A HOPF STRUCTURE FOR DOWN-UP ALGEBRAS

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§1. Introduction

Down-up algebras were introduced in [BR] as a generalization of the algebra generated by the down and up operators on a partially ordered set. They have a presentation by generators and relations. Let α, β, γ be fixed elements of the field $\mathbb{K}$. The unital associative $\mathbb{K}$-algebra $A = A(\alpha, \beta, \gamma)$ with generators d, u and defining relations

$$\begin{aligned} d^2u &= \alpha dud + \beta ud^2 + \gamma d \\ du^2 &= \alpha udu + \beta u^2 d + \gamma u \end{aligned} \tag{1.1}$$

is a **down-up algebra**.

Examples of down-up algebras include the universal enveloping algebra $U(\mathfrak{sl}_2)$ of $\mathfrak{sl}_2$, which is the down-up algebra $A(2, -1, -2)$, and the universal enveloping algebra $U(H)$ of the Heisenberg Lie algebra H, which is $A(2, -1, 0)$. The quantized enveloping algebra $U_q(\mathfrak{sl}_3)$ of $\mathfrak{sl}_3$ has generators $E_i, F_i, K_i^{\pm 1}$, $i = 1, 2$, and defining relations which can be found in [J], for example. The subalgebra $U_q^+(\mathfrak{sl}_3)$ generated by E_1, E_2 is the down-up algebra $A([2], -1, 0)$, where $[2] = (q^2 - q^{-2})/(q - q^{-1}) = q + q^{-1}$. The down-up algebras $A([2], -1, 0)$ and $A(2q, -q^2, 0)$ are respectively the algebras H_q and H_q' studied in [KS] in connection with quantum harmonic oscillators and q-analogues of the universal enveloping algebra of the Heisenberg Lie algebra. Any complex down-up algebra $A(\alpha, \beta, \gamma)$ with not both α and β equal to zero is isomorphic to one of Witten's 7-parameter deformations of $U(\mathfrak{sl}_2)$ (see [B], [W1], [W2]).

If $\gamma \neq 0$, then the down-up algebra $A(\alpha, \beta, \gamma)$ is isomorphic to $A(\alpha, \beta, 1)$. Thus, there are essentially two different cases: $\gamma = 0$ and $\gamma = 1$. In this work we focus on the case that $\gamma = 0$. We consider the subgroup G of the automorphism group of A generated by two particular commuting semisimple automorphisms ω_1 and ω_2 which are defined using the roots r, s (which we assume are nonzero and live in $\mathbb{K}$) of the equation

$$t^2 - \alpha t - \beta = (t - r)(t - s) = 0. \tag{1.2}$$

This enables us to form the skew group algebra $B = B(\alpha, \beta, 0) = A * G$ whenever r and s are nonzero. We show that B has a Hopf algebra structure.

We study in detail the finite-dimensional simple modules of the algebra $B = A * G$, for $A = A(r + s, -rs, 0)$ over an algebraically closed field of characteristic zero. The down-up

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algebras $A(r+s,-rs,0)$ with r a primitive ath root of unity and s a primitive bth root of unity have the richest assortment of finite-dimensional simple modules. They are natural generalizations of the algebras $U_q^+(\mathfrak{sl}_3) = A(q+q^{-1},-1,0)$ at q a root of unity. For these down-up algebras, the automorphisms ω_1 and ω_2 have order ℓ where $\ell = \mathrm{lcm}(a,b)$.

Since G is a finite group when r and s are roots of unity, Clifford theory is particularly well-suited for determining the simple B-modules in that case. A generalization of classical Clifford theory to skew group algebras (due to Dade) allows us to give an explicit construction of finite-dimensional simple B-modules from finite-dimensional simple A-modules and simple modules for twisted group algebras of subgroups of G. We outline the necessary Clifford theory results in §3. In §§4–6, we assemble all the information needed to apply Clifford theory to $B = B(\alpha,\beta,0)$, including a classification of the finite-dimensional simple A-modules for arbitrary α and $\beta \neq 0$ (Theorem 4.18) and the structure of their stabilizer subgroups (Lemma 5.2 and Theorem 6.6). The simple A-modules have been studied before from various points of view in [BR], [CM], [Jo1], [Jo2], [Ku1], and [Ku2]. For our investigations we require a more explicit description of them. We determine all the finite-dimensional simple B-modules in the non root of unity case in §5 and for the root of unity case in §7. In the final section, we describe tensor products of the finite-dimensional simple B-modules on which d and u act nilpotently.

§2. A Hopf structure for $A(\alpha,\beta,0)$

Definition 2.1. *Assume $A = A(\alpha,\beta,0)$, where $\alpha = r+s$, $\beta = -rs$, and r,s are nonzero elements of $\mathbb{K}$. Let G be the subgroup of $\mathrm{Aut}(A)$ generated by ω_1, ω_2, which act on A by the following rules:*

$$\omega_1.d = rs^{-1}d \qquad \omega_1.u = su$$
$$\omega_2.d = r^{-1}d \qquad \omega_2.u = rs^{-1}u. \tag{2.2}$$

The algebra $B = B(r+s,-rs,0)$ is the skew group algebra $B = A * G$. That is, $B = \{\sum_{g\in G} a_g g \mid a_g \in A\}$, the free A-module with basis G, and with multiplication $(ag)(bh) = a(g.b)gh$ for all $a,b \in A$ and $g,h \in G$.

Since the relations defining $A = A(r+s,-rs,0)$ are homogeneous, it is easy to see that ω_1, ω_2 do in fact belong to the automorphism group of A.

Proposition 2.3. *For $B = B(r+s,-rs,0)$, ($rs \neq 0$) define*

$$\begin{array}{lll} \Delta(d) = 1\otimes d + d\otimes \omega_1 & S(d) = -d\omega_1^{-1} & \epsilon(d) = 0 \\ \Delta(u) = 1\otimes u + u\otimes \omega_2 & S(u) = -u\omega_2^{-1} & \epsilon(u) = 0 \\ \Delta(\omega_i^{\pm 1}) = \omega_i^{\pm 1}\otimes \omega_i^{\pm 1} & S(\omega_i^{\pm 1}) = \omega_i^{\mp 1} & \epsilon(\omega_i^{\pm 1}) = 1, \quad i = 1,2. \end{array} \tag{2.4}$$

Then (B,Δ,ϵ,S) is a (non-cocommutative) Hopf algebra.

We leave it to the reader to verify that if the maps Δ,ϵ,S are first defined on the free associative algebra generated by $d,u,\omega_i,\omega_i^{-1}$, $i = 1,2$, according to (2.4), that they preserve the relations coming from (1.1) and (2.2), and so induce algebra homomorphisms (and in the case of S, an algebra antihomomorphism) on B. Coassociativity, the co-unit property, and the property $m \circ (S \otimes \mathrm{id}) \circ \Delta = \imath \circ \epsilon = m \circ (\mathrm{id} \otimes S) \circ \Delta$ (where m is the multiplication on B and $\imath : \mathbb{K} \to B$ is the natural embedding) are straightforward to check.

An alternate approach is to note that $A = A(r+s, -rs, 0)$ is a coalgebra under $\Delta'(d) = d \otimes 1 + 1 \otimes d$ and $\Delta'(u) = u \otimes 1 + 1 \otimes u$, but it is not a bialgebra because these are not algebra homomorphisms. The algebra A is graded by the group G, and the group action preserves the grading. So one can form the biproduct, in the sense of Radford (see for example, [M, Chap. 10]), and the result is exactly the Hopf algebra $B = B(r+s, -rs, 0)$. We thank S. Montgomery for pointing out this other interpretation.

Examples 2.5. Consider the down-up algebra $A(q+q^{-1}, -1, 0) \cong U_q^+(\mathfrak{sl}_3)$, and observe for $r = q$ and $s = q^{-1}$ that $rs^{-1} = q^2$ and $r^{-1} = q^{-1} = s$. The relations in (2.2) imply that

$$\begin{aligned} \omega_1 d &= q^2 d\omega_1 & \qquad \omega_1 u &= q^{-1} u\omega_1 \\ \omega_2 d &= q^{-1} d\omega_2 & \qquad \omega_2 u &= q^2 u\omega_2. \end{aligned}$$

Therefore, via the mapping $d \mapsto E_1,\ u \mapsto E_2,\ \omega_i^{\pm 1} \mapsto K_i^{\pm 1},\ i = 1, 2$, the algebra $B(q + q^{-1}, -1, 0)$ is isomorphic (as a Hopf algebra) to the subalgebra of $U_q(\mathfrak{sl}_3)$ generated by $E_i, K_i^{\pm 1}$, $i = 1, 2$, when q is not a root of unity. When q is an ℓth root of unity and $\mathbb{K}$ has characteristic 0, then $B(q+q^{-1}, -1, 0)$ is isomorphic to the subalgebra of $U_q(\mathfrak{sl}_3)$ generated by $E_i, K_i^{\pm 1}$, modulo the relations $K_i^\ell = 1$, $i = 1, 2$, $\big($and $(K_1^{-1}K_2)^{\ell/3} = 1$ when 3 divides ℓ (see Example 7.2) $\big)$.

When $q = 1$, then $A(2, -1, 0) \cong U(H)$ (the enveloping algebra of the Heisenberg Lie algebra). In this particular case, ω_1 and ω_2 are the identity automorphisms, and $B(2, -1, 0) \cong A(2, -1, 0)$. We may identify these automorphisms with the unit element 1 of $A(2, -1, 0)$. Then the formulas for Δ reduce to the usual comultiplication for a universal enveloping algebra in which the elements d and u are primitive: $\Delta(d) = d \otimes 1 + 1 \otimes d,\ \ \Delta(u) = u \otimes 1 + 1 \otimes u$. These examples illustrate that the Hopf structure we have introduced on down-up algebras is a natural extension of known Hopf structures.

Remark 2.6. There is another Hopf structure associated to $A(r+s, -rs, 0)$ which can be gotten by interchanging the roles of r and s. Thus, if we define

$$\begin{aligned} \omega_1^\dagger.d &= r^{-1}sd & \qquad \omega_1^\dagger.u &= ru \\ \omega_2^\dagger.d &= s^{-1}d & \qquad \omega_2^\dagger.u &= r^{-1}su, \end{aligned} \tag{2.7}$$

then these automorphisms may be used in place of ω_1, ω_2 in Proposition 2.3 to obtain a Hopf algebra $B^\dagger(r+s, -rs, 0)$. The algebra $B^\dagger(r+s, -rs, 0)$ is isomorphic to $B(r^{-1} + s^{-1}, -r^{-1}s^{-1}, 0)$ as a Hopf algebra by the mapping: $d \mapsto u'$, $u \mapsto d'$, $\omega_1^\dagger \mapsto \omega_2'$, and $\omega_2^\dagger \mapsto \omega_1'$, which extends the isomorphism from $A(r+s, -rs, 0)$ to $A(r^{-1}+s^{-1}, -r^{-1}s^{-1}, 0)$ given by $d \mapsto u'$, $u \mapsto d'$. This shows that the apparent lack of symmetry in the relations in (2.2) in fact does not exist.

§3. Clifford theory

Suppose $B = A * G = \{\sum_{g \in G} a_g g \mid a_g \in A\}$ is a skew group algebra formed from an algebra A and a group G of automorphisms of A, so that the multiplication in B is given by $(ag)(bh) = a(g.b)gh$ for all $a, b \in A$ and $g, h \in G$. (A standard reference for skew-group algebras is [P].)

Assume V is a simple A-module for which Schur's Lemma holds, i.e. $\mathrm{End}_A(V) = \mathbb{K}\,\mathrm{Id}$, and let $\varrho : A \longrightarrow \mathrm{End}_{\mathbb{K}}(V)$ be the associated representation. For each $g \in G$, we can define a new A-module ${}^gV$, the *conjugate* of V by g, which is V as a vector space, but has A-action given by $a \cdot_g v = (g^{-1}.a) \cdot v$ for all $a \in A$, and $v \in V$. Thus, if ${}^g\varrho : A \to \mathrm{End}_{\mathbb{K}}(V)$ denotes the corresponding representation, then ${}^g\varrho(a) = \varrho(g^{-1}.a)$ for all $a \in A$. The *stabilizer* (or *inertia subgroup*) of V in G is

$$H = H_V = \{g \in G \mid {}^gV \cong V\}.$$

For each $g \in H$, let $t_g : {}^gV \to V$ be an A-module isomorphism. That is, $t_g \in \mathrm{End}_{\mathbb{K}}(V)$ and

$$\varrho(a)t_g = t_g\, {}^g\varrho(a) = t_g\, \varrho(g^{-1}.a) \tag{3.1}$$

for all $a \in A$ and $g \in H$. Note that t_g is unique up to a scalar multiple by Schur's Lemma. Since

$$\varrho(a)t_gt_h = t_g\, \varrho(g^{-1}.a)t_h = t_gt_h\, \varrho(h^{-1}g^{-1}.a) = t_gt_h\, \varrho((gh)^{-1}.a),$$

we see that $t_gt_h = \chi(g,h)t_{gh}$ for some 2-cocycle $\chi : H \times H \to \mathbb{K}^*$. Let $\psi = \chi^{-1}$ and let $\mathbb{K}^\psi H = \mathrm{span}_{\mathbb{K}}\{s_h \mid h \in H\}$ be the *twisted group algebra* with multiplication given by

$$s_gs_h = \psi(g,h)s_{gh}.$$

The structure of $\mathbb{K}^\psi H$ depends only on the image of ψ in $\mathrm{H}^2(H, \mathbb{K}^*)$ [K, Lemma 3.2.2].

The basic idea of Clifford theory is that simple modules for $B = A * G$ can be constructed from simple A-modules V and simple $\mathbb{K}^\psi H$-modules for $H = H_V$ using induction. We will assume that G is finite for the rest of this section; later we will see that this is the case whenever B has finite-dimensional simple modules of dimension larger than one (in case $\mathbb{K}$ is algebraically closed of characteristic 0). We will also assume all modules are finite-dimensional for the rest of this section.

Let V be as above (a simple A-module whose stabilizer is H and whose associated representation is $\varrho : A \to \mathrm{End}_{\mathbb{K}}(V)$), and suppose Y is a simple $\mathbb{K}^\psi H$-module. There is an $A * H$-module action on $V \otimes_{\mathbb{K}} Y$ defined by

$$(ah)(v \otimes y) = \varrho(a)t_hv \otimes s_hy. \tag{3.2}$$

(See [CR, Thm. 11.17(ii)].)

Clifford theory provides the following one-to-one correspondences on simple modules, which may be deduced from [CR, Prop. 11.16 and Thm. 11.17]. The assumption in [CR] that B is finitely generated over $\mathbb{K}$ is not necessary for these results.

I. Simple $\mathbb{K}^\psi H$-modules correspond one-to-one (up to isomorphism) with simple $A * H$-modules which contain V upon restriction to A. The map is $Y \mapsto V \otimes_{\mathbb{K}} Y$, where the $A * H$-action on $V \otimes_{\mathbb{K}} Y$ is specified by (3.2).

II. Simple $A * H$-modules containing V on restriction to A correspond one-to-one (up to isomorphism) with simple $A*G$-modules containing V on restriction to A. The correspondence is given by sending the $A * H$-module M to the induced module $(A * G) \otimes_{A*H} M$.

If we apply I and II of Clifford theory to a conjugate ${}^gV$ of V, we obtain a set of modules isomorphic to the ones gotten from V. Consequently, choosing a representative simple module from each isomorphism class and applying I and II to them will produce all the distinct finite-dimensional simple B-modules up to isomorphism.

Therefore, in order to determine the finite-dimensional simple B-modules, we first describe the finite-dimensional simple A-modules V and their stabilizers H in G. Then we analyze the structure of the twisted group algebras $\mathbb{K}^{\psi}H$ for the purpose of obtaining information about the simple $\mathbb{K}^{\psi}H$-modules Y.

§4. The module theory for $A(\alpha, \beta, 0)$

Verma modules and their simple quotients.

Henceforth we will assume that $\mathbb{K}$ is an algebraically closed field of characteristic zero. In this section we will temporarily suspend our assumption that r and s are nonzero.

In our discussion of the modules for $A = A(\alpha, \beta, 0)$, the Verma modules, which were introduced in [BR], play a critical role. For each $\lambda \in \mathbb{K}$, the corresponding Verma module $V(\lambda) = \text{span}_{\mathbb{K}}\{v_n \mid n = 0, 1, \dots\}$ has A-action given by

$$u \cdot v_n = v_{n+1} \qquad d \cdot v_n = \lambda_{n-1} v_{n-1},$$

where $v_{-1} = 0$, and the coefficients λ_n satisfy the recursion $\lambda_{-1} = 0$, $\lambda_0 = \lambda$, and

$$\lambda_n = \alpha\lambda_{n-1} + \beta\lambda_{n-2}, \quad n \geq 1. \tag{4.1}$$

The solutions to this recursion were determined explicitly in [BR, Prop. 2.12], and from that result, we have

Proposition 4.2. *Suppose $t^2 - \alpha t - \beta = (t - r)(t - s)$.*

(i) *If $r \neq s$, then for all $n \geq -1$,*

$$\lambda_n = [n + 1]\lambda \qquad \text{where} \qquad [n + 1] = \frac{s^{n+1} - r^{n+1}}{s - r}.$$

(ii) *If $r = s$, then for all $n \geq -1$,*

$$\lambda_n = (n + 1)r^n\lambda.$$

(It is interesting to note that if we view s as a variable and r as a constant in (i) and take the limit as s goes to r, then by L'Hôpital's rule, the expression for λ_n in (i) becomes the expression in (ii).)

A Verma module is simple if and only if $\lambda_n \neq 0$ for any $n \geq 0$ ([BR, Prop. 2.4]). From Proposition 4.2 we see that $V(\lambda)$ is simple unless:

(4.3)
(i) $\lambda = 0$, (and r, s are arbitrary);
(ii) $r \neq s$ but $s^k = r^k$ for some $k \geq 2$,
(iii) $r = s = 0$, and λ is arbitrary.

We suppose that $m \geq 1$ is minimal such that $\lambda_{m-1} = 0$ and let

$$M(\lambda) = \mathrm{span}_{\mathbb{K}}\{v_n \mid n \geq m\}. \tag{4.4}$$

If no such m exists, $V(\lambda)$ is simple, and we set $M(\lambda) = (0)$. In either event, $M(\lambda)$ is a maximal submodule, and

$$L(\lambda) = V(\lambda)/M(\lambda) \tag{4.5}$$

is a simple A-module.

In a down-up algebra A, the elements du and ud commute (see [BR]), and, as shown in [KMP], they generate a polynomial algebra if and only if $\beta \neq 0$. The basis vectors of a Verma module are simultaneous eigenvectors for du and ud with

$$du \cdot v_n = \lambda_n v_n \qquad \text{and} \qquad ud \cdot v_n = \lambda_{n-1} v_n.$$

It is helpful to think of $\mathfrak{h} = \mathbb{K}du \oplus \mathbb{K}ud$ as the "Cartan subalgebra" of A, and to say that the linear functional $\Lambda_n \in \mathfrak{h}^*$ with $\Lambda_n(du) = \lambda_n$ and $\Lambda_n(ud) = \lambda_{n-1}$ is the *weight* of the vector v_n. As an abbreviation it is convenient to write $\Lambda_n = (\lambda_n, \lambda_{n-1})$.

Theorem 2.13 of [BR] determines when two weights Λ_n and $\Lambda_{n'}$ for an arbitrary down-up algebra $A(\alpha, \beta, \gamma)$ are the same. As in [BR, Prop. 2.23], this information facilitates the determination of the maximal submodules. When $\gamma = 0$, the calculations reduce considerably, and the expressions in Proposition 4.2 above (or Theorem 2.13 of [BR]) can be used to deduce the following:

Proposition 4.6. *Suppose $t^2 - \alpha t - \beta = (t-r)(t-s)$. Assume for $A = A(\alpha, \beta, 0)$ and $\lambda \neq 0$ that two weights of a Verma module $V(\lambda)$ are equal, that is $\lambda_n = \lambda_{n'}$ and $\lambda_{n-1} = \lambda_{n'-1}$ for some n, n'. Then one of the following cases holds:*

(i) *$r \neq s$ and $r^\ell = 1 = s^\ell$ for some ℓ;*
(ii) *$r = 0$ and $s^\ell = 1$, or $s = 0$ and $r^\ell = 1$ for some $\ell \geq 1$;*
(iii) *$r = s = 0$.*

Proposition 4.7. *For $A = A(r+s, -rs, 0)$, the simple quotients of the Verma module $V(\lambda)$ are the following:*

(1) *when $\lambda = 0$,*

$$L(0, \xi) = V(0)/\mathrm{span}_{\mathbb{K}}\{v_{n+1} - \xi v_n \mid n = 0, 1, \dots\}, \qquad \xi \in \mathbb{K}$$

(2) *when $\lambda \neq 0$,*

(i) *$r = s = 0$:* $L(\lambda) = V(\lambda)/\mathrm{span}_{\mathbb{K}}\{v_n \mid n \geq 2\}$
(ii) *$r = s \neq 0$:* $L(\lambda) = V(\lambda)$
(iii) *$r \neq s$ and s/r is not a root of unity:* $L(\lambda) = V(\lambda)$
(iv) *s/r a primitive mth root of unity for $m \geq 2$, but neither r nor s a root of unity:* $L(\lambda) = V(\lambda)/\mathrm{span}_{\mathbb{K}}\{v_n \mid n \geq m\}$
(v) *$r \neq s$, r and s primitive ath and bth roots of unity respectively:*

$$L(\lambda,\xi) = V(\lambda)/\mathrm{span}_{\mathbb{K}}\{v_{n+\ell} - \xi^\ell v_n \mid n \geq 0\}, \qquad \xi \in \mathbb{K}^*$$
$$L(\lambda) = V(\lambda)/\mathrm{span}_{\mathbb{K}}\{v_n \mid n \geq m\}, \qquad \textit{where}$$

$$\ell = \mathrm{lcm}(a,b), \ \ \textit{and} \ \ m \geq 2 \ \ \textit{is minimal so} \ \ r^m = s^m.$$

Proof. When $\lambda = 0$, the maximal submodules of $V(\lambda)$ have the form $M_\xi = \mathrm{span}_{\mathbb{K}}\{v_{n+1} - \xi v_n \mid n \geq 0\}$, where $\xi \in \mathbb{K}$, so that the simple quotients are $L(0,\xi) = V(0)/M_\xi$ as in (1).

We suppose in what follows that $\lambda \neq 0$. We consider the three cases in Proposition 4.6. First, if $r = s = 0$, there are three distinct weights in any $V(\lambda)$: $\Lambda_0 = (\lambda, 0)$, $\Lambda_1 = (0, \lambda)$, and $(0,0) = \Lambda_2 = \Lambda_3 = \dots$. If M is a submodule, M decomposes into weight spaces. If Λ_1 is a weight of M, then M contains v_1. Applying d we see M contains v_0, hence it equals $V(\lambda)$. Similarly, if Λ_0 is a weight, then $M = V(\lambda)$. Thus, every proper submodule is contained in $M(\lambda) = \mathrm{span}_{\mathbb{K}}\{v_n \mid n \geq 2\}$, and $L(\lambda) = V(\lambda)/M(\lambda)$ is the unique simple quotient. When $r = s \neq 0$ as in (2)(ii), we have $\lambda_n = (n+1)r^n\lambda$, and every Verma module $V(\lambda)$ with $\lambda \neq 0$ is simple.

Finally, consider the case that $r \neq s$, where $\lambda_n = \lambda(s^{n+1} - r^{n+1})/(s - r)$. Here $V(\lambda)$ is not simple if and only if $s^m = r^m$ for some $m \geq 2$ (which can be taken to be minimal with that property). The conditions $s^m = r^m$ and $s \neq r$ force s, r to be nonzero, and $\rho = s/r$ is a primitive mth root of 1. Then $\lambda_n = \lambda r^n(\rho^{n+1} - 1)/(\rho - 1)$. Two weights Λ_n and $\Lambda_{n'}$ are equal if and only if $\lambda_n = \lambda_{n'}$ and $\lambda_{n-1} = \lambda_{n'-1}$. Multiplying the second equation by $r\rho$ and subtracting it from the first, we determine that $r^{n-n'}(\rho - 1) = (\rho - 1)$. Since $\rho \neq 1$, this implies r must be a root of unity. (This is (i) of Proposition 4.6.)

If r is not a root of unity, the weights are distinct. A submodule M contains a weight vector, hence some v_j. If $j < m$, then by applying d^j we get $v_0 \in M$ so that $M = V(\lambda)$. Otherwise, $M \subseteq M(\lambda) = \mathrm{span}_{\mathbb{K}}\{v_n \mid n \geq m\}$. This shows that when r is not a root of unity as in (2)(iv), $M(\lambda)$ is the unique maximal submodule and $L(\lambda)$ is the unique simple quotient.

When r is a root of unity (say a primitive ath root of unity), then $s = r\rho$ is a root of unity (say a primitive bth root). Set $\ell = \mathrm{lcm}(a,b)$. Then $\lambda_{\ell-1} = 0$ so that $\ell \geq m$. The weights $\Lambda_0, \Lambda_1, \dots, \Lambda_{\ell-1}$ are distinct, and $\Lambda_{n+\ell} = \Lambda_n$ for all $n \geq 0$. Suppose M is a submodule of $V(\lambda)$ not contained in $M(\lambda) = \mathrm{span}_{\mathbb{K}}\{v_n \mid n \geq m\}$. Then M contains some weight vector $v_j + a_1 v_{\ell+j} + a_2 v_{2\ell+j} + \cdots + a_k v_{k\ell+j}$ with $0 \leq j \leq m-1$. Applying d^j we obtain $v_0 + a_1 v_\ell + a_2 v_{2\ell} + \cdots + a_k v_{k\ell} \in M$. Thus, for $p(t) = a_k t^k + \cdots + a_1 t + 1$, we have $p(u^\ell)v_0 \in M$. Hence, $f(u)p(u^\ell)v_0 \in M$ for all polynomials $f(t)$. Moreover, if $p(t)$ is chosen to have minimal degree so that $p(u^\ell)v_0 \in M$, then every vector in M has this form. The module M will be maximal if and only if $p(t)$ is linear - say $p(t) = t - \xi^\ell$ for some $\xi \in \mathbb{K}^*$. Thus, $M(\lambda,\xi) = \mathrm{span}_{\mathbb{K}}\{v_{n+\ell} - \xi^\ell v_n \mid n = 0, 1, \dots\}$ is a maximal submodule and $L(\lambda,\xi) = V(\lambda)/M(\lambda,\xi)$ is a simple ℓ-dimensional quotient for any choice of $\xi \in \mathbb{K}^*$ (see [BR, Add]). The other simple quotient of $V(\lambda)$ is $L(\lambda) = V(\lambda)/M(\lambda)$ where $M(\lambda) = \mathrm{span}_{\mathbb{K}}\{v_n \mid n \geq m\}$ and m is the minimal value such that $s^m = r^m$. This is the assertion in (2)(v). □

This theorem tell us that when $\lambda \neq 0$, the simple quotients of $V(\lambda)$ are just the modules $L(\lambda) = V(\lambda)/M(\lambda)$ except when r is a primitive ath root of unity and s is a primitive bth root of unity for some a and b.

Example 4.8. In (v) it may happen that $m < \ell$. For example, suppose that r is a primitive 6th root of unity and s is a primitive 10th root of unity in the complex numbers. Then $\ell = 30$. Assume $\theta = e^{2\pi i/30}$. If $r = \theta^5$ and $s = \theta^3$, then $m = 15$. In this instance, $L(\lambda)$ is 15-dimensional for all $\lambda \neq 0$, while all the modules $L(\lambda, \xi)$ with λ, ξ nonzero have dimension 30.

Any quotient V of a Verma module is a highest weight module, i.e. it has a vector v such that $d \cdot v = 0$, $du \cdot v = \lambda v$ for some $\lambda \in \mathbb{K}$, and $V = Av$; and any highest weight module is such a quotient. There are lowest weight analogues of the modules $V(\lambda)$ in which u annihilates a vector. These are described in [BR, Prop. 2.30]:

Definition 4.9. *Assume $\kappa \in \mathbb{K}$ and let $W(\kappa)$ be the $\mathbb{K}$-vector space having basis $\{w_n \mid n = 0, 1, \dots\}$. Then $W(\kappa)$ is an $A(\alpha, \beta, 0)$-module under the action,*

$$d \cdot w_n = w_{n+1} \qquad u \cdot w_n = \kappa_{n-1} w_{n-1} \qquad (w_{-1} = 0), \tag{4.10}$$

where the κ_n values are given as follows:

(a) $\beta \neq 0$: $\quad \kappa_{-1} = 0,\ \kappa_0 = \kappa,$ *and* $\kappa_n = \beta^{-1}(-\alpha\kappa_{n-1} + \kappa_{n-2})$ *for* $n \geq 1$

(b) $\beta = 0$: $\quad \kappa_n = 0$ *for all* $n \geq -1$. *(Here $\kappa = 0$ must hold.)*

When $\beta \neq 0$ there is an isomorphism between the down-up algebras $A(\alpha, \beta, 0)$ and $A(-\beta^{-1}\alpha, \beta^{-1}, 0)$ given by $d \mapsto u'$ and $u \mapsto d'$ (see Remark 2.6). Under this isomorphism the lowest weight module $W(\kappa)$ for $A(\alpha, \beta, 0)$ becomes a highest weight module for $A(-\beta^{-1}\alpha, \beta^{-1}, 0)$. This allows Proposition 4.7 to be used to determine the simple quotients of $W(\kappa)$.

Finite-dimensional simple modules.

We begin by quoting a result from [BR, Prop. 4.1]:

Proposition 4.11. *Suppose M is a module for $A = A(\alpha, \beta, \gamma)$ and $v \in M$ has weight $\nu = (\nu', \nu'')$ relative to $\mathfrak{h} = \mathbb{K}du \oplus \mathbb{K}ud$.*

(i) *Then $u \cdot v$ is a vector of weight $\mu(\nu) = (\mu(\nu)', \mu(\nu)'')$ where*

$$\mu(\nu)' = \alpha\nu' + \beta\nu'' + \gamma, \qquad \mu(\nu)'' = \nu'.$$

(ii) *If $\beta \neq 0$, then $d \cdot v$ is a vector of weight $\delta(\nu) = (\delta(\nu)', \delta(\nu)'')$ where*

$$\delta(\nu)' = \nu'', \qquad \delta(\nu)'' = \beta^{-1}(\nu' - \alpha\nu'' - \gamma).$$

(iii) $\delta(\mu(\nu)) = \nu = \mu(\delta(\nu))$.

We will use Proposition 4.11 to provide an explicit description of all the finite-dimensional simple modules for $A = A(\alpha, \beta, 0)$ where $\beta \neq 0$. Our approach follows that of [BR, Sec. 5], and our main result here is Theorem 4.18. Carvalho and Musson [CM, Prop. 2.4] have described the finite-dimensional simple modules for $A = A(\alpha, \beta, 0)$, $\beta \neq 0$, as quotients of Verma modules or in terms of maximal ideals in $\mathbb{K}[du, ud]$. (Compare also [Jo1], [Jo2], [Ku1],

[Ku2].) Our later work requires more detailed information about the simple modules, which we derive along the way using the approach in [BR].

Let X be a finite-dimensional simple module for $A(\alpha,\beta,0)$, $\beta \neq 0$. Then X has a nonzero weight vector relative to $\mathfrak{h}$. The span of all the weight vectors in X is a nonzero submodule by Proposition 4.11, hence it equals X. Let x be an eigenvector for d in X corresponding to the eigenvalue ξ. Consider first the possibility that $\xi = 0$. It is easy to see using the first defining relation for A in (1.1) that the eigenspace corresponding to the eigenvalue 0 is du-invariant. Thus, there exists some $0 \neq y \in X$ such that $d \cdot y = 0$ and $du \cdot y = \lambda y$. Since $Ay = X$ by simplicity, X is a highest weight module and so is a homomorphic image of $V(\lambda)$. Since we have already determined those simple quotients (see Proposition 4.7), we can assume $\xi \neq 0$ in what follows. Then $x = \sum_\nu x_\nu$, where x_ν belongs to the ν-weight space. Equating weight components on both sides of $d \cdot x = \xi x$, we have by Proposition 4.11,

$$d \cdot x_\nu = \xi x_{\delta(\nu)} \tag{4.12}$$

Thus, if S is the $\mathbb{K}$-span of the vectors x_ν, then $dS = S$. Now $\xi u \cdot x = (ud) \cdot x = \sum_\nu \nu'' x_\nu$, which implies that

$$u \cdot x_\nu = \xi^{-1} \mu(\nu)'' x_{\mu(\nu)}. \tag{4.13}$$

It follows that S is an A-submodule of X, and so $X = S$ by simplicity.

By (4.12) we must have $\delta^j(\nu) = \delta^i(\nu)$ for some $j > i$. Applying μ^i to both sides we get that $\delta^{j-i}(\nu) = \nu$. Thus, there is some least value ℓ such that $\delta^\ell(\nu) = \nu$ and $x_\nu, x_{\delta(\nu)}, \dots, x_{\delta^{\ell-1}(\nu)}$ are linearly independent since they correspond to different weights. Their $\mathbb{K}$-span S' is invariant under d by (4.12). Using the fact that $\mu(\delta^i(\nu)) = \delta^{i-1}(\nu)$ for $i \geq 1$ and $\mu(\nu) = \mu(\delta^\ell(\nu)) = \delta^{\ell-1}(\nu)$, along with (4.13), we see that $uS' \subseteq S'$ also. Thus, $X = S'$. The set $\{\nu, \delta(\nu), \dots, \delta^{\ell-1}(\nu)\}$ is invariant under δ and μ, and any weight in it generates the whole set under δ and μ.

If some weight $\delta^i(\nu)$ has its first component equal to 0, then by replacing ν by $\delta^i(\nu)$, we can assume it is ν. It is convenient to write $\nu = (\lambda, \kappa)$. Setting $\lambda_0 = \lambda$ and $\lambda_{-1} = \kappa$, we have $\delta^{\ell-1}(\nu) = \mu(\lambda_0, \lambda_{-1}) = (\alpha\lambda_0 + \beta\lambda_{-1}, \lambda_0)$. Let $\lambda_1 = \alpha\lambda_0 + \beta\lambda_{-1}$. Applying μ we obtain $\delta^{\ell-i}(\nu) = (\lambda_i, \lambda_{i-1})$. Thus, we obtain the recursion $\lambda_n = \alpha\lambda_{n-1} + \beta\lambda_{n-2}$ with the initial conditions $\lambda_0 = \lambda$ and $\lambda_{-1} = \kappa$.

Case 1: $\alpha = r + s$ and $\beta = -rs$ and $s \neq r$.

The solutions to the recursion can be found using standard methods: $\lambda_n = c_1 r^n + c_2 s^n$, and $c_1 = -r(\lambda - s\kappa)/(s-r)$ and $c_2 = s(\lambda - r\kappa)/(s-r)$. Thus,

$$\lambda_n = [n+1]\lambda - rs[n]\kappa \qquad \text{where } [n] = \frac{s^n - r^n}{s - r}. \tag{4.14}$$

Case 2: $\alpha = 2r$, $\beta = -r^2$, where $r \neq 0$.

Then $\lambda_n = c_1 r^n + c_2 n r^n$ where $c_1 = \lambda$ and $c_2 = \lambda - r\kappa$. Thus,

$$\lambda_n = (n+1)r^n\lambda - nr^{n+1}\kappa. \tag{4.15}$$

Using the relations $\lambda_\ell = \lambda_0 = \lambda$ and $\lambda_{\ell-1} = \lambda_{-1} = \kappa$ in the expressions with the the constants c_i (they are a bit easier to work with than (4.14) and (4.15)), we see that:

Case 1: $\lambda = r\kappa$ and $r^\ell = 1$, $\lambda = s\kappa$ and $s^\ell = 1$, or $r^\ell = 1 = s^\ell$,
Case 2: $\lambda = \kappa = 0$, or $\lambda = r\kappa$, $r^\ell = 1$, and $\lambda_n = r^n\lambda$.

Assume first $\lambda = 0$. If we are in Case 2 or in Case 1 with $\kappa = 0$, then for $w_0 = x_\nu$, we have $d \cdot w_0 = \xi w_0$, $u \cdot w_0 = 0$. If $W(0) = \text{span}_{\mathbb{K}}\{w_n \mid n = 0, 1, \dots\}$ is the universal lowest weight module with lowest weight 0 as in Definition 4.9, then X is isomorphic to $L'(0,\xi) \overset{\text{def}}{=} W(0)/\text{span}_{\mathbb{K}}\{w_{n+1} - \xi w_n \mid n = 0, 1 \dots\}$. When $\lambda = 0$ and $\kappa \neq 0$ (so $r \neq s$ and $r^\ell = 1 = s^\ell$ must hold), then $\lambda_n = -rs[n]\kappa$. Setting $w_n = \xi^n x_{\delta^n(\nu)}$ for $n = 0, 1, \dots, \ell - 1$, we have by (4.12) and (4.13) that

$$
\begin{aligned}
&u \cdot w_n = \delta^{n-1}(\nu)'' w_{n-1} \quad (1 \le n \le \ell - 1) && u \cdot w_0 = 0,\\
&d \cdot w_n = w_{n+1} \quad (0 \le n \le \ell - 2) && d \cdot w_{\ell-1} = \xi^\ell w_0.
\end{aligned} \tag{4.16}
$$

Thus, $u \cdot w_n = \lambda_{\ell-n} w_{n-1}$ where

$$\lambda_{\ell-n} = -rs\frac{s^{\ell-n} - r^{\ell-n}}{s - r}\kappa = (sr)^{-(n-1)}[n]\kappa$$

Therefore,

$$u \cdot w_n = (sr)^{-(n-1)}[n]\kappa w_{n-1} \quad \text{for} \quad n = 0, 1 \dots, \ell - 1.$$

The quantity $(sr)^{-(n-1)}[n]\kappa$ is κ_{n-1} in Definition 4.9 (this can be computed directly or seen from [BR, Prop. 2.32]). So in this case, X is isomorphic to $W(\kappa)/\text{span}_{\mathbb{K}}\{w_{n+\ell} - \xi^\ell w_n \mid n \ge 0\}$.

We may assume that no weight has first component equal to 0. Thus, $\lambda_0, \lambda_1, \dots, \lambda_{\ell-1}$ are all nonzero. Suppose

$$x_0 = x_\nu, \qquad x_n = \lambda_0 \cdots \lambda_{n-1}\xi^{-n} x_{\delta^{\ell-n}(\nu)} \quad \text{for} \quad n = 1, \dots, \ell - 1.$$

Then

$$
\begin{aligned}
u \cdot x_n &= \lambda_0 \cdots \lambda_{n-1}\xi^{-n} u \cdot x_{\delta^{\ell-n}(\nu)}\\
&= \lambda_0 \cdots \lambda_{n-1}\xi^{-(n+1)}\delta^{\ell-n-1}(\nu)'' x_{\delta^{\ell-n-1}(\nu)}\\
&= \lambda_0 \cdots \lambda_{n-1}\xi^{-(n+1)}\lambda_n x_{\delta^{\ell-(n+1)}(\nu)} = x_{n+1} \quad (0 \le n \le \ell - 2)\\
u \cdot x_{\ell-1} &= (\lambda_0 \cdots \lambda_{\ell-1})\xi^{-\ell} x_0\\
d \cdot x_n &= \lambda_0 \cdots \lambda_{n-1}\xi^{-(n-1)} x_{\delta^{\ell-(n-1)}(\nu)} = \lambda_{n-1} x_{n-1} \quad (1 \le n \le \ell - 1)\\
d \cdot x_0 &= (\lambda_0 \cdots \lambda_{\ell-2})^{-1}\xi^\ell x_{\ell-1}.
\end{aligned} \tag{4.13}
$$

When $\lambda = r\kappa$, we have $\lambda_n = [n+1]r\kappa - rs[n]\kappa = r^{n+1}\kappa = r^n\lambda$. Here r is a primitive ath root of unity, and the A-action is given by (4.13) with $\ell = a$. Similarly, when $\lambda = s\kappa$, we have $\lambda_n = s^n\lambda$, s is a primitive bth root of unity, and (4.13) holds with $\ell = b$.

We claim that if $M = M(\lambda, \kappa, \xi)$ is the span of the vectors x_i, $i = 0, 1, \dots, \ell - 1$ (where $\ell = a$ if $\lambda = r\kappa$ and r is a primitive ath root of 1, or $\ell = b$ if $\lambda = s\kappa$ and s is a primitive bth root of 1), then $M(\lambda, \kappa, \xi)$ is a simple A-module. Indeed, any submodule must decompose into weight spaces relative to du and ud. Because of our choice of ℓ, the weights of the vectors x_i will be distinct and so a nonzero submodule must contain some x_i. But it is easy to see that each x_i generates M when all the λ_i are nonzero.

Finally, consider the case that $r = s$. As we noted earlier this implies $\lambda = r\kappa$ and $\lambda_n = r^n\lambda$. This is identical to the $\lambda = r\kappa$ case considered above. To summarize these results we have the following.

Theorem 4.18. *Assume M is a finite-dimensional simple module for the down-up algebra $A(r+s,-rs,0)$ with $rs \neq 0$. Then M is isomorphic to one of the following:*

(i) *(for all r, s)*
$L(0,\xi) = V(0)/\mathrm{span}_{\mathbb{K}}\{v_{n+1} - \xi v_n \mid n \geq 0\}, \qquad \xi \in \mathbb{K},$
$L'(0,\xi) = W(0)/\mathrm{span}_{\mathbb{K}}\{w_{n+1} - \xi w_n \mid n \geq 0\}, \qquad \xi \in \mathbb{K}.$

(ii) *(when s/r is a primitive mth root of unity for some $m \geq 2$)*
$L(\lambda) = V(\lambda)/\mathrm{span}_{\mathbb{K}}\{v_n \mid n \geq m\}, \qquad \lambda \in \mathbb{K}^*.$

(iii) *(when $r^\ell = 1 = s^\ell$ for some least $\ell \geq 2$ and $r \neq s$)*
(a) $L(\lambda,\xi) = V(\lambda)/\mathrm{span}_{\mathbb{K}}\{v_{n+\ell} - \xi^\ell v_n \mid n \geq 0\}$ *for* $\lambda, \xi \in \mathbb{K}^*$, *where $V(\lambda)$ is the Verma module.*
(b) $L'(\kappa,\xi) = W(\kappa)/\mathrm{span}_{\mathbb{K}}\{w_{n+\ell} - \xi^\ell w_n \mid n \geq 0\}$ *for* $\kappa, \xi \in \mathbb{K}^*$, *where $W(\kappa)$ is as in Definition 4.9.*
(c) $M(\lambda,\kappa,\xi) = \mathrm{span}_{\mathbb{K}}\{x_n \mid n = 0, 1, \ldots, \ell-1\}$, *with*

$$u \cdot x_n = x_{n+1} \quad (0 \leq n \leq \ell-2), \qquad u \cdot x_{\ell-1} = (\lambda_0 \cdots \lambda_{\ell-1})\xi^{-\ell} x_0,$$

$$d \cdot x_n = \lambda_{n-1} x_{n-1} \quad (1 \leq n \leq \ell-1), \qquad d \cdot x_0 = (\lambda_0 \cdots \lambda_{\ell-2})^{-1}\xi^\ell x_{\ell-1},$$

where $\lambda_n = [n+1]\lambda - rs[n]\kappa,\ [n] = \dfrac{s^n - r^n}{s - r},\ \lambda_n \neq 0$ *for any n and $\lambda \neq r\kappa$ or $s\kappa$.*

(iv) *(when r is a primitive ath root of unity and s is arbitrary, and $\lambda, \xi \in \mathbb{K}^*$)*
$M(\lambda, r^{-1}\lambda, \xi)$, *which has A-action given as in* (iii)(c) *with ℓ replaced by a and with $\lambda_n = r^n\lambda$.*

(v) *(when s is a primitive bth root of unity and r is arbitrary, and $\lambda, \xi \in \mathbb{K}^*$)*
$M(\lambda, s^{-1}\lambda, \xi)$, *which has A-action given as in* (iii)(c) *with ℓ replaced by b and with $\lambda_n = s^n\lambda$.*

Remark. When $r = s$, the finite-dimensional simple A-modules are the 1-dimensional modules $L(0,\xi)$ and $L'(0,\xi)$ ($\xi \in \mathbb{K}$), together with the modules $M(\lambda, r^{-1}\lambda, \xi)$ ($\lambda, \xi \in \mathbb{K}^*$) of dimension a in case $r = s$ is a primitive ath root of unity.

§5. r, s NOT BOTH ROOTS OF UNITY

In this section we assume that at least one of r or s is not a root of unity. We show that in this case, all finite-dimensional simple B-modules are 1-dimensional, with d and u acting as multiplication by 0. The case where r and s are both roots of unity will be handled separately.

We first prove that $G \cong \mathbb{Z} \times \mathbb{Z}$. Note that $\omega_1^i \omega_2^j$ is the trivial automorphism if and only if $\omega_1^i \omega_2^j.d = d$ and $\omega_1^i \omega_2^j.u = u$, that is

$$r^{i-j}s^{-i} = 1 \quad \text{and} \quad s^{i-j}r^j = 1. \tag{5.1}$$

Raising the first equation to the $i-j$ power and the second to the $-i$ power, we obtain $r^{(i-j)^2} = s^{i(i-j)} = r^{-ij}$, that is $r^{(i-j)^2+ij} = 1$. If r is not a root of unity, this forces $(i-j)^2 + ij = 0$. We may write $(i-j)^2 + ij = (i+j)^2 - 3ij$ to see that this quantity is positive except when $i = j = 0$. If r is a root of unity, but s is not a root of unity, a similar argument shows that $i = j = 0$. Therefore $G \cong \mathbb{Z} \times \mathbb{Z}$.

Lemma 5.2. *Suppose that at least one of r or s is not a root of unity. Let V be a finite-dimensional simple A-module, and let H be the stabilizer of V in G. If $V \cong L(0) = L(0,0)$, then $H = G$, and otherwise $[G:H] = \infty$.*

Proof. Clearly the stabilizer of $L(0)$ is G. Assuming V is not isomorphic to $L(0)$, let $\varrho : A \to \mathrm{End}_{\mathbb{K}}(V)$ be the associated representation. By (3.1), $\omega_1^i\omega_2^j \in H$ if and only if there is a map $t \in \mathrm{End}_{\mathbb{K}}(V)$ such that

$$\varrho(a)t = t\varrho(\omega_1^{-i}\omega_2^{-j}.a) \tag{5.3}$$

for all $a \in A$. In particular, letting $a = u$ and then $a = d$, we see that $\omega_1^i\omega_2^j \in H$ if and only if

$$\varrho(u)t = r^{-j}s^{j-i}t\varrho(u) \quad \text{and} \quad \varrho(d)t = r^{j-i}s^{i}t\varrho(d).$$

Suppose that V is 1-dimensional, that is $V \cong L(0,\xi)$ or $V \cong L'(0,\xi)$ for some $\xi \in \mathbb{K}^*$, in the notation of Theorem 4.18. Then either $\varrho(u) = 0$ or $\varrho(d) = 0$ and the other is multiplication by the nonzero scalar ξ. Therefore i and j must satisfy either

$$r^{j}s^{i-j} = 1 \quad \text{or} \quad r^{i-j}s^{-i} = 1.$$

In the first case, let j be the smallest positive integer such that $r^js^{i-j} = 1$ for some i. (If no such j exists, then $H \le \langle\omega_1\rangle$ automatically satisfies $[G:H] = \infty$.) Suppose $r^{j'}s^{i'-j'} = 1$, and write $j' = jq + c$ for some q and $0 \le c < j$. Multiply $r^{j'}s^{i'-j'} = 1$ by $r^{-jq}s^{(j-i)q} = 1$ to obtain $r^cs^{-c+i'-iq} = 1$. By minimality of j, we have $c = 0$ and $s^{i'-iq} = 1$. If s is not a root of unity, this implies $i' = iq$, so that $\omega_1^{i'}\omega_2^{j'} = (\omega_1^i\omega_2^j)^q$, and so $H = \langle\omega_1^i\omega_2^j\rangle$ is of infinite index in G. If s is a root of 1 and r is not a root of 1, a similar argument shows that $[G:H] = \infty$.

When V is not 1-dimensional, we set $a = du$ in (5.3):

$$\varrho(du)t = r^{-i}s^{j}t\varrho(du).$$

Suppose that $s/r = \rho$ is a primitive mth root of unity, and $V \cong L(\lambda)$ for $\lambda \neq 0$ in Theorem 4.18(ii). We apply the above equation to the basis vector $v_0 \in V$ to see that $t(v_0)$ is a vector of weight $(r^{-i}s^j\lambda, 0)$, which must equal $(\lambda_n, \lambda_{n-1})$ for some n. As $\lambda_{n-1} = \lambda(s^n - r^n)/(s-r) = 0$ and $\lambda \neq 0$, this implies that n is a multiple of m, that is $n = 0$ and $1 = r^{-i}s^j = r^{-i+j}\rho^j$. Because r is not a root of unity in this case, it must be that $i = j$, and i and j are multiples of m. Therefore $H \subseteq \langle\omega_1^m\omega_2^m\rangle$, which has infinite index in G.

If r is a primitive ath root of unity and $V \cong M(\lambda, r^{-1}\lambda, \xi)$ from Theorem 4.18(iv), analogous arguments show that $t(v_0)$ is a vector of weight $(r^{-i}s^j\lambda, r^{-i-1}s^j\lambda)$, so that $r^{-i}s^j\lambda = r^n\lambda$ for some n, or $r^{-i-n}s^j = 1$. As s is not a root of unity, this implies $j = 0$. Thus $H \subseteq \langle\omega_1\rangle$ and $[G:H] = \infty$. The case $V \cong M(\lambda, s^{-1}\lambda, \xi)$ is similar. □

Now we are ready to describe all finite-dimensional simple B-modules.

Theorem 5.4. *Suppose that at least one of r or s is not a root of unity, and let M be a finite-dimensional simple B-module. Then M is 1-dimensional, with d and u acting as multiplication by 0.*

Proof. The restriction of M to A contains a finite-dimensional simple A-module V. For each $g \in G$, the subspace $g \cdot V$ (that is, the set of all $g \cdot v$, $v \in V$) of M is an A-submodule of M isomorphic to the conjugate module ${}^gV$. The stabilizer H of V in G must have finite index, as otherwise M contains a sum of infinitely many nonisomorphic simple A-modules of the form ${}^gV$, necessarily a direct sum. By Lemma 5.2, this forces $V = L(0,0) = L(0)$. As G is abelian, this implies $M = L(0)$, with ω_1 and ω_2 acting as arbitrary elements of $\mathbb{K}^*$. □

§6. The stabilizers

In this section we assume that r and s are roots of unity. We compute the stabilizers H in G of the finite-dimensional simple modules V for $A = A(r+s, -rs, 0)$. Throughout we adopt the following notation:

(6.1) r and s are primitive ath and bth roots of unity, respectively
$\ell = \operatorname{lcm}(a,b)$ and $\theta \in \mathbb{K}$ is a primitive ℓth root of unity
$r = \theta^y$ and $s = \theta^z$
$e = \gcd(y^2 + z^2 - yz, \ell)$
$g = \gcd(a,b)$ and $a = ga'$, $b = gb'$
$m \geq 1$ is the smallest positive integer so that $r^m = s^m$

Without loss of generality, *we may assume θ, y, and z are chosen so that* $\gcd(y,z) = 1$: Let $h = \gcd(y,z)$, and suppose that p divides h and ℓ. Then $r^{\ell/p} = \theta^{y\ell/p} = 1 = \theta^{z\ell/p} = s^{\ell/p}$. By minimality of ℓ we must have $p = 1$. Thus, $\vartheta = \theta^h$ is a primitive ℓth root of 1, and if $y = hy_1$, $z = hz_1$, then $r = \vartheta^{y_1}$ and $s = \vartheta^{z_1}$ where $\gcd(y_1, z_1) = 1$.

Our computations of the structure of the group G and of the stabilizers require us to know some arithmetic properties of the quantities defined in (6.1), as well as solutions to the equation $r^{-i}s^j = 1$. The proofs of the following two lemmas are straightforward.

Lemma 6.2. *Assume (6.1). Then the integer solutions i, j to the equation $r^{-i}s^j = 1$ are $i = kz \pmod{\ell}$ and $j = ky \pmod{\ell}$, where k is an arbitrary integer.*

Lemma 6.3. *Assuming (6.1) holds, then*
(i) $\gcd(e,y) = 1 = \gcd(e,z)$,
(ii) $m = \ell/\gcd(y-z,\ell)$, *where we define* $\gcd(0,\ell) = \ell$,
(iii) $\gcd(y, \ell/m) = 1 = \gcd(z, \ell/m)$,
(iv) *$a'b'$ divides m (consequently, ℓ/m divides g),*
(v) *e divides g,*
(vi) $\gcd(e, \ell/m) = 1$ *(consequently, e divides m).*

These results enable us to describe the structure of the group G.

Lemma 6.4. *With assumptions as in (6.1),*

$$G \cong \langle \omega_1, \omega_2 \mid \omega_1\omega_2 = \omega_2\omega_1,\ \omega_1^\ell = \omega_2^\ell = (\omega_1^z\omega_2^y)^{\ell/e} = 1\rangle$$

as an abstract group, and the order of G is ℓ^2/e.

Proof. The element $\omega_1^i\omega_2^j$ is the trivial automorphism if and only if (5.1) holds, that is $r^{i-j}s^{-i} = 1$ and $s^{i-j}r^j = 1$. Therefore, both ω_1 and ω_2 have order ℓ in G. Solving each equation for s^i, we find that $r^{i-j} = s^i = r^{-j}s^j$, that is $r^i = s^j$. By Lemma 6.2, $\omega_1^i\omega_2^j = (\omega_1^z\omega_2^y)^k$ for some power k (where i and j are read modulo ℓ). Substituting $r = \theta^y$, $s = \theta^z$, $i = kz$, and $j = ky$ into (5.1), we see that the equations are both equivalent to $k(y^2+z^2-yz) = 0 \pmod{\ell}$. Hence k must be a multiple of ℓ/e, and G has the structure claimed.

To determine the order of G, we note that $G \cong G'/\langle(\omega_1^z\omega_2^y)^{\ell/e}\rangle$ where G' is the abstract group given by $G' = \langle \omega_1, \omega_2 \mid \omega_1\omega_2 = \omega_2\omega_1,\ \omega_1^\ell = \omega_2^\ell = 1\rangle$. Apply Lemma 6.3(i) to see that $(\omega_1^z\omega_2^y)^{\ell/e}$ has order e in G'. □

Example 6.5. Suppose that r is a primitive 30th root of unity and s is a primitive 42nd root of unity in the complex numbers. Let $\theta = e^{2\pi i/210}$, a primitive 210th root of unity where $210 = \ell = \mathrm{lcm}(30, 42)$. If, for example, $r = \theta^7$ and $s = \theta^5$, then $m = 105$ and $e = 3$, so that G has order $(210)^2/3 = 14,700$. On the other hand, if $r = \theta^{77}$ and $s = \theta^5$, then $m = 35$ and $e = 1$, so that G has order $(210)^2 = 44,100$.

We are now in a position to describe the stabilizers in G of each of the simple A-modules given in Theorem 4.18.

Theorem 6.6. *Assume (6.1). For the finite-dimensional simple modules of $A = A(r+s, -rs, 0)$, the stabilizers H in $G = \langle \omega_1, \omega_2\rangle$ are as follows:*

$$L(0) = L(0,0) = L'(0,0): \quad H = G,$$
$$L(0,\xi),\ \xi \neq 0: \quad H = \langle \omega_1^{z-y}\omega_2^z\rangle,\ |H| = \ell/e,$$
$$L'(0,\xi),\ \xi \neq 0: \quad H = \langle \omega_1^y\omega_2^{y-z}\rangle,\ |H| = \ell/e,$$
$$L(\lambda)\ (r \neq s): \quad H = \langle \omega_1^z\omega_2^y\rangle,\ |H| = \ell/e,$$
$$L(\lambda,\xi) \text{ or } L'(\kappa,\xi),\ \lambda,\kappa,\xi \neq 0\ (r \neq s), \text{ or } M(\lambda,\kappa,\xi),\ \lambda \neq r\kappa, s\kappa\ (r \neq s):$$
$$H = \langle \omega_1^z\omega_2^y,\ \omega_1^m\rangle \cong \langle \omega_1^z\omega_2^y\rangle \times \langle \omega_1^m\rangle,\ |H| = \ell^2/em,$$
$$M(\lambda, r^{-1}\lambda, \xi): \quad H = \langle \omega_1^y,\ \omega_2^y\rangle,\ |H| = a^2/e,$$
$$M(\lambda, s^{-1}\lambda, \xi): \quad H = \langle \omega_1^z,\ \omega_2^z\rangle,\ |H| = b^2/e.$$

Proof. The proof is organized as follows. For each of the modules in the statement of the theorem, we compute its conjugates under elements of G, and then determine which of the conjugates are isomorphic to the original module. This gives us the stabilizer H.

The underlying vector space of the conjugate module ${}^{\omega_1^i\omega_2^j}L(0,\xi)$ is the same as that of the 1-dimensional module $L(0,\xi)$, but the action is given by $(\omega_1^{-i}\omega_2^{-j}.d)\cdot v_0 = 0$ and $(\omega_1^{-i}\omega_2^{-j}.u)\cdot$

$v_0 = r^{-j}s^{j-i}u \cdot v_0 = r^{-j}s^{j-i}\xi v_0$. Therefore ${}^{\omega_1^i\omega_2^j}L(0,\xi) \cong L(0, r^{-j}s^{j-i}\xi)$, which is isomorphic to $L(0,\xi)$ if and only if $\xi = 0$ or $r^{-j}s^{j-i} = 1$. Thus $H = G$ in case $\xi = 0$, and otherwise

$$H = \{\omega_1^i\omega_2^j \mid r^{-j}s^{j-i} = 1\} = \{\omega_1^{i-j}\omega_2^i \mid r^{-i}s^j = 1\}.$$

By Lemma 6.2, $H = \langle\omega_1^{z-y}\omega_2^z\rangle$. To determine the order of H, note that by (5.1), $(\omega_1^{z-y}\omega_2^z)^p$ is the trivial automorphism if and only if $p(y^2+z^2-yz) = 0 \pmod{\ell}$. Therefore $|H| = \ell/e$. Similar arguments apply to the modules $L'(0,\xi)$.

In the conjugate module ${}^{\omega_1^i\omega_2^j}L(\lambda)$, we have

$$(\omega_1^{-i}\omega_2^{-j}.(du)) \cdot v_0 = r^{-i}s^j du \cdot v_0 = r^{-i}s^j\lambda v_0,$$

and $(\omega_1^{-i}\omega_2^{-j}.ud) \cdot v_0 = 0$. Thus v_0 has weight $(r^{-i}s^j\lambda, 0)$. Let

$$v_n' = (\omega_1^{-i}\omega_2^{-j}.u)^n \cdot v_0 = r^{-jn}s^{(j-i)n}v_n \ (0 \le n \le m-1).$$

By Proposition 4.11, v_n' is a vector of weight $(r^{-i}s^j\lambda_n, r^{-i}s^j\lambda_{n-1})$ in ${}^{\omega_1^i\omega_2^j}L(\lambda)$. In addition, $(\omega_1^{-i}\omega_2^{-j}.u)^m \cdot v_0' = s^{-im}u^m \cdot v_0 = 0$. It may be checked that the map $f : {}^{\omega_1^i\omega_2^j}L(\lambda) \to L(r^{-i}s^j\lambda)$ given by $f(v_n') = v_n$, or equivalently $f(v_n) = r^{jn}s^{(i-j)n}v_n$ $(0 \le n \le m-1)$, also commutes with the action of d, and so is an A-module isomorphism. (We will need the function f in the sequel.) Next we argue that $L(\lambda) \cong L(\lambda')$ if and only if $\lambda = \lambda'$. If $f : L(\lambda) \to L(\lambda')$ is an isomorphism, then $f(v_0)$ must have weight $(\lambda, 0)$. This forces $\lambda_n' = \lambda$ and $\lambda_{n-1}' = 0$ for some n, that is $\lambda_{n-1}' = \lambda'(s^n - r^n)/(s-r) = 0$ by Proposition 4.2(i). As $\lambda' \ne 0$, this implies that n is a multiple of m, that is $n = 0$ and $\lambda = \lambda'$. Since ${}^{\omega_1^i\omega_2^j}L(\lambda) \cong L(r^{-i}s^j\lambda)$, we see that this is isomorphic to $L(\lambda)$ if and only if $r^{-i}s^j = 1$. Applying Lemma 6.2, we have that the stabilizer of $L(\lambda)$ is $H = \langle\omega_1^z\omega_2^y\rangle$. Thus $|H| = \ell/e$, the order of $\omega_1^z\omega_2^y$ in G by Lemma 6.4.

Similarly, ${}^{\omega_1^i\omega_2^j}L(\lambda,\xi) \cong L(r^{-i}s^j\lambda, \xi)$. Note that $(\omega_1^{-i}\omega_2^{-j}.u)^\ell \cdot v_0 = u^\ell \cdot v_0 = \xi^\ell v_0$, so that the parameter ξ remains unchanged. However, there are some nontrivial isomorphisms among these modules. We argue that $L(\lambda,\xi) \cong L(r^m\lambda,\xi)$, from which it follows that $L(\lambda,\xi) \cong L(r^{km}\lambda,\xi)$ for any integer k. By Proposition 4.2(i), if $\lambda' = r^m\lambda$, then $\lambda_n' = \lambda_{n+m}$, where subscripts may be read modulo ℓ as $r^\ell = s^\ell = 1$. Define $f : L(r^m\lambda,\xi) \to L(\lambda,\xi)$ by $f(v_n) = v_{n+m}$ $(0 \le n \le \ell-1)$, where $v_{n+\ell} = \xi^\ell v_n$ in accordance with the definition of $L(\lambda,\xi)$. A calculation shows that f preserves the actions of d and u. Now suppose there is an A-module isomorphism $\phi : L(\lambda',\xi) \to L(\lambda,\xi)$. As $\phi(v_0')$ must be a vector of weight $(\lambda', 0)$, we have $\lambda_n = \lambda'$, $\lambda_{n-1} = 0$ for some n. Then $\lambda_{n-1} = \lambda(s^n - r^n)/(s-r) = 0$, where $\lambda \ne 0$, so $n = km$ for some k and $\lambda' = \lambda_{km} = r^{km}\lambda$. Consequently, $L(\lambda,\xi) \cong L(\lambda',\xi)$ if and only if $\lambda' = r^{km}\lambda$ for some k. This shows that the conjugate module ${}^{\omega_1^i\omega_2^j}L(\lambda,\xi)$ is isomorphic to $L(\lambda,\xi)$ if and only if $r^{-i}s^j = r^{km}$ for some k. It follows by Lemma 6.2 that the stabilizer H of $L(\lambda,\xi)$ consists of the elements $(\omega_1^z\omega_2^y)^p(\omega_1^m)^k$ for some p, k. That is, $H = \langle\omega_1^z\omega_2^y, \omega_1^m\rangle$. Direct calculation shows that there are no relations between the generators $\omega_1^z\omega_2^y$ and ω_1^m, so that in fact H is a direct product with order $(\ell/e)(\ell/m) = \ell^2/em$. Similar arguments apply to $L'(\kappa,\xi)$. In this case an A-module isomorphism $f : {}^{\omega_1^i\omega_2^j}L'(\kappa,\xi) \to L'(r^{-i}s^j\kappa,\xi)$ is given by $f(w_n) = r^{(i-j)n}s^{-in}w_n$ $(0 \le n \le \ell-1)$.

We may argue as before that ${}^{\omega_1^i\omega_2^j}M(\lambda,\kappa,\xi) \cong M(r^{-i}s^j\lambda, r^{-i}s^j\kappa,\xi)$, letting $x_n' = (\omega_1^{-i}\omega_2^{-j}.u)^n \cdot x_0 = r^{-jn}s^{(j-i)n}x_n$ in the conjugate module. This may be realized by the isomorphism $f : {}^{\omega_1^i\omega_2^j}M(\lambda,\kappa,\xi) \to M(r^{-i}s^j\lambda, r^{-i}s^j\kappa,\xi)$ given by $f(x_n) = r^{jn}s^{(i-j)n}x_n$ $(0 \le n \le \ell-1)$. Next we show that $M(\lambda,\kappa,\xi) \cong M(\lambda_1,\lambda,\xi)$, which implies that $M(\lambda,\kappa,\xi) \cong M(\lambda_k,\lambda_{k-1},\xi)$

for any positive integer k. By Theorem 4.18(iii)(c), if λ is replaced by $\lambda' = \lambda_1$, and κ by $\kappa' = \lambda$, then $\lambda'_k = \lambda_{k+1}$ $(-1 \le k \le \ell - 2)$. Define $f : M(\lambda_1, \lambda, \xi) \to M(\lambda, \kappa, \xi)$ by $f(x_n) = x_{n+1}$ $(0 \le n \le \ell - 1)$, where we let $x_{p+\ell} = (\lambda_0 \cdots \lambda_{\ell-1})\xi^{-\ell} x_p$. It may be checked that f is an A-module isomorphism. Now suppose $\phi : M(\lambda', \kappa', \xi) \to M(\lambda, \kappa, \xi)$ is any A-module isomorphism. Then $\phi(x_0)$ has weight (λ', κ'), which implies $(\lambda', \kappa') = (\lambda_k, \lambda_{k-1})$ for some k. We have shown that $M(\lambda', \kappa', \xi) \cong M(\lambda, \kappa, \xi)$ if and only if $(\lambda', \kappa') = (\lambda_k, \lambda_{k-1})$ for some integer k. Therefore the stabilizer H of $M(\lambda, \kappa, \xi)$ consists of the elements $\omega_1^i \omega_2^j$ such that $r^{-i}s^j\lambda = \lambda_n$ and $r^{-i}s^j\kappa = \lambda_{n-1}$ for some n, $0 \le n \le \ell - 1$. Recall that the solutions to the recurrence relation have the form $\lambda_n = c_1 r^n + c_2 s^n$ where $c_1 = -r(\lambda - s\kappa)/(s - r)$ and $c_2 = s(\lambda - r\kappa)/(s - r)$ (see (4.14)). Therefore,

$$r^{-i}s^j\lambda = c_1 r^n + c_2 s^n \quad \text{and} \quad r^{-i}s^j\kappa = c_1 r^{n-1} + c_2 s^{n-1}.$$

Multiplying the second equation by r, subtracting the first equation, and substituting c_2, we find that $r^{-i}s^j(r\kappa - \lambda) = s^n(r\kappa - \lambda)$. Since $\lambda \neq r\kappa$ by Theorem 4.18(iii)(c), it must be that $r^{-i}s^j = s^n$. Substituting this back into the original equations yields

$$s^n\lambda = c_1 r^n + c_2 s^n \quad \text{and} \quad s^n\kappa = c_1 r^{n-1} + c_2 s^{n-1}.$$

Now multiplying the second equation by s, subtracting the first, and substituting c_1, we have $s^n(s\kappa - \lambda) = r^n(s\kappa - \lambda)$. This forces $s^n = r^n$ since $\lambda \neq s\kappa$. Therefore $n = km$ for some integer k. If i, j satisfy $r^{-i}s^j = s^{km}$, then $\omega_1^i\omega_2^j$ does indeed stabilize $M(\lambda, \kappa, \xi)$, as in that case, $r^{-i}s^j\lambda = s^{km}\lambda = \lambda_{km}$ and $r^{-i}s^j\kappa = s^{km}\kappa = \lambda_{km-1}$. As in the last case, we obtain $H = \langle \omega_1^z\omega_2^y,\, \omega_1^m \rangle$.

To determine conjugates of $M(\lambda, r^{-1}\lambda, \xi)$, the only required change from $M(\lambda, \kappa, \xi)$ is to replace ℓ with a, which affects the following calculations:

$$(\omega_1^{-i}\omega_2^{-j}.u) \cdot x'_{a-1} = (r^{-i}s^j\lambda_1) \cdots (r^{-i}s^j\lambda_{a-1})(s^i\xi)^{-a}x'_0 \text{ and}$$
$$(\omega_1^{-i}\omega_2^{-j}.d) \cdot x'_0 = ((r^{-i}s^j\lambda_0) \cdots (r^{-i}s^j\lambda_{a-2}))^{-1}(s^i\xi)^a x'_{a-1}.$$

Thus ${}^{\omega_1^i\omega_2^j}M(\lambda, r^{-1}\lambda, \xi) \cong M(r^{-i}s^j\lambda, r^{-i-1}s^j\lambda, s^i\xi)$. As in the previous case it can be shown, using $\lambda_n = r^n\lambda$, that $M(\lambda, r^{-1}\lambda, \xi) \cong M(\lambda', r^{-1}\lambda', \xi')$ if and only if $\lambda' = r^k\lambda$ for some integer k and $(\xi')^a = \xi^a$. Therefore $\omega_1^i\omega_2^j$ stabilizes $M(\lambda, r^{-1}\lambda, \xi)$ if and only if $r^{-i}s^j = r^k$ for some k and $s^{ia} = 1$. Clearly then i must be a multiple of b' where $\ell = b'a$, and by Lemma 6.2, j is independently a multiple of y. Therefore $H = \langle \omega_1^{b'}, \omega_2^y \rangle$. To see $\omega_1^{b'}$ has order a observe by (6.1) that $\ell = ab'$ divides ay, so $y = b'y'$ for some y'. Then $ay = ab'y' = 0 \pmod{\ell}$, with a the smallest positive integer having this property. Consequently $\gcd(y', a) = 1$, and so $(\omega_1^{b'})^{y'} = \omega_1^y$ has order a as well. Therefore $\omega_1^{b'}$ and ω_1^y generate the same subgroup of G, and we have shown that $H = \langle \omega_1^y, \omega_2^y \rangle$. (Note that in case $r = s$, this yields $H = G$.) A calculation shows that $(\omega_1^z\omega_2^y)^{\ell/e} \in H$, and so by Lemma 6.4, $|H| = a^2/e$.

Similar calculations prove that ${}^{\omega_1^i\omega_2^j}M(\lambda, s^{-1}\lambda, \xi) \cong M(r^{-i}\lambda, r^{-i}s^{-1}\lambda, r^{-i+j}\xi)$. The stabilizer of $M(\lambda, s^{-1}\lambda, \xi)$ is $H = \langle \omega_1^z, \omega_2^z \rangle$, of order b^2/e. □

§7. Finite-dimensional simple B-modules.

We assume here that r and s are both roots of unity. Our goal in this section is to apply Clifford theory as outlined in §3 to determine the finite-dimensional simple modules for $B = B(r+s, -rs, 0)$. Each finite-dimensional simple B-module W contains a simple A-submodule V, and Clifford theory then gives the structure of W as $W \cong B \otimes_{A*H} (V \otimes_{\mathbb{K}} Y)$, where H is the stabilizer of V in G, and Y is a simple $\mathbb{K}^{\psi}H$-module (the 2-cocycle ψ, which depends on V, is as described in §3). We find the dimensions of the simple $\mathbb{K}^{\psi}H$-modules Y, allowing us to give the dimensions of all the finite-dimensional simple B-modules. In §8 we determine more explicitly the structure of the simple B-modules on which d and u act nilpotently in order to describe tensor products of these modules.

Theorem 7.1. *Assume (6.1). The finite-dimensional simple B-modules are of the form $W = B \otimes_{A*H} (V \otimes_{\mathbb{K}} Y)$, where V, H, and Y are as follows:*

(i) *$V = L(0,\xi)$, $L'(0,\xi)$, or $L(\lambda)$, or $m = \ell$ and $V = L(\lambda,\xi), L'(\kappa,\xi)$, or $M(\lambda,\kappa,\xi)$, H is the stabilizer of V as in Theorem 6.6, and Y is any one of the $|H|$ 1-dimensional simple $\mathbb{K}H$-modules. Then* $\dim W = [G:H]\cdot \dim V$, *so that* $\dim W = 1$ *when $V = L(0)$;* $\dim W = \ell$ *when $V = L(0,\xi)$, $L'(0,\xi)$, $\xi \neq 0$;* $\dim W = \ell m$ *when $V = L(\lambda)$, $\lambda \neq 0$; and* $\dim W = \ell^2$ *in the remaining cases.*

(ii) *$m \neq \ell$ and $V = L(\lambda,\xi)$, $L'(\lambda,\xi)$, or $M(\lambda,\kappa,\xi)$, $H = \langle \omega_1^z\omega_2^y, \omega_1^m\rangle$, Y is any one of the m/e simple $\mathbb{K}^{\psi}H$-modules, each of dimension ℓ/m, and* $\dim W = \ell^2$.

(iii) *$V = M(\lambda, r^{-1}\lambda,\xi)$ (resp. $M(\lambda, s^{-1}\lambda,\xi)$), $H = \langle \omega_1^y, \omega_2^y\rangle$, (resp., $H = \langle \omega_1^z,\omega_2^z\rangle$), Y is any one of the e simple $\mathbb{K}^{\psi}H$-modules, each of dimension a/e (resp. b/e), and* $\dim W = \ell^2/e$.

Proof. (i) If $V = L(0)$, then $H = G$ and $\mathbb{K}^{\psi}H = \mathbb{K}G$, so the result is clear. For each of the other cases, the stabilizer H is cyclic, from which it follows that $\mathrm{H}^2(H,\mathbb{K}^*) = 1$, and $\mathbb{K}^{\psi}H \cong \mathbb{K}H$ [K, Thm. 2.3.1 and Lemma 3.2.2(i)]. In this case, there are $|H|$ simple $\mathbb{K}^{\psi}H$-modules Y, each of dimension one. For each such Y, the dimension of the simple B-module $W = B \otimes_{A*H} (V \otimes_{\mathbb{K}} Y)$ is $[G:H]\cdot \dim V$, as B is free of rank $[G:H]$ over $A*H$.

(ii) In case $m \neq \ell$, the stabilizer H of $L(\lambda,\xi)$ is the direct product of cyclic groups, $\langle \omega_1^z\omega_2^y\rangle \times \langle \omega_1^m\rangle$, according to Theorem 6.6. Let $\sigma = \omega_1^z\omega_2^y$ and $\tau = \omega_1^m$. To determine the structure of the twisted group algebra $\mathbb{K}^{\psi}H$, we choose isomorphisms of A-modules $t_h : {}^hL(\lambda,\xi) \to L(\lambda,\xi)$ for each $h \in H$, as prescribed in §3. These functions are provided in the proof of Theorem 6.6. First define $t_\sigma = t_{\omega_1^z\omega_2^y}$ by $t_\sigma(v_n) = s^{(z-y)n}r^{yn}v_n = \eta^n v_n$ $(0 \le n \le \ell-1)$, where $\eta = r^y s^{z-y} = \theta^{y^2+z^2-yz}$. Let $t_\tau = t_{\omega_1^m}$ be the composition of $f' : {}^{\omega_1^m}L(\lambda,\xi) \to L(r^{-m}\lambda,\xi)$, $f'(v_n) = s^{mn}v_n$, and $f : L(r^{-m}\lambda,\xi) \to L(\lambda,\xi)$, $f(v_n) = v_{n-m}$. (Here we use the relation $v_{k+\ell} = \xi^{\ell}v_k$, for all k, in $L(\lambda,\xi)$.) Therefore, $t_\tau(v_n) = s^{mn}v_{n-m}$ $(0 \le n \le \ell-1)$. Since each element of H may be expressed uniquely as $\sigma^i\tau^j$ for some i,j $(0 \le i \le \ell/e-1,\ 0 \le j \le \ell/m-1)$, we may define $t_{\sigma^i\tau^j} = (t_\sigma)^i(t_\tau)^j$. The cocycle χ may be computed using $t_ht_{h'} = \chi(h,h')t_{hh'}$ for all $h,h' \in H$. Let $\psi = \chi^{-1}$ and $\mathbb{K}^{\psi}H = \mathrm{span}_{\mathbb{K}}\{s_h \mid h \in H\}$ with $s_hs_{h'} = \psi(h,h')s_{hh'}$ for all $h,h' \in H$, as in §3.

By [K, Thms. 3.2.10 and 7.9.3], the number of simple $\mathbb{K}^{\psi}H$-modules is equal to the number of *ψ-regular* elements of H, where $h \in H$ is ψ-regular if and only if $s_hs_{h'} = s_{h'}s_h$ for all $h' \in H$ (as H is abelian). Since $\psi = \chi^{-1}$, this is true if and only if $t_ht_{h'} = t_{h'}t_h$ for all $h' \in H$. Note that

$$t_\tau t_\sigma(v_n) = t_\tau(\eta^n v_n) = s^{mn}\eta^n v_{n-m},$$
$$\text{and}\quad t_\sigma t_\tau(v_n) = t_\sigma(s^{mn}v_{n-m}) = \eta^{n-m}s^{mn}v_{n-m}.$$

Therefore $t_\tau t_\sigma = \eta^m t_\sigma t_\tau$. Now $\sigma^i\tau^j$ is ψ-regular if and only if $t_{\sigma^i\tau^j}$ commutes with all $t_{\sigma^k\tau^{k'}}$. Equivalently, it must commute with t_σ and t_τ. From

$$t_{\sigma^i\tau^j}t_\sigma = t_\sigma^i t_\tau^j t_\sigma = \eta^{jm}t_\sigma^{i+1}t_\tau^j \quad\text{and}\quad t_\sigma t_{\sigma^i\tau^j} = t_\sigma^{i+1}t_\tau^j,$$

we see that $\eta^{jm} = 1$. As $\eta = \theta^{y^2+z^2-yz}$, this implies $(y^2+z^2-yz)jm = 0 \pmod{\ell}$, or $(y^2+z^2-yz)j = 0 \pmod{\ell/m}$. But ℓ/m and y^2+z^2-yz have no factors in common by Lemma 6.3(vi), so this implies that j is a multiple of ℓ/m. Similarly, from the requirement that $t_{\sigma^i\tau^j}t_\tau = t_\tau t_{\sigma^i\tau^j}$, we obtain $\eta^{im} = 1$, and so i must be a multiple of ℓ/m as well. Because $\tau = \omega_1^m$ has order ℓ/m, and $\sigma = \omega_1^z\omega_2^y$ has order ℓ/e, it follows that the ψ-regular elements of H are the m/e distinct powers of $(\omega_1^z\omega_2^y)^{\ell/m}$. As a result, there are m/e simple $\mathbb{K}^\psi H$-modules Y, up to isomorphism. By [K, Thm. 7.9.5], these modules all have the same dimension. Since $\mathbb{K}^\psi H$ is semisimple, dimension counting shows that each simple module Y has dimension the square root of $|H|/(m/e)$, which is ℓ/m. The dimension of a simple B-module of the form $W = B\otimes_{A*H}(L(\lambda,\xi)\otimes_{\mathbb{K}} Y)$ is therefore $[G:H]\cdot \dim L(\lambda,\xi)\cdot \dim Y = m\cdot\ell\cdot\ell/m = \ell^2$.

Similar calculations apply to the modules $L'(\kappa,\xi)$ and $M(\lambda,\kappa,\xi)$, proving (ii).

(iii) By Theorem 6.6, the stabilizer H of $M(\lambda, r^{-1}\lambda,\xi)$ is $\langle\omega_1^y,\omega_2^y\rangle$. Let $\sigma = \omega_1^y$ and $\tau = \omega_2^y$. Assume $t_\sigma : {}^\sigma M(\lambda,r^{-1}\lambda,\xi)\to M(\lambda,r^{-1}\lambda,\xi)$ is defined by $t_\sigma(x_n) = s^{yn}x_{n-y}$ $(0\le n\le \ell-1)$, a scalar multiple of the function provided in the proof of Theorem 6.6. (Here we define $x_{n-a} = (\lambda_0\cdots\lambda_{a-1})^{-1}\xi^a x_n$.) To construct t_τ, we note that $s^y = \theta^{zy} = r^z$, and use the proof of Theorem 6.6 to define $t_\tau : {}^\tau M(\lambda,r^{-1}\lambda,\xi)\to M(\lambda,r^{-1}\lambda,\xi)$ by $t_\tau(x_n) = r^{yn}s^{-yn}x_{n+z}$ $(0\le n\le\ell-1)$.

We claim that every element of H may be expressed uniquely as $\sigma^i\tau^j$ $(0\le i\le a-1,\ 0\le j\le a/e-1)$. To see this, write $y = b'y'$ (as $\ell = a'b$ divides ay) and observe that by Lemma 6.3(i), $\gcd(b',e) = \gcd(y,e) = 1$, so that b' has an inverse $(b')^{-1}$ modulo e. Thus

$$\tau^{a/e} = \omega_2^{ya/e} = (\omega_2^{y\ell/e})^{(b')^{-1}} = (\omega_1^{-z\ell/e})^{(b')^{-1}} = (\omega_1^{b'})^{-z(b')^{-1}a/e},$$

which is a power of $\sigma = \omega_1^y$ as $\omega^{b'}$ and ω_1^y generate the same group. As a consequence, in every expression $\sigma^i\tau^j$, j may be reduced modulo a/e. Since there are exactly $a^2/e = |H|$ such expressions, they must be unique.

As before, we define $t_{\sigma^i\tau^j} = t_\sigma^i t_\tau^j$ $(0\le i\le a-1,\ 0\le j\le a/e-1)$. Note that

$$\begin{aligned} t_\tau t_\sigma(x_n) &= t_\tau(s^{yn}x_{n-y}) = r^{y(n-y)}s^{y^2}x_{n-y+z}\\ \text{and}\quad t_\sigma t_\tau(x_n) &= t_\sigma(r^{yn}s^{-yn}x_{n+z}) = r^{yn}s^{yz}x_{n-y+z}.\end{aligned}$$

Therefore $t_\tau t_\sigma = r^{-y^2}s^{y^2-yz}t_\sigma t_\tau$. The coefficient of $t_\sigma t_\tau$ is $r^{-y^2}s^{y^2-yz} = \eta^{-y}$, where $\eta = \theta^{y^2+z^2-yz}$. Its order k must satisfy $k(y^2+z^2-yz) = 0 \pmod{a}$, using $y = b'y'$ and $\gcd(a,y') = 1$ (see (6.1)). Therefore $k = a/e$. As in the proof of (ii), $\sigma^i\tau^j$ is a ψ-regular element if and only if i and j are both multiples of a/e. In particular, $j = 0$, and the ψ-regular elements of H are exactly the e distinct powers of $\sigma^{a/e}$.

As there are e ψ-regular elements in H, there are e simple $\mathbb{K}^\psi H$-modules, each of dimension the square root of $|H|/e$, which is a/e. The dimension of a simple B-module of the form $W = B\otimes_{A*H}(M(\lambda,r^{-1}\lambda,\xi)\otimes_{\mathbb{K}} Y)$ is therefore $[G:H]\cdot\dim M(\lambda,r^{-1}\lambda,\xi)\cdot\dim Y = (\ell/a)^2\cdot a\cdot a/e = \ell^2/e$.

Analogous computations apply to the modules $M(\lambda,s^{-1}\lambda,\xi)$, proving (iii). $\square$

Example 7.2. In the special case that $r = q$, $s = q^{-1}$, where q is an ℓth root of unity, $A(q+q^{-1},-1,0)\cong U_q^+(\mathfrak{sl}_3)$, the subalgebra of the quantized enveloping algebra of $\mathfrak{sl}_3$ generated by E_1, E_2, and B is isomorphic to a quotient of the subalgebra of $U_q(\mathfrak{sl}_3)$ generated by E_i, $K_i^{\pm1}, i = 1,2$. If ℓ is odd, then $m = \ell$; whereas if ℓ is even, then $m = \ell/2$. We have $y = 1$ and

$z = -1$, so that $y^2 + z^2 - yz = 3$. Thus, if 3 divides ℓ, then $e = 3$; however if 3 does not divide ℓ, then $e = 1$. Theorem 7.1 gives the dimensions of the finite-dimensional simple B-modules. These dimensions are 1, ℓ, ℓ^2, $\ell^2/2$ (when ℓ is even), and $\ell^2/3$ (when ℓ is divisible by 3).

Remark. When $r = s = 1$, all the finite-dimensional simple modules for $A(2, -1, 0)$ (which is the universal enveloping algebra of the Heisenberg Lie algebra) have dimension 1 (see Theorem 4.18). The Heisenberg algebra is nilpotent, and it is a well-known result that all finite-dimensional simple modules of a nilpotent Lie algebra (hence of its enveloping algebra) are 1-dimensional. Theorem 4.18 can be viewed as a generalization of the Heisenberg algebra result. In this special case $G = \langle 1 \rangle$ and $B \cong A$. Theorem 7.1 just says that all the simple finite-dimensional B-modules are 1-dimensional, which is apparent in this case.

§8. Tensor Products

We suppose $B = B(r+s, -rs, 0)$, where r and s are roots of unity and adopt the conventions in (6.1). We describe the tensor products of simple B-modules on which d and u act nilpotently. Thus, we focus on the A-modules $L(\lambda)$. When $\lambda \neq 0$, the stabilizer subgroup H of G is cyclic, with generator $\sigma = \omega_1^z \omega_2^y$, by Theorem 6.6. Let p and q be integers such that $py - qz = 1$. Note that p and q exist as we have assumed that $\gcd(y, z) = 1$. We show next that H is complemented in G by the subgroup $\langle \tau \rangle$, where $\tau = \omega_1^p \omega_2^q$. We use $\langle \tau \rangle$ as a set of coset representatives for H in G in order to describe explicitly the simple B-modules containing $L(\lambda)$.

Lemma 8.1. *$G \cong \langle \sigma \rangle \times \langle \tau \rangle$, where $\sigma = \omega_1^z \omega_2^y$ and $\tau = \omega_1^p \omega_2^q$.*

Proof. First we argue that τ has order ℓ in G. The element $(\omega_1^p \omega_2^q)^k$ is the trivial automorphism if and only if $r^{pk-qk} s^{-pk} = 1$ and $s^{pk-qk} r^{qk} = 1$. In particular $r^{pk-qk} = s^{pk} = s^{qk} r^{-qk}$, or $r^{pk} = s^{qk}$. Substituting $r = \theta^y$ and $s = \theta^z$, we obtain $\theta^{k(py-qz)} = 1$, or $\theta^k = 1$. Therefore k must be a multiple of ℓ, and the order of τ is ℓ.

Next we verify that the intersection of the subgroups $\langle \sigma \rangle$ and $\langle \tau \rangle$ of G is $\langle 1 \rangle$, from which the lemma follows by order counting and Lemma 6.4. Suppose that $(\omega_1^p \omega_2^q)^k = (\omega_1^z \omega_2^y)^c$ for some integers k, c. Then $i = kp - cz, j = kq - cy$ is a solution to $r^{i-j} s^{-i} = 1$ and $s^{i-j} r^j = 1$, and in particular to $r^i = s^j$ (by solving each equation for s^i as before). That is, $r^{kp-cz} = s^{kq-cy}$. But $r^{cz} = s^{cy}$, and therefore $r^{kp} = s^{kq}$. As above, this forces k to be a multiple of ℓ, the order of $\tau = \omega_1^p \omega_2^q$. □

The B-modules $\widehat{L}^n(\lambda)$.

As noted in the proof of Theorem 7.1(i), because $H = \langle \sigma \rangle$ is cyclic, there are $|H| = \ell/e$ simple modules for $\mathbb{K}^\psi H \cong \mathbb{K}H$, each of dimension one, with the generator $\sigma = \omega_1^z \omega_2^y$ acting by an (ℓ/e)th root of unity. It follows from $e = \gcd(y^2 + z^2 - yz, \ell)$, that $\eta = \theta^{y^2+z^2-yz}$ is a primitive (ℓ/e)th root of unity. Assume σ acts as η^n on the 1-dimensional module Y^n. Let $\lambda \in \mathbb{K}^*$ and set

$$L^n(\lambda) = L(\lambda) \otimes_{\mathbb{K}} Y^n. \tag{8.2}$$

Since Y^n is 1-dimensional, we may identify $L^n(\lambda)$ as a vector space with $L(\lambda)$ and use the basis v_i, $i = 0, 1, \ldots, m-1$, from Proposition 4.7(2)(v). By the proof of Theorem 6.6, we may define an A-module isomorphism $t_\sigma : {}^{\sigma}L(\lambda) \to L(\lambda)$ by $t_\sigma(v_i) = \eta^i v_i$ $(i = 0, 1, \ldots, m-1)$, and $t_{\sigma^j} : {}^{\sigma^j}L(\lambda) \to L(\lambda)$ by $t_{\sigma^j} = (t_\sigma)^j$. Note that $(t_\sigma)^{\ell/e}$ is the identity map, as $\eta^{\ell/e} = 1$. Thus, the cocycles χ and $\psi = \chi^{-1}$ defined in §3 are trivial, and the basis elements s_h, $h \in H$, form a group basis for $\mathbb{K}^\psi H = \mathbb{K}H$. By (3.2), $L^n(\lambda)$ is an $A * H$-module with

$$(a\sigma) \cdot v_i = \eta^{n+i} a \cdot v_i, \tag{8.3}$$

for all $a \in A$ and $i = 0, 1, \ldots, m-1$. Clifford theory (§3) tells us that the induced module

$$\widehat{L}^n(\lambda) = (A * G) \otimes_{A*H} L^n(\lambda) \tag{8.4}$$

is a simple module for $B = A * G$ with basis

$$x_{j,k} = \tau^j \otimes v_k, \qquad j = 0, 1, \ldots, \ell - 1, \quad k = 0, 1, \ldots, m-1, \tag{8.5}$$

$\tau = \omega_1^p \omega_2^q$. The action of $A * G$ is specified by

$$\begin{aligned} \sigma \cdot x_{j,k} &= \eta^{n+k} x_{j,k} \\ \tau \cdot x_{j,k} &= x_{j+1,k} \\ d \cdot x_{j,k} &= (r^{(q-p)} s^p)^j \lambda_{k-1} x_{j,k-1} = \zeta^j \lambda_{k-1} x_{j,k-1} \\ u \cdot x_{j,k} &= (r^{-q} s^{(q-p)})^j x_{j,k+1} = (\theta\zeta)^{-j} x_{j,k+1} \end{aligned} \tag{8.6}$$

where $\eta = \theta^{y^2+z^2-yz} = r^y s^{(z-y)}$ and $\zeta = r^{(q-p)} s^p$. In (8.6) we are adopting the conventions that for $x_{j,k}$, the subscript j is read modulo ℓ, but $x_{j,k} = 0$ if $k \notin \{0, 1, \ldots, m-1\}$.

Note

$$\begin{aligned} &\omega_1 = \sigma^{-q}\tau^y, \qquad \omega_2 = \sigma^p \tau^{-z}, \qquad \text{so that} \\ &\omega_1 \cdot x_{j,k} = \eta^{-(n+k)q} x_{j+y,k} \\ &\omega_2 \cdot x_{j,k} = \eta^{(n+k)p} x_{j-z,k}. \end{aligned} \tag{8.7}$$

Now $A * H$ is a right coideal subalgebra of $B = A * G$, that is $\Delta(A * H) \subseteq (A * H) \otimes (A * G)$, so we may apply the following result to determine tensor products of $(A * G)$-modules. This proposition is analogous to a version of Frobenius reciprocity for finite groups, and the proof is a straightforward generalization. For a related result, see [T, Lemma 3.3]. The tensor identities for quantum groups (see [PW, Thm. 2.7]) are also similar, though they are phrased in the language of comodules.

Proposition 8.8. *Let B be a Hopf algebra and C a right coideal subalgebra of B. If M is a C-module and N is a B-module, then $(B \otimes_C M) \otimes N \cong B \otimes_C (M \otimes N)$ as B-modules.*

The B-modules $L^{n,t}(0)$.

Recall from Theorem 6.6 that $L(0) = L(0,0)$ has stabilizer G, so Clifford theory gives 1-dimensional B-modules on which d and u act as multiplication by 0, and σ and τ act as multiplication by (ℓ/e)th and ℓth roots of unity, respectively. Let $L^{n,t}(0)$ be such a module on which $\sigma = \omega_1^z\omega_2^y$ acts as multiplication by η^n and $\tau = \omega_1^p\omega_2^q$ acts as multiplication by θ^t. As the stabilizer of $L(0)$ is $H = G$, $L^{n,t}(0)$ is already a B-module and no induction to B is required. For this reason we have chosen not to use the notation $\widehat{L}^{n,t}(0)$ for these modules.

Lemma 8.9. *Let $n, n' \in \mathbb{Z}$ and $\lambda \in \mathbb{K}^*$. Then*

(i) $L^{n,t}(0) \otimes L^{n',t'}(0) \cong L^{n+n',t+t'}(0)$, *and*

(ii) $L^{n',t'}(0) \otimes \widehat{L}^n(\lambda) \cong \widehat{L}^{n+n'}(\lambda) \cong \widehat{L}^n(\lambda) \otimes L^{n',t'}(0)$.

Proof. (i) Let v_0 and v_0' be nonzero vectors in $L^{n,t}(0)$ and $L^{n',t'}(0)$, respectively. By the formulas (2.4) for coproducts, $d \cdot (v_0 \otimes v_0') = u \cdot (v_0 \otimes v_0') = 0$, $\sigma \cdot (v_0 \otimes v_0') = \eta^{n+n'} v_0 \otimes v_0'$, and $\tau \cdot (v_0 \otimes v_0') = \theta^{t+t'} v_0 \otimes v_0'$.

(ii) Let $\{x_{j,k} \mid j = 0, 1, \ldots, \ell-1; k = 0, 1, \ldots, m-1\}$ be the basis given in (8.5) for $\widehat{L}^n(\lambda)$ and let v_0' be a nonzero vector in $L^{n',t'}(0)$. Let $w_{j,k} = \theta^{jt'} v_0' \otimes x_{j,k}$ for $j = 0, \ldots, \ell-1$, $k = 0, \ldots, m-1$. Then by (8.6) and the coproduct formulas (2.4),

$$\begin{aligned}
\sigma \cdot w_{j,k} &= \theta^{jt'} \sigma v_0' \otimes \sigma \cdot x_{j,k} = \eta^{n+n'+k} w_{j,k},\\
\tau \cdot w_{j,k} &= \theta^{jt'} \tau \cdot v_0' \otimes \tau \cdot x_{j,k} = \theta^{(j+1)t'} v_0' \otimes x_{j+1,k} = w_{j+1,k},\\
d \cdot w_{j,k} &= (1 \otimes d + d \otimes \omega_1)(\theta^{jt'} v_0' \otimes x_{j,k}) = \zeta^j \lambda_{k-1} w_{j,k-1},\\
u \cdot w_{j,k} &= (1 \otimes u + u \otimes \omega_2)(\theta^{jt'} v_0' \otimes x_{j,k}) = (\theta\zeta)^{-j} w_{j,k+1}.
\end{aligned}$$

Therefore $L^{n',t'}(0) \otimes \widehat{L}^n(\lambda) \cong \widehat{L}^{n+n'}(\lambda)$ by (8.6).

For the second isomorphism in part (ii), our strategy is to determine $L^n(\lambda) \otimes L^{n',t'}(0)$ as an $A * H$-module, and then to apply Proposition 8.8. Again let v_0' be a nonzero vector in $L^{n',t'}(0)$, and let v_i, $i = 0, 1, \ldots, m-1$ be the standard basis for $L^n(\lambda)$. Then $ud \cdot (v_0 \otimes v_0') = 0$ as $d \cdot v_0 = 0 = d \cdot v_0'$, and

$$\begin{aligned}
(du) \cdot (v_0 \otimes v_0') &= (1 \otimes d + d \otimes \omega_1)(1 \otimes u + u \otimes \omega_2)(v_0 \otimes v_0')\\
&= (1 \otimes d + d \otimes \omega_1)(\eta^{n'p}\theta^{-t'z} v_1 \otimes v_0')\\
&= \eta^{n'(p-q)}\theta^{t'(y-z)}\lambda(v_0 \otimes v_0'),
\end{aligned}$$

by (8.7). Therefore $v_0 \otimes v_0'$ has weight $(\eta^{n'(p-q)}\theta^{t'(y-z)}\lambda, 0)$. By Proposition 4.11, $u^i \cdot (v_0 \otimes v_0')$ is a vector of weight $(\eta^{n'(p-q)}\theta^{t'(y-z)}\lambda_i, \eta^{n'(p-q)}\theta^{t'(y-z)}\lambda_{i-1})$. In addition, $u^m \cdot (v_0 \otimes v_0') = \Delta(u)^m(v_0 \otimes v_0') = 0$ as $u^m \cdot v_0 = 0$ and $u \cdot v_0' = 0$. Further,

$$\sigma \cdot (u^i \cdot (v_0 \otimes v_0')) = u^i \cdot \eta^i(\sigma \cdot v_0 \otimes \sigma \cdot v_0') = \eta^{n+n'+i} u^i \cdot (v_0 \otimes v_0').$$

Hence $L^n(\lambda) \otimes L^{n',t'}(0) \cong L^{n+n'}(\eta^{n'(p-q)}\theta^{t'(y-z)}\lambda)$ as $A * H$-modules. By Theorem 6.6, $L^{n+n'}(\eta^{n'(p-q)}\theta^{t'(y-z)}\lambda) \cong {}^hL^{n+n'}(\lambda)$ where $h = \omega_1^{-n'y(p-q)-t'}\omega_2^{n'(z-y)(p-q)-t'}$. Since conjugate $A * H$-modules induce to isomorphic B-modules, Proposition 8.8 yields

$$\begin{aligned}\widehat{L}^n(\lambda) \otimes L^{n',t'}(0) &\cong B \otimes_{A*H} \left(L^n(\lambda) \otimes L^{n',t'}(0)\right)\\ &\cong B \otimes_{A*H} \left(L^{n+n'}(\eta^{n'(p-q)}\theta^{t'(y-z)}\lambda)\right)\\ &\cong B \otimes_{A*H} L^{n+n'}(\lambda) = \widehat{L}^{n+n'}(\lambda). \quad \square\end{aligned}$$

Tensor decompositions.

Next we calculate the tensor products $\widehat{L}^0(\lambda) \otimes \widehat{L}^0(\mu)$ again using Proposition 8.8. Suppose $\{v_i\}_{i=0}^{m-1}$ is the basis for $L^0(\lambda)$ and $\{x_{j,k}\}_{j=0,k=0}^{\ell-1,m-1}$ is the basis for $\widehat{L}^0(\mu)$. From the expressions for the coproducts, we have

$$\begin{aligned}\sigma \cdot (v_i \otimes x_{j,k}) &= \eta^{i+k} v_i \otimes x_{j,k}\\ d \cdot (v_i \otimes x_{j,k}) &= \eta^{-kq}\lambda_{i-1} v_{i-1} \otimes x_{j+y,k} + \zeta^j \mu_{k-1} v_i \otimes x_{j,k-1} \qquad (8.10)\\ u \cdot (v_i \otimes x_{j,k}) &= (\theta\zeta)^{-j} v_i \otimes x_{j,k+1} + \eta^{kp} v_{i+1} \otimes x_{j-z,k}.\end{aligned}$$

Lemma 8.11. *For $c = 0, 1, \ldots, m-1$ and $j = 0, 1, \ldots, \ell-1$, suppose that*

$$T_j^c = \sum_{k=0}^{c} (-1)^k \mu^{c-k} \lambda^k \zeta^{-(kj+\frac{k(k+1)}{2}y)} \eta^{-\frac{k(k-1)}{2}q} \begin{bmatrix} c \\ k \end{bmatrix} v_{c-k} \otimes x_{j+ky,k}, \qquad (8.12)$$

where $[k] = (s^k - r^k)/(s-r)$; $[k]! = \prod_{n=1}^{k}[n]$ if $k \geq 1$ and $[0]! = 1$; and

$$\begin{bmatrix} c \\ k \end{bmatrix} = \frac{[c]!}{[k]![c-k]!}$$

Then $d \cdot T_j^c = 0$ and $(du) \cdot T_j^c = \theta^{-j}\mu T_j^c + s^c \lambda T_{j+y-z}^c$.

Proof. Showing that T_j^c is annihilated by d amounts to verifying that the identity

$$\eta^{-kq}\lambda_{c-k-1}\Gamma_{j,k}^c + \zeta^{j+(k+1)y}\mu_k \Gamma_{j,k+1}^c = 0 \qquad (8.13)$$

holds for $k = 0, 1, \ldots, c-1$, where $\Gamma_{j,k}^c = (-1)^k \mu^{c-k}\lambda^k \zeta^{-(kj+\frac{k(k+1)}{2}y)} \eta^{-\frac{k(k-1)}{2}q} \begin{bmatrix} c \\ k \end{bmatrix}$.

Using $T_j^c = \sum_{k=0}^{c} \Gamma_{j,k}^c v_{c-k} \otimes x_{j+ky,k}$ and (8.10), we compute that

$$\begin{aligned}(du) \cdot T_j^c &= \sum_{k=0}^{c} \Gamma_{j,k}^c (\theta\zeta)^{-(j+ky)} \lambda_{c-k-1} \eta^{-q(k+1)} v_{c-k-1} \otimes x_{j+(k+1)y,k+1}\\ &+ \sum_{k=0}^{c} \Gamma_{j,k}^c \theta^{-(j+ky)} \mu_k v_{c-k} \otimes x_{j+ky,k}\\ &+ \sum_{k=0}^{c} \Gamma_{j,k}^c \eta^{k(p-q)} \lambda_{c-k} v_{c-k} \otimes x_{j+(k+1)y-z,k} \qquad (8.14)\\ &+ \sum_{k=0}^{c} \Gamma_{j,k}^c \eta^{kp} \zeta^{j+ky-z} \mu_{k-1} v_{c-k+1} \otimes x_{j+ky-z,k-1}.\end{aligned}$$

The first two sums in (8.14) combine to give

$$\sum_{k=0}^{c}\Big(\Gamma^c_{j,k-1}(\theta\zeta)^{-(j+(k-1)y)}\lambda_{c-k}\eta^{-qk}+\Gamma^c_{j,k}\theta^{-(j+ky)}\mu_k\Big)v_{c-k}\otimes x_{j+ky,k},$$

where $\Gamma^c_{j,-1}=0$. Applying (8.13), we see that the coefficient of $v_{c-k}\otimes x_{j+ky,k}$ is

$$\begin{aligned}
&-(\theta\zeta)^{-(j+(k-1)y)}\eta^{-q}\zeta^{j+ky}\mu_{k-1}\Gamma^c_{j,k}+\theta^{-(j+ky)}\Gamma^c_{j,k}\mu_k\\
&\quad=\Gamma^c_{j,k}\theta^{-(j+ky)}\Big(\mu_k-(\theta\zeta)^y\eta^{-q}\mu_{k-1}\Big)\\
&\quad=\frac{\Gamma^c_{j,k}\theta^{-(j+ky)}\mu}{s-r}\Big(s^{k+1}-r^{k+1}-(r^{-q}s^{q-p})^{-y}(r^ys^{z-y})^{-q}(s^k-r^k)\Big)\\
&\quad=\frac{\Gamma^c_{j,k}\theta^{-(j+ky)}\mu}{s-r}\Big(s^{k+1}-r^{k+1}-s(s^k-r^k)\Big)\\
&\quad=\Gamma^c_{j,k}\theta^{-(j+ky)}\mu r^k=\Gamma^c_{j,k}\theta^{-j}\mu.
\end{aligned}$$

Note here we have used the relation $py-qz=1$.

Similarly, the coefficient of $v_{c-k}\otimes x_{j+y-z+ky,y}$ when the third and fourth sums in (8.14) are combined is

$$\begin{aligned}
&\Gamma^c_{j,k}\eta^{k(p-q)}\lambda_{c-k}+\Gamma^c_{j,k+1}\eta^{(k+1)p}\zeta^{j+(k+1)y-z}\mu_k\\
&\quad=\Gamma^c_{j,k}\eta^{k(p-q)}\lambda_{c-k}-\Gamma^c_{j,k}\eta^{(k+1)p-kq}\zeta^{-z}\lambda_{c-k-1}\\
&\quad=\Gamma^c_{j,k}\eta^{k(p-q)}\lambda\Big([c-k+1]-\eta^p\zeta^{-z}[c-k]\Big)\\
&\quad=\frac{\Gamma^c_{j,k}\eta^{k(p-q)}\lambda}{s-r}\Big(s^{c-k+1}-r^{c-k+1}-(r^ys^{(z-y)})^p(r^{(q-p)}s^p)^{-z}(s^{c-k}-r^{c-k})\Big)\\
&\quad=\frac{\Gamma^c_{j,k}\eta^{k(p-q)}\lambda}{s-r}\Big(s^{c-k+1}-r^{c-k+1}-r(s^{c-k}-r^{c-k})\Big)\\
&\quad=\Gamma^c_{j,k}\eta^{k(p-q)}\lambda s^{c-k}=\Gamma^c_{j+y-z,k}\zeta^{k(y-z)}\eta^{k(p-q)}\lambda s^{c-k}\\
&\quad=\Gamma^c_{j+y-z,k}(r^{(q-p)}s^p)^{k(y-z)}(r^ys^{(z-y)})^{k(p-q)}\lambda s^{c-k}=\Gamma^c_{j+y-z,k}\theta^{-kqzy+kpzy}\lambda s^{c-k}\\
&\quad=\Gamma^c_{j+y-z,k}\theta^{kz}\lambda s^{c-k}=\Gamma^c_{j+y-z,k}\lambda s^c.
\end{aligned}$$

Consequently, $(du)\cdot T^c_j=\theta^{-j}\mu T^c_j+s^c\lambda T^c_{j+y-z}$, as desired. □

Theorem 8.15. *Assume r and s are roots of unity as in (6.1). Suppose that $\lambda,\mu\in\mathbb{K}^*$. For each c $(0\le c\le m-1)$ and j $(0\le j\le \ell/m-1)$, assume that $(s^c\lambda)^m+(\theta^{-j}\mu)^m\neq 0$, and let $\nu_{c,j}$ be any mth root of $(s^c\lambda)^m+(\theta^{-j}\mu)^m$. Then*

$$\widehat{L}^0(\lambda)\otimes\widehat{L}^0(\mu)\cong\bigoplus_{c,j}m\widehat{L}^c(\nu_{c,j}),$$

where the sum is over c,j with $0\le c\le m-1$, and $0\le j\le \ell/m-1$. In particular, if $m=\ell$ and ν is any ℓth root of $\lambda^\ell+\mu^\ell$, then

$$\widehat{L}^0(\lambda)\otimes\widehat{L}^0(\mu)\cong\bigoplus_{c=0}^{\ell-1}\ell\ \widehat{L}^c(\nu).$$

Proof. By Proposition 8.8, $\widehat{L}^0(\lambda)\otimes\widehat{L}^0(\mu)\cong B\otimes_{A*H}\left(L^0(\lambda)\otimes\widehat{L}^0(\mu)\right)$. We show that $L^0(\lambda)\otimes\widehat{L}^0(\mu)\cong\bigoplus_{c,i,j}L^c(r^is^{-i}\nu_{c,j})$ as $A*H$-modules, where the sum is over c,i,j with $0\le c,i\le m-1$ and $0\le j\le \ell/m-1$. By the proof of Theorem 6.6, $L^c(r^is^{-i}\nu_{c,j})\cong {}^hL^c(\nu_{c,j})$, where $h=(\omega_1\omega_2)^{-i}$. As conjugate $A*H$-modules induce to isomorphic B-modules and induction of modules preserves direct sums, the theorem then follows.

For fixed c $(0\le c\le m-1)$, consider the action of du on $S^c=\mathrm{span}_{\mathbb{K}}\{T^c_j\mid j=0,1,\dots,\ell-1\}$. By Lemma 8.11, du preserves the subspace $S^c_j=\mathrm{span}_{\mathbb{K}}\{T^c_{j+k(y-z)}\mid k=0,1,\dots,m-1\}$ for each j. (The index k takes the indicated values as $m=\ell/\gcd(y-z,\ell)$.) Thus we obtain a decomposition $S^c=\oplus_{j=0}^{\ell/m-1}S^c_j$. On S^c_j, du acts by an $m\times m$ matrix whose characteristic polynomial is

$$\prod_{i=0}^{m-1}\left(t-\theta^{-j+i(z-y)}\mu\right)+(-1)^{m-1}(-s^c\lambda)^m.$$

Now $\theta^{i(z-y)}$, for $i=0,1,\dots,m-1$, runs over the distinct mth roots of 1. Therefore, the characteristic polynomial is $t^m-(\theta^{-j}\mu)^m-(s^c\lambda)^m$, with m distinct roots $\theta^{i(y-z)}\nu_{c,j}$ $(0\le i\le m-1)$ where $\nu_{c,j}$ is defined as in the statement of the lemma. Thus du is diagonalizable on S^c_j, with the specified eigenvalues. For c,i,j $(0\le c,i\le m-1,\ 0\le j\le \ell/m-1)$, let $w^c_{i,j}$ denote the eigenvector of du in S^c_j corresponding to the eigenvalue $\theta^{i(y-z)}\nu_{c,j}=r^is^{-i}\nu_{c,j}$, so that $w^c_{i,j}$ is a vector of weight $(r^is^{-i}\nu_{c,j},0)$ in $L^0(\lambda)\otimes\widehat{L}^0(\mu)$.

By (8.10), u acts nilpotently with nilpotent index no larger than $2m-1$. Applying powers of u to the vector $w^c_{i,j}$ generates a finite-dimensional quotient of the Verma module $V(r^is^{-i}\nu_{c,j})$ by Proposition 4.11. Such a quotient, on which u acts nilpotently, must be $V(r^is^{-i}\nu_{c,j})/\mathrm{span}_{\mathbb{K}}\{v_n\mid n\ge km\}$ for some k (see the proof of Proposition 4.7). As the nilpotent index of u is less than $2m$ in this case, the quotient is $L(r^is^{-i}\nu_{c,j})$. Since $\sigma\cdot w^c_{i,j}=\eta^cw^c_{i,j}$ (where $\sigma=\omega_1^z\omega_2^y$), the $A*H$-module generated by $w^c_{i,j}$ is $L^c(r^is^{-i}\nu_{c,j})$. Each such module has dimension m, and there are ℓm such submodules of $L^0(\lambda)\otimes\widehat{L}^0(\mu)$. As $L^0(\lambda)\otimes\widehat{L}^0(\mu)$ has dimension ℓm^2, it suffices to show that the sum of its distinct submodules $L^c(r^is^{-i}\nu_{c,j})$ is a direct sum, and it will follow that $L^0(\lambda)\otimes\widehat{L}^0(\mu)$ has the decomposition claimed. Assume that some $L^c(r^is^{-i}\nu_{c,j})$ has nonzero intersection with the sum of the remaining submodules $L^{c'}(r^{i'}s^{-i'}\nu_{c',j'})$. That is, there are scalars γ_k and $\delta_{k,c',i',j'}$ such that

$$\sum_k\gamma_ku^kw^c_{i,j}=\sum_{k,c',i',j'}\delta_{k,c',i',j'}u^kw^{c'}_{i',j'}.$$

Suppose that k' is largest with either $\gamma_{k'}\ne0$ or some $\delta_{k',c',i',j'}\ne0$. Apply $d^{k'}$ to obtain a nontrivial relation among the vectors $w^c_{i,j},w^{c'}_{i',j'}$. But these vectors are linearly independent by their definition. Therefore any such relation must be trivial, and so each $L^c(r^is^{-i}\nu_{c,j})$ must intersect the remaining $L^{c'}(r^{i'}s^{-i'}\nu_{c',j'})$ trivially. As a result, the sum is direct. □

Remark 8.16. When $(s^c\lambda)^m+(\theta^{-j}\mu)^m=0$ for some c,j, then the proof of Theorem 8.15 shows that du has characteristic polynomial t^m on S^c_j. By Lemma 8.11, as $m\ge2$, du does not act as multiplication by 0 on S^c_j. Therefore as an A-module, $L^0(\lambda)\otimes\widehat{L}^0(\mu)$ is not a sum of weight spaces, and therefore is not completely reducible. As G is finite, it follows that $\widehat{L}^0(\lambda)\otimes\widehat{L}^0(\mu)$ is not a completely reducible B-module.

Although it is not immediately apparent from the expression in Theorem 8.15, it is true that

$$\widehat{L}^0(\lambda)\otimes\widehat{L}^0(\mu)\cong\widehat{L}^0(\mu)\otimes\widehat{L}^0(\lambda).$$

To see this, fix c, $0\le c\le m-1$. For each j, $0\le j\le \ell/m-1$, we have $L^c(s^c\theta^j\nu_{c,j})\cong {}^hL^c(\nu_{c,j})$ for $h=\omega_1^{-jp}\omega_2^{c-jq}$ by the proof of Theorem 6.6, as $py-qz=1$. Therefore $\widehat{L}^c(\nu_{c,j})\cong\widehat{L}^c(s^c\theta^j\nu_{c,j})$. For each j, let j' $(0\le j'\le\ell/m-1)$ satisfy $j'=-j-2cz \pmod{\ell/m}$. Letting $\nu'_{c,j'}=s^c\theta^j\nu_{c,j}$, we find that

$$\begin{aligned}(\nu'_{c,j'})^m&=(s^c\theta^j\nu_{c,j})^m=s^{cm}\theta^{jm}\big((s^c\lambda)^m+(\theta^{-j}\mu)^m\big)\\&=(s^{2c}\theta^j\lambda)^m+(s^c\mu)^m=(\theta^{2cz-j'-2cz}\lambda)^m+(s^c\mu)^m\\&=(\theta^{-j'}\lambda)^m+(s^c\mu)^m.\end{aligned}$$

That is, $\nu'_{c,j'}$ is an mth root of $(s^c\mu)^m+(\theta^{-j'}\lambda)^m$, and we have shown that $\widehat{L}^c(\nu_{c,j})\cong\widehat{L}^c(\nu'_{c,j'})$. As j runs through the integers $0,1,\dots,\ell/m-1$, so does j', and thus

$$\bigoplus_{j=0}^{\ell/m-1} m\widehat{L}^c(\nu_{c,j})\cong\bigoplus_{j=0}^{\ell/m-1} m\widehat{L}^c(\nu'_{c,j}).$$

Applying Theorem 8.15 we obtain $\widehat{L}^0(\lambda)\otimes\widehat{L}^0(\mu)\cong\widehat{L}^0(\mu)\otimes\widehat{L}^0(\lambda)$.

Corollary 8.17. *Let $n,n'\in\mathbb{Z}$. Under the same hypotheses as in Theorem 8.15,*

$$\widehat{L}^n(\lambda)\otimes\widehat{L}^{n'}(\mu)\cong\bigoplus_{c,j} m\widehat{L}^{n+n'+c}(\nu_{c,j})$$

where the sum is over c,j with $0\le c\le m-1,\ \ 0\le j\le\ell/m-1$. In particular, if $m=\ell$ and ν is any ℓth root of $\lambda^\ell+\mu^\ell$, then

$$\widehat{L}^n(\lambda)\otimes\widehat{L}^{n'}(\mu)\cong\bigoplus_{c=0}^{\ell-1}\ell\widehat{L}^{n+n'+c}(\nu)\cong\bigoplus_{k=0}^{\ell-1}\ell\widehat{L}^k(\nu)\cong\widehat{L}^0(\lambda)\otimes\widehat{L}^0(\mu).$$

Proof. We apply coassociativity, Lemma 8.9, and Theorem 8.15 to obtain

$$\begin{aligned}\widehat{L}^n(\lambda)\otimes\widehat{L}^{n'}(\mu)&\cong\big(\widehat{L}^0(\lambda)\otimes L^{n,0}(0)\big)\otimes\big(\widehat{L}^0(\mu)\otimes L^{n',0}(0)\big)\\&\cong\big(\widehat{L}^0(\lambda)\otimes\widehat{L}^0(\mu)\big)\otimes\big(L^{n,0}(0)\otimes L^{n',0}(0)\big)\\&\cong\Big(\bigoplus_{c,j} m\widehat{L}^c(\nu_{c,j})\Big)\otimes L^{n+n',0}(0)\\&\cong\bigoplus_{c,j} m\widehat{L}^{c+n+n'}(\nu_{c,j}).\qquad\square\end{aligned}$$

Example 8.18. Suppose that $r=q$ and $s=q^{-1}$, where q is a primitive ℓth root of unity and ℓ is odd (i.e. suppose $A\cong U_q^+(\mathfrak{sl}_3)$ at an ℓth root of unity for ℓ odd). Then $m=\ell$, and in this particular example, Corollary 8.17 reduces to

$$\widehat{L}^n(\lambda)\otimes\widehat{L}^{n'}(\mu)\cong\bigoplus_{k=0}^{\ell-1}\ell\widehat{L}^k(\nu),$$

where ν is any ℓth root of $\lambda^\ell+\mu^\ell$.

References

[B] G. Benkart, *Down-up algebras and Witten's deformations of the universal enveloping algebra of* $\mathfrak{sl}_2$*;* ; *Recent Progress in Algebra*, Contemp. Math., vol. 224, Amer. Math. Soc., 1998, pp. 29-45.

[BR] G. Benkart and T. Roby, *Down-up algebras*, J. Algebra, **209** (1998), 305-344; ; Addendum **213** (1999), 378.

[Be] D.J. Benson, *Representations and Cohomology*, Volume I, Cambridge University Press, 1991.

[CM] P.A.A.B. Carvalho and I.M. Musson, *Down-up algebras and their representation theory*, J. Algebra (to appear).

[CR] C.W. Curtis and I. Reiner, *Methods of Representation Theory with Applications to Finite Groups and Orders*, vol. I, Wiley, New York, 1981.

[J] J.C. Jantzen, *Lectures on Quantum Groups*, vol. 6, Graduate Studies in Math. Amer. Math. Soc., Providence, 1996.

[Jo1] D.A. Jordan, *Finite-dimensional simple modules over certain iterated skew polynomial rings*, J. Pure Appl. Algebra **98** (1995), 45–55.

[Jo2] D.A. Jordan, *Down-up algebras and ambiskew polynomial rings*, J. Algebra (to appear).

[K] G. Karpilovsky, *Projective Representations of Finite Groups*, Marcel Dekker, New York, 1985.

[KMP] E.E. Kirkman, I. Musson, and D. Passman, *Noetherian down-up algebras*, Proc. Amer. Math. Soc. **127** (1999), 3161-3167.

[KS] E.E. Kirkman and L.W. Small, *q-analogs of harmonic oscillators and related rings*, Israel J. Math. **81** (1993), 111–127.

[Ku1] R.S. Kulkarni, *Irreducible representations of Witten's deformations of* $U(sl_2)$, J. Algebra **214** (1999), 64-91.

[Ku2] R.S. Kulkarni, *Down-up algebras and their representations* (to appear).

[M] S. Montgomery, *Hopf Algebras and Their Actions on Rings. CBMS Conf. Math. Publ.,*, vol. 82, Amer. Math. Soc. Providence, 1993.

[P] D.S. Passman, *Infinite Crossed Products*, Academic Press, Boston, 1989.

[PW] B. Parshall and J. Wang, *Quantum Linear Groups*, vol. 439, Memoirs Amer. Math. Soc., Providence, 1991.

[T] L.F. Tokoly, *Frobenius Reciprocity and Grothendieck Groups of Hopf Galois Extensions*, Ph.D. thesis, Temple University, 1999.

[W1] E. Witten, *Gauge theories, vertex models, and quantum groups*, Nuclear Phys. B **330** (1990), 285–346.

[W2] E. Witten, *Quantization of Chern-Simons gauge theory with complex gauge group*, Comm. Math. Phys. **137** (1991), 29–66.

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