-Reynolds Theorem [15, V.2.5].

Lemma 1.2 *The blocks of A lying over B correspond one-to-one with the blocks of A_{G_B} lying over B. Further, this correspondence is given by induction of blocks of A_{G_B}, as $A_{G_B}^{op}\otimes A_{G_B}$-modules, to $A^{op}\otimes A$-modules.*

Proof: We prove the first statement by showing that there is an isomorphism of algebras

$$\mathrm{End}_{A_{G_B}^{op}\otimes A_{G_B}}(B_{\Delta_{G_B}}\uparrow^{A_{G_B}^{op}\otimes A_{G_B}}) \cong \mathrm{End}_{A^{op}\otimes A}(B_{\Delta_{G_B}}\uparrow^{A^{op}\otimes A}),$$

and applying Lemma 1.1. By [1, Proposition 2.8.3] we have two natural isomorphisms

$$\mathrm{End}_{A_{G_B}^{op}\otimes A_{G_B}}(B_{\Delta_{G_B}}\uparrow^{A_{G_B}^{op}\otimes A_{G_B}}) \cong \mathrm{Hom}_{\Delta_{G_B}}(B_{\Delta_{G_B}}, B_{\Delta_{G_B}}\uparrow^{A_{G_B}^{op}\otimes A_{G_B}}\downarrow_{\Delta_{G_B}})$$

and

$$\mathrm{End}_{A^{op}\otimes A}(B_{\Delta_{G_B}}\uparrow^{A^{op}\otimes A}) \cong \mathrm{Hom}_{\Delta_{G_B}}(B_{\Delta_{G_B}}, B_{\Delta_{G_B}}\uparrow^{A^{op}\otimes A}\downarrow_{\Delta_{G_B}}).$$

We may consider $B_{\Delta_{G_B}}\uparrow^{A_{G_B}^{op}\otimes A_{G_B}}\downarrow_{\Delta_{G_B}}$ as a direct summand of $B_{\Delta_{G_B}}\uparrow^{A^{op}\otimes A}\downarrow_{\Delta_{G_B}}$ by Mackey's Theorem [16]. In order to achieve the desired isomorphism, we need only show that all Δ_{G_B}-homomorphisms from $B_{\Delta_{G_B}}$ to $B_{\Delta_{G_B}}\uparrow^{A^{op}\otimes A}\downarrow_{\Delta_{G_B}}$ in fact have image contained in $B_{\Delta_{G_B}}\uparrow^{A_{G_B}^{op}\otimes A_{G_B}}\downarrow_{\Delta_{G_B}}$. Consider a summand $A_sBA_t = A_sBA_{s^{-1}}A_{st} = B^{s^{-1}}A_{st}$ of $B_{\Delta_{G_B}}\uparrow^{A^{op}\otimes A}\downarrow_{\Delta_{G_B}}$ as an $A_1^{op}\otimes A_1$-module. If $s \notin G_B$, then the primitive central idempotent e of A_1 associated with B yields the identity map on B but 0 on $B^{s^{-1}} \neq B$. In this case there are no $A_1^{op}\otimes A_1$-homomorphisms, and so no Δ_{G_B}-homomorphisms from $B_{\Delta_{G_B}}$ to A_sBA_t. A similar argument works if $s \in G_B$ and $t \notin G_B$. Therefore we have proved the first statement of the lemma.

By the proof of Lemma 1.1, blocks of A_{G_B} lying over B are the indecomposable $A_{G_B}^{op}\otimes A_{G_B}$-direct summands of $B_{\Delta_{G_B}}\uparrow^{A_{G_B}^{op}\otimes A_{G_B}}$, while blocks of A lying over B are the indecomposable $A^{op}\otimes A$-direct summands of $B_{\Delta_{G_B}}\uparrow^{A^{op}\otimes A} \cong (B_{\Delta_{G_B}}\uparrow^{A_{G_B}^{op}\otimes A_{G_B}})\uparrow^{A^{op}\otimes A}$. Suppose that $B_{\Delta_{G_B}}\uparrow^{A_{G_B}^{op}\otimes A_{G_B}} \cong \widetilde{B}_1 \oplus \cdots \oplus \widetilde{B}_n$ is a decomposition into indecomposable $A_{G_B}^{op}\otimes A_{G_B}$-modules. Then

$$B_{\Delta_{G_B}}\uparrow^{A^{op}\otimes A} \cong \widetilde{B}_1\uparrow^{A^{op}\otimes A} \oplus \cdots \oplus \widetilde{B}_n\uparrow^{A^{op}\otimes A}$$

as $A^{op}\otimes A$-modules. If some $\widetilde{B}_i\uparrow^{A^{op}\otimes A}$ were not indecomposable, there would be a contradiction to the first statement of the lemma. Therefore each $\widetilde{B}_i\uparrow^{A^{op}\otimes A}$ is a block of A. □

2 Another endomorphism algebra

In this section we discuss a Clifford correspondence for *indecomposable* modules which is implicit in Dade's work in [13]. This provides a starting point for the Clifford correspondence for blocks. We show how an endomorphism algebra arising in this indecomposable module situation is related to the endomorphism algebra of Lemma 1.1. We assume B is a G-invariant block of A_1, so that $G_B = G$.

Let V be an indecomposable right A_1-module, and $\mathcal{E} = \mathrm{End}_A(V\uparrow^A)$. The induced module $V\uparrow^A = V \otimes_{A_1} A$ is a graded module [18], and so $\mathcal{E}$ is a G-graded algebra with identity component $\mathcal{E}_1 \cong \mathrm{End}_{A_1}(V)$. One way to see this is to consider the isomorphisms of vector spaces

$$\mathcal{E} \cong \mathrm{Hom}_{A_1}(V, V\uparrow^A\downarrow_{A_1}) \cong \sum_{g\in G} \mathrm{Hom}_{A_1}(V, V^g)$$

that follow from [1, Proposition 2.8.3] and Mackey's Theorem [16]. Here V^g is the A_1-module $V \otimes_{A_1} A_g$, and $\mathcal{E}_g \cong \mathrm{Hom}_{A_1}(V, V^g)$ as a vector space. If V is G-invariant, so that $V^g \cong V$ for all $g \in G$, then $\mathcal{E}$ is *fully* G-graded, as may be seen from the above decomposition of $\mathcal{E}$ or [18, I.5.2]. If V is also irreducible then Schur's Lemma implies that $\mathrm{End}_A(V\uparrow^A)$ is a twisted group algebra. This is the twisted group algebra of the classical Clifford correspondence. In case V is indecomposable but not irreducible, or in case k is replaced by a complete discrete valuation ring R, the fact that $\mathcal{E}_1 \cong \mathrm{End}_{A_1}(V)$ is local [5, Proposition 6.10] replaces Schur's Lemma as follows: Let $J_G(\mathcal{E})$ denote the *graded Jacobson radical* of $\mathcal{E}$, the intersection of all maximal graded right ideals. By [4, Theorem 4.4] $J_G(\mathcal{E})$ is contained in the ordinary Jacobson radical $J(\mathcal{E})$. In case V is G-invariant, [13, Proposition 2.19], [5, Proposition 5.22], and [7, Lemma 14.2] imply that $\mathcal{E}/J_G(\mathcal{E})$ is a twisted group algebra for a subgroup of G. For the sake of completeness, we will give these arguments in greater detail later for the special case where V is a block.

Whether or not V is G-invariant, there is a one-to-one correspondence between indecomposable A-direct summands of $V\uparrow^A$ and indecomposable right $\mathcal{E}/J_G(\mathcal{E})$-direct summands of $\mathcal{E}/J_G(\mathcal{E})$: Indecomposable summands of $V\uparrow^A$ correspond one-to-one with indecomposable right summands of $\mathcal{E}$ by sending a summand of $V\uparrow^A$ to the corresponding projection endomorphism, an idempotent of $\mathcal{E}$ [5, Proposition 6.3]. As $J_G(\mathcal{E}) \subseteq J(\mathcal{E})$, indecomposable right $\mathcal{E}$-direct summands of $\mathcal{E}$ correspond with indecomposable right $\mathcal{E}/J_G(\mathcal{E})$-direct summands of $\mathcal{E}/J_G(\mathcal{E})$ [5, Corollaries 6.22 and 6.25].

In the block situation, we replace A by the fully G-graded algebra $A_1^{op} \otimes A$, A_1 by $A_1^{op} \otimes A_1$, and V by the block B of A_1. The relevant G-graded endomorphism algebra is then

$$\mathcal{E} = \mathrm{End}_{A_1^{op}\otimes A}(B\uparrow^{A_1^{op}\otimes A}),$$

and the Clifford correspondence for indecomposable modules described above applies here. It will need to be modified to yield the Clifford correspondence for blocks given in Theorem 3.5. We caution that, although B is G-invariant as a block, it may not be $1 \times G$-invariant as an $A_1^{op} \otimes A_1$-module. Thus it does not follow from the above arguments that $\mathcal{E}/J_G(\mathcal{E})$ is a twisted group algebra; in Section 3 we show that this fact follows from a characterization of the graded Jacobson radical given in [13]. In addition, we are interested in $A^{op} \otimes A$-modules rather than $A_1^{op} \otimes A$-modules. This is where a

G-action on $\mathcal{E}$ arises. The following proposition is Theorem 1.3 of [17]; see also [10, Theorem 2.1].

Proposition 2.1 (Miyashita action) *Let G be a finite group and S a fully G-graded algebra. Let M and N be S-modules, $g \in G$, and $\phi \in \mathrm{Hom}_{S_1}(M, N)$. Then there is a unique element $\phi^g \in \mathrm{Hom}_{S_1}(M, N)$ such that $\phi^g(ms_g) = \phi(m)s_g$ for all $m \in M$ and $s_g \in S_g$. If $M = N$, then G acts as algebra automorphisms of $\mathrm{End}_{S_1}(M)$.*

We give the definition of ϕ^g, which will be needed in the next section. As S is fully G-graded, we have $S_{g^{-1}}S_g = S_1$ for all $g \in G$. As $1 \in S_1$ [18], there are elements $\alpha_i \in S_{g^{-1}}$, $\beta_i \in S_g$ such that $\sum_{i=1}^{n} \alpha_i\beta_i = 1$. Then for all $m \in M$,

$$\phi^g(m) = \sum_{i=1}^{n} \phi(m\alpha_i)\beta_i.$$

In the block situation, we let $S = A^{op} \otimes A$ be the fully G-graded algebra with components $S_g = (A^{op})_g \otimes A$. The proposition gives a G-action on $\mathrm{End}_{A_1^{op}\otimes A}(B_\Delta\!\uparrow^{A^{op}\otimes A})$, where $\Delta = \sum_{g\in G}(A^{op})_g \otimes A_g$, and B_Δ is the Δ-module B. This provides the connection between $\mathcal{E} = \mathrm{End}_{A_1^{op}\otimes A}(B\!\uparrow^{A_1^{op}\otimes A})$ and the endomorphism algebra $\mathrm{End}_{A^{op}\otimes A}(B_\Delta\!\uparrow^{A^{op}\otimes A})$ of Lemma 1.1, as we see next. We note that $B_\Delta\!\uparrow^{A^{op}\otimes A}\downarrow_{A_1^{op}\otimes A} \cong B\!\uparrow^{A_1^{op}\otimes A}$, where the latter module is induced from $A_1^{op} \otimes A_1$. This follows from an application of Mackey's Theorem [16], as there is only one $\delta(G), 1\times G$-double coset in $G\times G$, and $B_\Delta\!\downarrow_{A_1^{op}\otimes A_1} = B$. Therefore

$$\mathrm{End}_{A_1^{op}\otimes A}(B_\Delta\!\uparrow^{A^{op}\otimes A}) \cong \mathrm{End}_{A_1^{op}\otimes A}(B\!\uparrow^{A_1^{op}\otimes A}) = \mathcal{E}.$$

Thus the above proposition yields an action of G as automorphisms of $\mathcal{E}$. The fixed point subalgebra is $\mathcal{E}^G \cong \mathrm{End}_{A^{op}\otimes A}(B_\Delta\!\uparrow^{A^{op}\otimes A})$, the endomorphism algebra of Lemma 1.1. We mention that if we had taken $S_g = A^{op} \otimes A_g$ instead, we would have gotten a different G-graded endomorphism algebra, but upon taking G-fixed points, we would have obtained the same $A^{op} \otimes A$-endomorphism algebra.

We now give the relationship between our endomorphism algebras and the centralizer algebras of [8]. Let $C_A(A_1)$ denote the centralizer in A of A_1. Let e be the primitive central idempotent of A_1 corresponding to the block B. Then as graded algebras, $\mathcal{E}$ is isomorphic to $eC_A(A_1)$: We identify $B\!\uparrow^{A_1^{op}\otimes A}$ with $\sum_{g\in G} BA_g$ as discussed in the proof of Lemma 1.1. Given $\phi \in \mathcal{E}$, ϕ is determined by $\phi(e)$, as $\phi(b) = \phi(eb) = \phi(e)b$ for any $b \in B\!\uparrow^{A_1^{op}\otimes A}$. Further, $\phi(e) \in eC_A(A_1)$ as $e\phi(e) = \phi(e^2) = \phi(e)$ and $\phi \in \mathcal{E}$ implies that $\phi(e)$ commutes with elements of A_1. Conversely, any element of $eC_A(A_1)$ defines an element of $\mathcal{E}$ in this way. The G-grading on $eC_A(A_1)$ inherited from A corresponds to that of $\mathcal{E}$, and the G-action on $\mathcal{E}$ provided by Proposition 2.1 gives rise to a G-action on $eC_A(A_1)$.

Lemma 2.2 *The subalgebras $\mathcal{E}^G$ and $\mathcal{E}_1$ of $\mathcal{E}$ are both contained in the center of $\mathcal{E}$.*

Proof: Let $\phi \in \mathcal{E}$ and $\psi \in \mathcal{E}_g$ for some $g \in G$, and suppose $\psi(e) = r_g \in eC_A(A_1)$, so that $\psi(b) = r_g b$ for all $b \in B{\uparrow}^{A_1^{op}\otimes A}$. Applying Proposition 2.1, $\phi^{g^{-1}} \circ \psi(e) = \phi^{g^{-1}}(r_g e) = r_g\phi(e) = \psi \circ \phi(e)$, so that $\phi^{g^{-1}} \circ \psi = \psi \circ \phi$. If $\phi \in \mathcal{E}^G$ then, ϕ is in the center of $\mathcal{E}$. If $\psi \in \mathcal{E}_1$, then $\phi \circ \psi = \psi \circ \phi$ for all $\phi \in \mathcal{E}$, so that ψ is in the center of $\mathcal{E}$. □

Next we prove that blocks of $\mathcal{E}^G$ correspond one-to-one with G-conjugacy classes of blocks of $\mathcal{E}$. Combined with Lemma 1.1 for $G = G_B$, this shows that blocks of A lying over B correspond one-to-one with G-conjugacy classes of blocks of $\mathcal{E}$.

Lemma 2.3 *The blocks of $\mathcal{E}^G \cong \mathrm{End}_{A^{op}\otimes A}(B_\Delta{\uparrow}^{A^{op}\otimes A})$ correspond one-to-one with G-conjugacy classes of blocks of $\mathcal{E}$.*

Proof: A G-conjugacy class of blocks of $\mathcal{E}$ corresponds to a G-conjugacy class $e_1, \ldots, e_k$ of primitive central idempotents of $\mathcal{E}$. Their sum $e' = e_1 + \cdots + e_k$ is a central idempotent of $\mathcal{E}^G$. If e' were not primitive, then it would decompose into primitive central idempotents of $\mathcal{E}^G$, each of which is a central idempotent of $\mathcal{E}$ by Lemma 2.2. Thus each decomposes into primitive central idempotents of $\mathcal{E}$, and by uniqueness these must be the $e_1, \ldots, e_k$. This contradicts the assumption that $e_1, \ldots, e_k$ is a single G-conjugacy class. Conversely, a block of $\mathcal{E}^G$ corresponds to a primitive central idempotent of $\mathcal{E}^G$, which is a central idempotent of $\mathcal{E}$, and decomposes into primitive central idempotents of $\mathcal{E}$. The G-action must permute these primitive central idempotents transitively. □

3 The Clifford correspondence

Let B be a G-invariant block of A_1. In this section we show that blocks of $\mathcal{E} = \mathrm{End}_{A_1^{op}\otimes A}(B{\uparrow}^{A_1^{op}\otimes A})$ correspond one-to-one with blocks of $\mathcal{E}/J_G(\mathcal{E})$, a version of [8, Lemma 3.1 and Theorem 3.5]. Together with the results of Sections 1 and 2, this yields the Clifford correspondence in Theorem 3.5. In order to achieve this correspondence, it seems necessary to introduce a *fully* group-graded subalgebra $\mathcal{E}[B]$ of $\mathcal{E}$, as is done for centralizer algebras in [8].

Let $G[B]$ be the subgroup of G defined by $g \in G[B]$ if $\mathcal{E}_g$ contains a unit, and $\mathcal{E}[B] = \mathcal{E}_{G[B]} = \sum_{g\in G[B]} \mathcal{E}_g$. Then $\mathcal{E}[B]$ is a fully $G[B]$-graded algebra, as for each $g \in G[B]$, $\mathcal{E}_g = u_g\mathcal{E}_1$ for a unit $u_g \in \mathcal{E}_g$. There are examples for which $G[B] \neq G$ [14, Lemma 3.11].

The graded Jacobson radical $J_G(\mathcal{E})$ of any group-graded algebra $\mathcal{E}$ is a graded two-sided ideal of $\mathcal{E}$ [18, Lemma I.7.4]. We give the characterization of $J_G(\mathcal{E})$ in [13], providing details for the sake of completeness. The components of $J_G(\mathcal{E})$ are given by

$$J_G(\mathcal{E})_g = \{\phi \in \mathcal{E}_g \mid \phi\mathcal{E}_{g^{-1}} \subseteq J(\mathcal{E}_1)\}.$$

This follows immediately from the one-to-one correspondence between all maximal G-graded right ideals M of $\mathcal{E}$ and all maximal right ideals N of $\mathcal{E}_1$ given by sending M to $N = M_1$, and N to M where $M_g = \{\phi \in \mathcal{E}_g \mid \phi\mathcal{E}_{g^{-1}} \subseteq N\}$. This correspondence is straightforward to verify. As B is an indecomposable $A_1^{op} \otimes A_1$-module, $\mathcal{E}_1 \cong \mathrm{End}_{A_1^{op}\otimes A_1}(B)$ is local, and so $\mathcal{E}_g - J_G(\mathcal{E})_g$ is the set of units in $\mathcal{E}_g$ [13, Lemma 2.18]. It follows that

$$J_G(\mathcal{E})_g = \begin{cases} J(\mathcal{E}_1)\mathcal{E}_g & \text{if } g \in G[B] \\ \mathcal{E}_g & \text{if } g \notin G[B] \end{cases},$$

as in [13, Proposition 2.19]. These arguments also apply to $\mathcal{E}[B]$, so in that case $J_{G[B]}(\mathcal{E}[B]) = J(\mathcal{E}_1)\mathcal{E}[B]$, and $\mathcal{E}[B]/J_{G[B]}(\mathcal{E}[B])$ is a twisted group algebra for $G[B]$ by [7, Lemma 14.2]. The above characterizations of $J_G(\mathcal{E})$ and $J_{G[B]}(\mathcal{E}[B])$ also immediately imply the following lemma.

Lemma 3.1 *There is an isomorphism of twisted group algebras*

$$\mathcal{E}/J_G(\mathcal{E}) \cong \mathcal{E}[B]/J_{G[B]}(\mathcal{E}[B]).$$

The proof of the next lemma, a reinterpretation of that of [8, Lemma 3.1], uses $\mathcal{E}_1 \subseteq Z(\mathcal{E}[B])$ and the fact that $J_{G[B]}(\mathcal{E}[B]) = J(\mathcal{E}_1)\mathcal{E}[B]$ in that Nakayama's Lemma is applied to $\mathcal{E}_1$-submodules of $\mathcal{E}[B]$. This shows the necessity of dealing with $\mathcal{E}[B]$ rather than $\mathcal{E}$.

Lemma 3.2 *The blocks of $\mathcal{E}[B]$ correspond one-to-one with blocks of $\mathcal{E}[B]/J_{G[B]}(\mathcal{E}[B])$.*

Proof: Let $\mathcal{E}[B]/J_{G[B]}(\mathcal{E}[B]) = b_1 \oplus \cdots \oplus b_k$ be a decomposition into blocks. Consider each b_i as a *right* $\mathcal{E}[B]$-module via the canonical map from $\mathcal{E}[B]$ to $\mathcal{E}[B]/J_{G[B]}(\mathcal{E}[B])$. Let $P(b_i)$ be the projective cover of b_i as a right $\mathcal{E}[B]$-module. As $J_{G[B]}(\mathcal{E}[B]) \subseteq J(\mathcal{E}[B])$ [4, Theorem 4.4], $\mathcal{E}[B]$ itself is the projective cover of $\mathcal{E}[B]/J_{G[B]}(\mathcal{E}[B])$ as a right $\mathcal{E}[B]$-module [5, Corollary 6.22]. As projective covers preserve direct sums we have $\mathcal{E}[B] \cong P(b_1) \oplus \cdots \oplus P(b_k)$ as a right $\mathcal{E}[B]$-module. We identify the $P(b_i)$ with right ideals of $\mathcal{E}[B]$ via this isomorphism. We will show that the $P(b_i)$ are also left ideals, and that they are indecomposable $\mathcal{E}[B]^{op} \otimes \mathcal{E}[B]$-modules; that is, the $P(b_i)$ are blocks of $\mathcal{E}[B]$.

We let $P'(b_i)$ be the projective cover of b_i as a *left* $\mathcal{E}[B]$-module, so that $\mathcal{E}[B] \cong P'(b_1) \oplus \cdots \oplus P'(b_k)$ as a left $\mathcal{E}[B]$-module. Consider the sum m_i of all blocks b_j with $j \neq i$. We have $\mathcal{E}[B] \cong P(b_i) \oplus P(m_i)$ as a right $\mathcal{E}[B]$-module, and $\mathcal{E}[B] \cong P'(b_i) \oplus P'(m_i)$ as a left $\mathcal{E}[B]$-module. Restrict these $\mathcal{E}[B]$-modules to $\mathcal{E}_1$-modules. As $\mathcal{E}_1$ is central in $\mathcal{E}[B]$ by Lemma 2.2, we do not distinguish between left and right modules. We have $P(b_i) = P(b_i)\mathcal{E}[B] \cong P(b_i)P'(b_i) \oplus P(b_i)P'(m_i)$ and $P(m_i) \cong P(m_i)P'(b_i) \oplus P(m_i)P'(m_i)$ as $\mathcal{E}_1$-modules, since clearly this is true without the direct sums, but for example $P(b_i)P'(b_i) \subseteq P'(b_i)$, $P(b_i)P'(m_i) \subseteq P'(m_i)$ and $P'(b_i) \cap P'(m_i) = 0$. Therefore

$$\mathcal{E}[B] \cong P(b_i)P'(b_i) \oplus P(b_i)P'(m_i) \oplus P(m_i)P'(b_i) \oplus P(m_i)P'(m_i)$$

as $\mathcal{E}_1$-modules. As $J_{G[B]}(\mathcal{E}[B]) = J(\mathcal{E}_1)\mathcal{E}[B]$, the right $\mathcal{E}_1$-module $P(b_i)P'(m_i)$ becomes 0 upon passing to the quotient $\mathcal{E}[B]/J_{G[B]}(\mathcal{E}[B])$, and so $P(b_i)P'(m_i)J(\mathcal{E}_1) = P(b_i)P'(m_i)$. By Nakayama's Lemma, $P(b_i)P'(m_i) = 0$, and similarly $P(m_i)P'(b_i) = 0$. Therefore

$$\mathcal{E}[B] \cong P(b_i)P'(b_i) \oplus P(m_i)P'(m_i)$$

as $\mathcal{E}_1$-modules, where $P(b_i) = P(b_i)P'(b_i) = P'(b_i)$, and $P(m_i) = P(m_i)P'(m_i) = P'(m_i)$. It follows that each $P(b_i)$ is both a left and a right ideal of $\mathcal{E}[B]$.

Suppose $P(b_i) = \widetilde{B}_1 \oplus \widetilde{B}_2$ as $\mathcal{E}[B]^{op} \otimes \mathcal{E}[B]$-modules. Under the canonical map from $\mathcal{E}[B]$ to $\mathcal{E}[B]/J_{G[B]}(\mathcal{E}[B])$, the image of one of $\widetilde{B}_1$ or $\widetilde{B}_2$ must be 0. Considering $\widetilde{B}_1$ and $\widetilde{B}_2$ as $\mathcal{E}_1$-modules, another application of Nakayama's Lemma implies that one of $\widetilde{B}_1$ or $\widetilde{B}_2$ is 0. □

Next we see that G-conjugacy classes of blocks of $\mathcal{E}/J_G(\mathcal{E})$ correspond one-to-one with G-conjugacy classes of blocks of $\mathcal{E}$, resulting in the Clifford correspondence. For this, we need a lemma.

Lemma 3.3 *The subgroup $G[B]$ is normal in G, and $(\mathcal{E}_h)^g = \mathcal{E}_{h^g}$ for all $g, h \in G$.*

Proof: Let $g, h \in G$, and $\phi \in \mathcal{E}_h$. As stated in the text following Proposition 2.1, $\phi^g(b) = \sum_i \beta_i \phi(\alpha_i b)$ for all $b \in B{\uparrow}^{A_1^{op}\otimes A}$, where $\alpha_i \in A_g$, $\beta_i \in A_{g^{-1}}$, and $\sum_i \beta_i\alpha_i = 1$. Let e be the primitive central idempotent of A_1 corresponding to the block B, so $\phi(e) \in A_h$ as discussed in the text preceding Lemma 2.2. As e is a G-invariant element of $C_A(A_1)$, e is central in A_1 so $\phi^g(e) = \sum_i \beta_i\phi(\alpha_i e) = \sum_i \beta_i \phi(e)\alpha_i$, since ϕ is an $A_1^{op} \otimes A$-map. But $\sum_i \beta_i\phi(e)\alpha_i \in A_{g^{-1}}A_hA_g = A_{h^g}$. Therefore $\phi^g \in \mathcal{E}_{h^g}$, and $(\mathcal{E}_h)^g \subseteq \mathcal{E}_{h^g}$. Conversely, let $\psi \in \mathcal{E}_{h^g}$, and let $\phi = \psi^{g^{-1}} \in (\mathcal{E}_{h^g})^{g^{-1}} \subseteq \mathcal{E}_h$ by the above argument. Then $\psi = \phi^g \in (\mathcal{E}_h)^g$, and so $\mathcal{E}_{h^g} = (\mathcal{E}_h)^g$.

To see that $G[B]$ is normal in G, let $h \in G[B]$, $g \in G$, and u_h a unit in $\mathcal{E}_h$. Then $(u_h)^g$ is a unit in $\mathcal{E}_{h^g}$. □

By the lemma, the G-action on $\mathcal{E}$ yields a G-action on $\mathcal{E}[B]$. Further, G fixes the ideal $J_{G[B]}(\mathcal{E}[B]) = J(\mathcal{E}_1)\mathcal{E}[B]$, as G fixes $\mathcal{E}_1$ by the lemma and so permutes the maximal ideals of $\mathcal{E}_1$. Therefore the G-action on $\mathcal{E}[B]$ induces a G-action on $\mathcal{E}[B]/J_{G[B]}(\mathcal{E}[B])$.

Lemma 3.4 *The G-conjugacy classes of blocks of $\mathcal{E}$ correspond one-to-one with G-conjugacy classes of blocks of $\mathcal{E}/J_G(\mathcal{E})$.*

Proof: By Lemmas 3.1 and 3.2, we need only show that G-conjugacy classes of blocks of $\mathcal{E}[B]$ correspond one-to-one with G-conjugacy classes of blocks of $\mathcal{E}$. Given a G-conjugacy class of blocks of $\mathcal{E}[B]$, there is a corresponding G-invariant central idempotent e of $\mathcal{E}[B]$, which is also then a G-invariant central idempotent of $\mathcal{E}$, as $\mathcal{E}^G$ is central in

$\mathcal{E}$. Consider the image $\overline{e}$ of e in $\mathcal{E}/J_G(\mathcal{E}) \cong \mathcal{E}[B]/J_{G[B]}(\mathcal{E}[B])$. If e may be decomposed in $\mathcal{E}$ as the sum of two nontrivial G-invariant central idempotents, this decomposition gives such a decomposition of $\overline{e}$, contradicting Lemma 3.2.

Conversely, a G-conjugacy class of blocks of $\mathcal{E}$ corresponds to a G-invariant central idempotent e of $\mathcal{E}$. This corresponds to an idempotent $\overline{e}$ of $\mathcal{E}/J_G(\mathcal{E}) \cong \mathcal{E}[B]/J_{G[B]}(\mathcal{E}[B])$, which by Lemma 3.2 corresponds to an idempotent e' of $\mathcal{E}[B]$. As $J_{G[B]}(\mathcal{E}[B]) \subseteq J_G(\mathcal{E})$, e and e' are central idempotents of $\mathcal{E}$ having the same image modulo $J_G(\mathcal{E})$, and so $e = e'$. □

The Clifford correspondence now follows immediately.

Theorem 3.5 *The blocks of A lying over B correspond one-to-one with G-conjugacy classes of blocks of the twisted group algebra $\mathcal{E}/J_G(\mathcal{E})$.*

Proof: By Lemma 1.1 for $G = G_B$ and Lemma 2.3, the blocks of A lying over B correspond one-to-one with G-conjugacy classes of blocks of $\mathcal{E}$. By Lemma 3.4, G-conjugacy classes of blocks of $\mathcal{E}$ correspond one-to-one with G-conjugacy classes of blocks of $\mathcal{E}/J_G(\mathcal{E})$. □

We close by returning to the general case where B is a block of A_1 that is not necessarily G-invariant, G_B is the set of all elements g of G such that $A_{g^{-1}}BA_g = B$, and $\mathcal{E} = \mathrm{End}_{A_1^{op}\otimes A_{G_B}}(B\uparrow^{A_1^{op}\otimes A_{G_B}})$. By Lemma 1.2 and Theorem 3.5 with G replaced by G_B, we have the following.

Corollary 3.6 *The blocks of A lying over B correspond one-to-one with G_B-conjugacy classes of blocks of the twisted group algebra $\mathcal{E}/J_{G_B}(\mathcal{E})$.*

References

[1] D. J. Benson, *Representations and Cohomology I: Basic representation theory of finite groups and associative algebras*, Cambridge University Press, 1991.

[2] P. R. Boisen, *The representation theory of fully group-graded algebras*, J. Algebra, 151 (1992), pp. 160–179.

[3] A. H. Clifford, *Representations induced in an invariant subgroup*, Ann. of Math., (2) 38 (1937), pp. 533–550.

[4] M. Cohen and S. Montgomery, *Group-graded rings, smash products, and group actions*, Trans. Amer. Math. Soc., 282 (1984), pp. 237–257.

[5] C. W. Curtis and I. Reiner, *Methods of Representation Theory with Applications to Finite Groups and Orders, Volume I*, Wiley, 1981.

[6] ——, *Methods of Representation Theory with Applications to Finite Groups and Orders, Volume II*, Wiley, 1987.

[7] E. C. Dade, *Compounding Clifford's theory*, Ann. of Math., 91 (1970), pp. 236–290.

[8] ——, *Block extensions*, Illinois J. Math., 17 (1973), pp. 198–272.

[9] ——, *Group-graded rings and modules*, Math. Z., 174 (1980), pp. 241–262.

[10] ——, *The equivalence of various generalizations of group rings and modules*, Math. Z., 181 (1982), pp. 335–344.

[11] ——, *Counting characters in blocks, I*, Inv. Math., 109 (1992), pp. 187–210.

[12] ——, *Counting characters in blocks, II*, J. reine angew. Math., 448 (1994), pp. 97–190.

[13] ——, *Clifford theory for blocks.* Preprint, 1995.

[14] H. Ellers, *Cliques of irreducible representations, quotient groups, and Brauer's theorems on blocks*, Can. J. Math., 47 (5) (1995), pp. 929–945.

[15] W. Feit, *The Representation Theory of Finite Groups*, North-Holland, 1982.

[16] J. Haefner, *Graded equivalence theory with applications*, J. Algebra, 172 (1995), pp. 385–424.

[17] Y. Miyashita, *On Galois extensions and crossed products*, J. Fac. Sci. Hokkaido Univ. Ser. I, 21 (1970), pp. 97–121.

[18] C. Năstăsescu and F. van Oystaeyen, *Graded Ring Theory*, North-Holland, 1982.