SUPPORT VARIETIES AND REPRESENTATION TYPE OF SMALL QUANTUM GROUPS

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ABSTRACT. In this paper we provide a wildness criterion for any finite dimensional Hopf algebra with finitely generated cohomology. This generalizes a result of Farnsteiner to not necessarily cocommutative Hopf algebras over ground fields of arbitrary characteristic. Our proof uses the theory of support varieties for modules, one of the crucial ingredients being a tensor product property for some special modules. As an application we prove a conjecture of Cibils stating that small quantum groups of rank at least two are wild.

1. INTRODUCTION

Rickard [29] discovered an important connection between the representation type of a finite dimensional self-injective algebra over an algebraically closed field and the complexity of its modules: If there is a module with complexity greater than two, then the algebra is expected to be wild. However there is a gap in the proof. Rickard relies on [29, Lemma 1], for which there is a counterexample that was found recently, namely the truncated polynomial algebra $k[x,y]/(x^2, y^2)$ (see [15]). Farnsteiner [15] later recovered Rickard's result for finite group schemes, using the theory of support varieties. Bergh and Solberg [7] adapted Farnsteiner's approach to prove Rickard's result for self-injective algebras under some finiteness conditions on Ext-algebras of modules.

In this paper, we show that Farnsteiner's geometric methods apply more directly to any finite dimensional Hopf algebra whose cohomology is finitely generated as an algebra over an algebraically closed field of arbitrary characteristic. (More precisely, see assumption (**fg**) below.) Under this assumption, one may define the cohomological support variety of any finite dimensional module. Many of the expected properties hold just as for finite groups and, more generally, finite group schemes (i.e., finite dimensional cocommutative Hopf algebras). Some important classes of non-cocommutative finite dimensional Hopf algebras are known to satisfy our finite generation assumption (**fg**) under some restrictions on the parameters: Lusztig's small quantum groups [4, 19] and, more generally, finite dimensional pointed Hopf algebras having abelian groups of group-like elements [24], or certain truncated quantized function algebras of simply-connected complex semisimple

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algebraic groups [20]. Moreover, Etingof and Ostrik [14] conjectured that the cohomology of any finite tensor category is finitely generated; in case the tensor category is the representation category of a finite dimensional Hopf algebra, this cohomology is the same as that of the Hopf algebra.

One useful property of cohomological support varieties for finite group schemes is the tensor product property: Friedlander and Pevtsova [17] used rank varieties to prove that the variety of the tensor product of two modules for a finite group scheme is the intersection of their varieties. Rank varieties have not been defined in our general setting, however we give in Theorem 2.5 a weaker tensor product property in the general setting that suffices for our purposes. That is, the property holds for any tensor product of an arbitrary module with a special type of module analogous to those originally constructed by Carlson [9] in the finite group setting, Carlson's modules L_ζ. This was proved for quantum elementary abelian groups by Pevtsova and the second author [28], and we show here that the proof is valid more generally. This result is precisely what is needed to generalize Farnsteiner's results on the representation type of finite group schemes to finite dimensional Hopf algebras satisfying (**fg**). We prove that if there exists a module having complexity at least 3 (or equivalently the support variety of the module has dimension at least 3), then the Hopf algebra is wild. In fact, we prove a stronger version for blocks in Theorem 3.1.

We apply our main theorem in particular to small quantum groups, whose cohomology was computed originally by Ginzburg and Kumar [19] and later by Bendel, Nakano, Parshall, and Pillen [4] under weaker conditions on the parameter. Exploiting [24] to weaken these conditions even further, we show in Theorems 4.3 and 4.6 that if the rank of the simple Lie algebra $\mathfrak{g}$ is at least 2, then both the small quantum group $u_q(\mathfrak{g})$ and its Borel-type subalgebra $u_q^+(\mathfrak{g})$ are wild. This proves a conjecture of Cibils for $u_q(\mathfrak{g})$ [10], who established the analogous result for $u_q^+(\mathfrak{g})$, in the simply laced case, using completely different methods.

Throughout the paper, we let k be an algebraically closed field of arbitrary characteristic. For the applications in Section 4, we will take $k = \mathbb{C}$. We will assume that all modules over Hopf algebras or their blocks are unital left modules and that all tensor products are over k unless indicated otherwise.

2. Complexity and varieties

In this section, we first give some basic definitions and results on complexity and varieties for modules of a finite dimensional Hopf algebra, and then prove the tensor product property for Carlson's modules L_ζ.

Let $V_\bullet$ be a graded vector space over k with finite dimensional homogeneous components. Define the *rate of growth* $\gamma(V_\bullet)$ to be the smallest non-negative integer c such that there is a number b for which $\dim_k V_n \leq bn^{c-1}$ for all positive integers n. If no such c exists, we define $\gamma(V_\bullet)$ to be ∞.

Let A be a finite dimensional Hopf algebra over k. We will also denote by k the ground field as an A-module via the counit (or augmentation) of A. The *complexity* $\mathrm{cx}_A(M)$ of an A-module M may be defined in the standard way (cf. [6, Definition 5.3.4]):

Let $P_\bullet : \cdots \to P_1 \to P_0 \to M \to 0$ be a minimal projective resolution of M. Then $\mathrm{cx}_A(M) := \gamma(P_\bullet)$.

We will need the following properties of complexity.

Proposition 2.1. *Let A be a finite dimensional Hopf algebra over k, let H be a Hopf subalgebra of A, and let M be any finite dimensional A-module. Then*

(1) $\mathrm{cx}_A(M) \leq \mathrm{cx}_A(k)$.
(2) $\mathrm{cx}_H(M) \leq \mathrm{cx}_A(M)$.

Proof. (1): Consider a minimal projective resolution $\cdots \to P_1 \to P_0 \to k \to 0$ of the trivial A-module k. Tensoring this resolution with M yields a projective resolution $\cdots \to M \otimes P_1 \to M \otimes P_0 \to M \to 0$ of M, since the tensor product of any module with a projective module is again projective. It is clear that the rate of growth of this resolution is also $\mathrm{cx}_A(k)$, and because $\mathrm{cx}_A(M)$ is the rate of growth of a *minimal* projective resolution of M, the desired inequality follows.

(2): Note that A is a free H-module by the Nichols-Zoeller Theorem (see [26, Theorem 7] or [25, Theorem 3.1.5]), and so every projective resolution of A-modules restricts to a projective resolution of H-modules. $\square$

We will use the following result, which can be established using a long exact cohomology sequence.

Proposition 2.2. *Let A be a finite dimensional Hopf algebra over k, and let $0 \to M_1 \to M_2 \to M_3 \to 0$ be a short exact sequence of finite dimensional A-modules. Then*

$$\mathrm{cx}_A(M_i) \leq \max\{\mathrm{cx}_A(M_j), \mathrm{cx}_A(M_k)\},$$

whenever $\{i, j, k\} = \{1, 2, 3\}$.

For any A-module M, let $\mathrm{H}^\bullet(A, M) := \mathrm{Ext}_A^\bullet(k, M)$. The vector space $\mathrm{H}^\bullet(A, k)$ is an associative graded k-algebra under cup product, or equivalently under Yoneda composition (see [5, Section 3.2]). If M and N are any two A-modules, then $\mathrm{H}^\bullet(A, k)$ acts on $\mathrm{Ext}_A^\bullet(M, N)$ via the cup product, or equivalently as $-\otimes N$ followed by Yoneda composition (see [5, Proposition 3.2.1]).

We use the notational convention that

$$\mathrm{H}^{\mathrm{ev}}(A, k) := \begin{cases} \displaystyle\bigoplus_{n=0}^{\infty} \mathrm{H}^n(A, k), & \text{if } \mathrm{char}\, k = 2, \\[2em] \displaystyle\bigoplus_{n=0}^{\infty} \mathrm{H}^{2n}(A, k), & \text{if } \mathrm{char}\, k \neq 2. \end{cases}$$

Since $H^\bullet(A, k)$ is graded commutative (see [19, Section 5.6] or [30, Section 2.2]), in either case, $H^{ev}(A, k)$ is a commutative k-algebra. We need the following assumption for the main results in this paper.

Assumption (**fg**):

Assume $H^{ev}(A, k)$ is finitely generated, and that for any two finite dimensional A-modules M and N, the $H^{ev}(A, k)$-module $\text{Ext}_A^\bullet(M, N)$ is finitely generated.

This is known to be the case, for example, if A is finite dimensional cocommutative [18], if A is a small quantum group $u_q(\mathfrak{g})$ under some restrictions on q [4, 19], or, more generally, if A is a finite dimensional pointed Hopf algebra with an abelian group of group-like elements under some restrictions on the parameters [24] or if A is a certain truncated quantized function algebra of a simply-connected complex semisimple algebraic group for the degree of the parameter being not too small [20]. We note that $\text{Ext}_A^\bullet(M, N) \cong \text{Ext}_A^\bullet(k, M^* \otimes N)$, an isomorphism of $H^\bullet(A, k)$-modules. Therefore the second part of the assumption (**fg**) may be replaced by the assumption that $H^\bullet(A, M)$ is a finitely generated $H^{ev}(A, k)$-module for every finite dimensional A-module M.

Under the assumption (**fg**), we may define varieties for modules in the usual way:

Let M and N be A-modules. Let $I_A(M, N)$ be the annihilator of the action of $H^{ev}(A, k)$ on $\text{Ext}_A^\bullet(M, N)$, a homogeneous ideal of $H^{ev}(A, k)$, and let $\mathcal{V}_A(M, N)$ denote the maximal ideal spectrum of the finitely generated commutative k-algebra $H^{ev}(A, k)/I_A(M, N)$. As the ideal $I_A(M, N)$ is homogeneous, the variety $\mathcal{V}_A(M, N)$ is conical. If $M = N$, we write $I_A(M) := I_A(M, M)$ and $\mathcal{V}_A(M) := \mathcal{V}_A(M, M)$. The latter is called the *support variety* of M.

We will need the following connection between complexity and varieties.

Proposition 2.3. *Let A be a finite dimensional Hopf algebra over k satisfying (**fg**), and let M be a finite dimensional A-module. Then*

$$\text{cx}_A(M) = \dim \mathcal{V}_A(M).$$

Proof. The proof of [6, Proposition 5.7.2] applies in this context. We include the details for the convenience of the reader. The cup product maps $H^{ev}(A, k)$ to $\text{Ext}_A^\bullet(M, M)$, with kernel $I_A(M)$. By assumption (**fg**), $\text{Ext}_A^\bullet(M, M)$ is a finitely generated module over $H^{ev}(A, k)$, and so it is a finitely generated module over the quotient $H^{ev}(A, k)/I_A(M)$. By the definition of $\mathcal{V}_A(M)$ and because Krull dimension and rate of growth of a finitely generated commutative graded algebra coincide, we have

$$\dim \mathcal{V}_A(M) = \dim(H^{ev}(A, k)/I_A(M)) = \gamma(H^{ev}(A, k)/I_A(M)) = \gamma(\text{Ext}_A^\bullet(M, M)).$$

It remains to prove that $\text{cx}_A(M) = \gamma(\text{Ext}_A^\bullet(M, M))$.

Let $P_{\bullet}$ be a minimal resolution of M. The multiplicity of the projective cover $P_A(S)$ of a simple A-module S, as a direct summand of P_n, is

$$\dim_k \operatorname{Hom}_A(P_n, S) = \dim_k \operatorname{Ext}_A^n(M, S).$$

Let $\operatorname{Irr}(A)$ denote a complete set of representatives of isomorphism classes of simple A-modules. It follows that

$$\dim_k P_n = \sum_{S \in \operatorname{Irr}(A)} \dim_k P_A(S) \cdot \dim_k \operatorname{Ext}_A^n(M, S).$$

This shows that

$$\gamma(P_{\bullet}) \leq \max\{\gamma(\operatorname{Ext}_A^{\bullet}(M, S)) \mid S \in \operatorname{Irr}(A)\}.$$

On the other hand, for each simple A-module S, since $\operatorname{Ext}^{\bullet}(M, S)$ is a finitely generated $\operatorname{H}^{\operatorname{ev}}(A, k)$-module, and the action of $\operatorname{H}^{\operatorname{ev}}(A, k)$ on $\operatorname{Ext}_A^{\bullet}(M, S)$ factors through $\operatorname{Ext}_A^{\bullet}(M, M)$, we have that $\operatorname{Ext}_A^{\bullet}(M, S)$ is a finitely generated $\operatorname{Ext}_A^{\bullet}(M, M)$-module. This implies $\gamma(\operatorname{Ext}_A^{\bullet}(M, S)) \leq \gamma(\operatorname{Ext}_A^{\bullet}(M, M))$. It is an immediate consequence of $\dim_k \operatorname{Ext}_A^n(M, M) \leq \dim_k \operatorname{Hom}_k(P_n, M) = (\dim_k M)(\dim_k P_n)$ for every non-negative integer n that $\gamma(\operatorname{Ext}_A^{\bullet}(M, M)) \leq \gamma(P_{\bullet})$. In conclusion, we obtain

$$
\begin{aligned}
\max\{\gamma(\operatorname{Ext}_A^{\bullet}(M, S)) \mid S \in \operatorname{Irr}(A)\} &\leq \gamma(\operatorname{Ext}_A^{\bullet}(M, M)) \\
&\leq \gamma(P_{\bullet}) \\
&\leq \max\{\gamma(\operatorname{Ext}_A^{\bullet}(M, S)) \mid S \in \operatorname{Irr}(A)\}.
\end{aligned}
$$

Thus all the inequalities above are equalities, and since $\operatorname{cx}_A(M) = \gamma(P_{\bullet})$, we have shown that $\operatorname{cx}_A(M) = \gamma(\operatorname{Ext}_A^{\bullet}(M, M))$, as desired. □

We will also need the following properties of support varieties for modules. Again let $\operatorname{Irr}(A)$ denote a complete set of representatives of isomorphism classes of simple A-modules.

Proposition 2.4. *Let A be a finite dimensional Hopf algebra over k satisfying* (**fg**), *and let M and N be finite dimensional A-modules. Then*

(1) $\mathcal{V}_A(M) = \{0\}$ *if and only if M is projective.*
(2) $\mathcal{V}_A(M \oplus N) = \mathcal{V}_A(M) \cup \mathcal{V}_A(N)$.
(3) $\mathcal{V}_A(M, N) \subseteq \mathcal{V}_A(M) \cap \mathcal{V}_A(N)$.
(4) $\mathcal{V}_A(M) = \cup_{S \in \operatorname{Irr}(A)} \mathcal{V}_A(M, S) = \cup_{S \in \operatorname{Irr}(A)} \mathcal{V}_A(S, M)$.
(5) $\mathcal{V}_A(M \otimes N) \subseteq \mathcal{V}_A(M) \cap \mathcal{V}_A(N)$.

Proof. The proofs of (1)–(4) are exactly analogous to the proofs in [6, Section 5.7].

The proof of (5) is standard, however we provide the details for the convenience of the reader: The action of $\operatorname{H}^{\operatorname{ev}}(A, k)$ on $\operatorname{Ext}_A^{\bullet}(M \otimes N, M \otimes N)$ factors through its action on $\operatorname{Ext}_A^{\bullet}(M, M)$, as we may first apply $- \otimes M$, then $- \otimes N$, and finally Yoneda composition. Thus $I_A(M, M) \subseteq I_A(M \otimes N, M \otimes N)$, which implies that $\mathcal{V}_A(M \otimes N) \subseteq \mathcal{V}_A(M)$. To show that $\mathcal{V}_A(M \otimes N) \subseteq \mathcal{V}_A(N)$, note that $\operatorname{Ext}_A^{\bullet}(M \otimes$

$N, M \otimes N) \cong \mathrm{Ext}^{\bullet}_A(N, M^* \otimes M \otimes N)$, so $\mathcal{V}_A(M \otimes N) = \mathcal{V}_A(N, M^* \otimes M \otimes N)$, which is contained in $\mathcal{V}_A(N)$ by (3). □

Ostrik conjectured in [27, Remark 3.7(ii)] that the containment in (5) is an equality. In Theorem 2.5 below, we will prove equality in a special case.

Let $\Omega_A(M)$ denote the kernel of an epimorphism from the projective cover of M to M, let $n > 0$, and let $0 \neq \zeta \in \mathrm{H}^n(A, k) \cong \mathrm{Hom}_A(\Omega_A^n(k), k)$. We will define a corresponding A-module L_ζ, following the original construction by Carlson [9] for finite groups: Let L_ζ be the kernel of a representative map $\widehat{\zeta} : \Omega_A^n(k) \to k$. So L_ζ is defined by the short exact sequence

$$0 \longrightarrow L_\zeta \longrightarrow \Omega_A^n(k) \xrightarrow{\hat{\zeta}} k \longrightarrow 0 \,.$$

For each $\zeta \in \mathrm{H}^{\mathrm{ev}}(A, k)$, denote by $\langle \zeta \rangle$ the ideal of $\mathrm{H}^{\mathrm{ev}}(A, k)$ generated by ζ. For each ideal I of $\mathrm{H}^{\mathrm{ev}}(A, k)$, denote by $Z(I)$ its zero set, that is the set of all maximal ideals of $\mathrm{H}^{\mathrm{ev}}(A, k)$ containing I.

The following theorem is a special case of the conjectured equality $\mathcal{V}_A(M \otimes N) = \mathcal{V}_A(M) \cap \mathcal{V}_A(N)$.

Theorem 2.5. *Let A be a finite dimensional Hopf algebra over k satisfying (**fg**), let M be a finite dimensional A-module, and let ζ be a non-zero homogeneous element of positive degree in $\mathrm{H}^{\mathrm{ev}}(A, k)$. Then*

$$\mathcal{V}_A(M \otimes L_\zeta) = \mathcal{V}_A(M) \cap Z(\langle \zeta \rangle) \,.$$

In particular, $\mathcal{V}_A(L_\zeta) = Z(\langle \zeta \rangle)$.

Proof. The proof is essentially that of [28, Proposition 3] where it is stated only in the case that A is a quantum elementary abelian group. This result is in fact valid under our more general assumptions.

Let N and N' be finite dimensional A-modules, and let $\mathfrak{m}$ be a maximal ideal in $\mathrm{H}^{\mathrm{ev}}(A, k)$. Since $\mathrm{Ext}^{\bullet}_A(N, N')$ is finitely generated over $\mathrm{H}^{\mathrm{ev}}(A, k)$ by assumption (**fg**), we have that $\mathfrak{m} \in \mathcal{V}_A(N, N')$ if and only if $I_A(N, N') \subseteq \mathfrak{m}$ if and only if $\mathrm{Ext}^{\bullet}_A(N, N')_{\mathfrak{m}} \neq 0$.

We will first show that $\mathcal{V}_A(M) \cap Z(\langle \zeta \rangle) \subseteq \mathcal{V}_A(M \otimes L_\zeta)$. By applying Proposition 2.4(4) to M and to $M \otimes L_\zeta$, we have

$$\mathcal{V}_A(M) = \cup_{S \in \mathrm{Irr}(A)} \mathcal{V}_A(M, S) \text{ and } \mathcal{V}_A(M \otimes L_\zeta) = \cup_{S \in \mathrm{Irr}(A)} \mathcal{V}_A(M \otimes L_\zeta, S) \,.$$

Thus it suffices to show that

$$\mathcal{V}_A(M, S) \cap Z(\langle \zeta \rangle) \subseteq \mathcal{V}_A(M \otimes L_\zeta, S)$$

for every simple A-module S. Let $\mathfrak{m}$ be a maximal ideal in $\mathcal{V}_A(M, S) \cap Z(\langle \zeta \rangle)$. Then $I_A(M, S) \subseteq \mathfrak{m}$ and $\langle \zeta \rangle \subseteq \mathfrak{m}$, and therefore $\mathfrak{m}$ contains the ideal generated by $I_A(M, S)$ and ζ. We must show that $\mathfrak{m} \in \mathcal{V}_A(M \otimes L_\zeta, S)$, that is $I_A(M \otimes L_\zeta, S) \subseteq \mathfrak{m}$.

Suppose $I_A(M \otimes L_\zeta, S) \not\subseteq \mathfrak{m}$. Then as noted above, $\mathrm{Ext}_A^\bullet(M \otimes L_\zeta, S)_\mathfrak{m} = 0$. Apply $M \otimes -$ and $\mathrm{Ext}_A^\bullet(-, S)$ to the exact sequence

$$0 \to L_\zeta \to \Omega_A^n(k) \xrightarrow{\hat{\zeta}} k \to 0 \,.$$

Now $\Omega_A^n(M)$ is isomorphic to $M \otimes \Omega_A^n(k)$ up to projective direct summands, and so $\mathrm{Ext}_A^i(M \otimes \Omega_A^n(k), S) \cong \mathrm{Ext}_A^{i+n}(M, S)$. Thus we obtain a long exact sequence

$$\cdots \to \mathrm{Ext}_A^i(M, S) \xrightarrow{\tilde{\zeta}} \mathrm{Ext}_A^{i+n}(M, S) \xrightarrow{\eta} \mathrm{Ext}_A^i(M \otimes L_\zeta, S) \xrightarrow{\delta} \mathrm{Ext}_A^{i+1}(M, S) \to \cdots$$

where the map $\tilde{\zeta} : \mathrm{Ext}_A^i(M, S) \to \mathrm{Ext}_A^{i+n}(M, S)$ is just the action of $\zeta \in \mathrm{H}^n(A, k)$ on $\mathrm{Ext}_A^i(M, S)$. The maps in the sequence are $\mathrm{H}^{\mathrm{ev}}(A, k)$-module homomorphisms. Let $z \in \mathrm{Ext}_A^{i+n}(M, S)$, and let a be any homogeneous element of $I_A(M \otimes L_\zeta, S)$ that is not in $\mathfrak{m}$. So we have $\eta(az) = a\eta(z) = 0$. Considering the above long exact sequence, since $\eta(az) = 0$, we have $az = \zeta y$ for some $y \in \mathrm{Ext}_A^{i+\deg(a)}(M, S)$, so that $z = \zeta a^{-1} y \in \zeta \, \mathrm{Ext}_A^\bullet(M, S)_\mathfrak{m}$. Therefore

$$\mathrm{Ext}_A^j(M, S)_\mathfrak{m} = \zeta \, \mathrm{Ext}_A^j(M, S)_\mathfrak{m}$$

for all $j > n$. We will show that this also holds for $j \leq n$. Assume $z \in \mathrm{Ext}_A^j(M, S)$ for $j \leq n$. Let b be a homogeneous element of positive degree in $\mathrm{H}^{\mathrm{ev}}(A, k)$ that is not in $\mathfrak{m}$. Multiply z by a large enough power of b so that $\deg(b^m z) > n$. Then $b^m z \in \mathrm{Ext}_A^\bullet(M, S)_\mathfrak{m} = \zeta \, \mathrm{Ext}_A^\bullet(M, S)_\mathfrak{m}$ as above. Now b is invertible in $\mathrm{Ext}_A^\bullet(M, S)_\mathfrak{m}$, so we obtain $z \in \zeta \, \mathrm{Ext}_A^\bullet(M, S)_\mathfrak{m}$.

Since $\zeta \in \mathfrak{m}$ and $\mathrm{Ext}_A^\bullet(M, S)$ is finitely generated over $\mathrm{H}^{\mathrm{ev}}(A, k)$ by assumption (**fg**), Nakayama's Lemma applied to the local ring $\mathrm{H}^{\mathrm{ev}}(A, k)_\mathfrak{m}$ implies that $\mathrm{Ext}_A^\bullet(M, S)_\mathfrak{m} = 0$. This contradicts the assumption $I_A(M, S) \subseteq \mathfrak{m}$. Therefore $I_A(M \otimes L_\zeta, S) \subseteq \mathfrak{m}$, and so $\mathcal{V}_A(M, S) \cap Z(\langle\zeta\rangle) \subseteq \mathcal{V}_A(M \otimes L_\zeta, S)$.

To prove the opposite inclusion $\mathcal{V}_A(M \otimes L_\zeta) \subseteq \mathcal{V}_A(M) \cap Z(\langle\zeta\rangle)$, by Proposition 2.4(5) it suffices to show that $\mathcal{V}_A(L_\zeta) \subseteq Z(\langle\zeta\rangle)$. Applying Proposition 2.4(4) again, it is enough to show that $\mathcal{V}_A(L_\zeta, S) \subseteq Z(\langle\zeta\rangle)$ for every simple A-module S. Thus we need to show that if $\mathfrak{m}$ is a maximal ideal of $\mathrm{H}^{\mathrm{ev}}(A, k)$ for which $\mathrm{Ext}_A^\bullet(L_\zeta, S)_\mathfrak{m} \neq 0$, then $\zeta \in \mathfrak{m}$. Assume to the contrary that $\zeta \notin \mathfrak{m}$. Then multiplication by ζ induces an isomorphism on $\mathrm{Ext}_A^\bullet(k, S)_\mathfrak{m}$, since ζ is invertible in $\mathrm{H}^{\mathrm{ev}}(A, k)_\mathfrak{m}$. As localization is exact, the existence of the short exact sequence defining L_ζ implies that $\mathrm{Ext}_A^\bullet(L_\zeta, S)_\mathfrak{m}$ is the kernel of the isomorphism $\zeta : \mathrm{Ext}_A^\bullet(k, S)_\mathfrak{m} \to \mathrm{Ext}_A^{\bullet+n}(k, S)_\mathfrak{m}$ and therefore $\mathrm{Ext}_A^\bullet(L_\zeta, S)_\mathfrak{m} = 0$.

Finally, by setting $M = k$ we obtain the second statement. $\qquad\square$

As a consequence of the theorem, we may find modules for blocks having prescribed varieties:

Corollary 2.6. *Let A be a finite dimensional Hopf algebra over k satisfying (**fg**). Let B be a block of A, let M be a finite dimensional B-module, and let d be a*

positive integer. Then for any non-zero $\zeta \in \mathrm{H}^{2d}(A, k)$, there exists a B-module N_ζ such that $\mathcal{V}_A(N_\zeta) = Z(\langle\zeta\rangle) \cap \mathcal{V}_A(M)$ and $\dim_k N_\zeta \leq (\dim_k M)(\dim_k \Omega_A^{2d}(k))$.

Proof. The proof is exactly the same as [15, Proposition 2.2], using Theorem 2.5 and Proposition 2.4(1),(2) in the last part of the argument. The module N_ζ is defined to be $e \cdot (M \otimes L_\zeta)$, where e is the primitive central idempotent associated to B. The crucial observation in the proof is that $(1 - e) \cdot (M \otimes L_\zeta)$ is projective since $\Omega_A^{2d}(M)$ and $M \otimes \Omega_A^{2d}(k)$ differ by a projective module. $\qquad\square$

Remark 2.7. A further consequence of the theorem is that any conical subvariety of $\mathcal{V}_A(k)$ can be realized as the support variety of some module: If $I = \langle\zeta_1, \ldots, \zeta_t\rangle$ is any homogeneous ideal in $\mathrm{H}^{\mathrm{ev}}(A, k)$, we may successively apply both parts of Theorem 2.5 to obtain the A-module $M = L_{\zeta_1} \otimes \cdots \otimes L_{\zeta_t}$ with the property that $\mathcal{V}_A(M) = Z(I)$. Similarly, we may use Corollary 2.6 to obtain a generalization of [15, Corollary 2.3], namely that any conical subvariety of the variety $\mathcal{V}_B := \cup_{S \in \mathrm{Irr}(B)} \mathcal{V}_A(S)$ of a block B can be realized as the support variety of some B-module. Avramov and Iyengar [3, Existence Theorem 5.4] proved a more general realizability result that does not use tensor products.

3. Complexity and representation type

Theorem 3.1 below generalizes [15, Theorem 3.1]. We show that Farnsteiner's proof works in this more general context, as an application of the general theory developed in Section 2. For a similar result in the setting of self-injective algebras, see [7, Corollary 4.2]. Here we develop and apply the theory for $\mathrm{H}^\bullet(A, k)$ directly, in the spirit of prior work on finite group schemes.

Recall that finite dimensional associative algebras over an algebraically closed field can be divided into three classes (see [11, Corollary C] or [5, Theorem 4.4.2]): An algebra A is *representation-finite* if there are only finitely many isomorphism classes of finite dimensional indecomposable A-modules. An algebra A is called *tame* if it is not representation-finite and if the isomorphism classes of indecomposable A-modules in any fixed dimension are almost all contained in a finite number of one-parameter families. An algebra A is said to be *wild* if the category of finite dimensional A-modules contains the category of finite dimensional modules over the free associative algebra in two indeterminates. (For more precise definitions we refer the reader to [5, Definition 4.4.1].) Note that the classification of indecomposable objects (up to isomorphism) of the latter category is a well-known unsolvable problem and so one is only able to classify the finite dimensional indecomposable modules of representation-finite or tame algebras.

Theorem 3.1. *Let A be a finite dimensional Hopf algebra over k for which assumption (**fg**) holds, and let B be a block of A. If there is a finite dimensional B-module M such that $\mathrm{cx}_A(M) \geq 3$, then B is wild.*

Proof. Let $\mathrm{Irr}(B)$ denote a complete set of representatives for the isomorphism classes of simple B-modules. Let $M_B := \oplus_{S \in \mathrm{Irr}(B)} S$, let $\mathcal{V}_B := \mathcal{V}_A(M_B)$ denote the support variety of M_B, and set $n := \dim \mathcal{V}_B$. It follows from Proposition 2.2 that $n \geq \mathrm{cx}_A(M) \geq 3$. By [15, Lemma 1.1], there are non-zero elements ζ_s $(s \in k)$ in $\mathrm{H}^{2d}(A, k)$ for some $d > 0$ such that

(i) $\dim Z(\langle \zeta_s \rangle) \cap \mathcal{V}_B = n - 1$ for all $s \in k$,
(ii) $\dim Z(\langle \zeta_s \rangle) \cap Z(\langle \zeta_t \rangle) \cap \mathcal{V}_B = n - 2$ for $s \neq t \in k$.

By applying Corollary 2.6 to M_B, there are B-modules N_{ζ_s} such that

$$\mathcal{V}_A(N_{\zeta_s}) = Z(\langle \zeta_s \rangle) \cap \mathcal{V}_B \quad \text{and} \quad \dim_k N_{\zeta_s} \leq (\dim_k M_B)(\dim_k \Omega_A^{2d}(k))$$

for all $s \in k$. Decompose N_{ζ_s} into a direct sum of indecomposable B-modules. By Proposition 2.4(2) and (i) above, there is an indecomposable direct summand X_s of N_{ζ_s} such that

$$\mathcal{V}_A(X_s) \subseteq \mathcal{V}_A(N_{\zeta_s}) = Z(\langle \zeta_s \rangle) \cap \mathcal{V}_B$$

and $\dim \mathcal{V}_A(X_s) = n - 1$. We will show that $\mathcal{V}_A(X_s) \neq \mathcal{V}_A(X_t)$ when $s \neq t$: Note that if $\mathcal{V}_A(X_s) = \mathcal{V}_A(X_t)$ then

$$\mathcal{V}_A(X_s) \subseteq Z(\langle \zeta_s \rangle) \cap Z(\langle \zeta_t \rangle) \cap \mathcal{V}_B\,,$$

and the latter variety has dimension $n - 2$ if $s \neq t$. Therefore the varieties $\mathcal{V}_A(X_s)$ are distinct for different values of s, implying that the indecomposable B-modules X_s are pairwise non-isomorphic. The dimensions of the modules X_s are all bounded by $(\dim_k M_B)(\dim_k \Omega_A^{2d}(k))$, since X_s is a direct summand of N_{ζ_s}. Consequently, there are infinitely many non-isomorphic indecomposable B-modules X_s of some fixed dimension. By Proposition 2.3, $\mathrm{cx}_A(X_s) = \dim \mathcal{V}_A(X_s) = n - 1 \geq 2$.

Suppose now that B is not wild. By [11, Corollary C], B is tame or representation-finite and it follows from [11, Theorem D] that only finitely many indecomposable B-modules of any dimension (up to isomorphism) are not isomorphic to their Auslander-Reiten translates. Since the Auslander-Reiten translation τ_A is the same as $\nu_A \circ \Omega_A^2 = \Omega_A^2 \circ \nu_A$ where ν_A denotes the Nakayama automorphism of A (see [2, Proposition IV.3.7(a)]) and the Nakayama automorphism of any finite dimensional Hopf algebra has finite order (see [16, Lemma 1.5]), it is clear that any module isomorphic to its Auslander-Reiten translate is periodic and thus has complexity 1. Hence in any dimension there are only finitely many isomorphism classes of indecomposable B-modules with complexity not equal to 1. This cannot be the case, as we have shown that for some dimension there are infinitely many non-isomorphic indecomposable B-modules of complexity greater than 1. Therefore B is wild. □

As an immediate consequence of Theorem 3.1 and the Trichotomy Theorem [11, Corollary C] one obtains as in [15, Corollary 3.2]:

Corollary 3.2. *Let A be a finite dimensional Hopf algebra over k for which assumption (**fg**) holds. If B is a tame block of A, then $\mathrm{cx}_A(M) \leq 2$ for every finite dimensional B-module M.*

4. Small quantum groups

Let $\mathfrak{g}$ be a finite dimensional complex simple Lie algebra, Φ its root system, h its Coxeter number, and $r := \mathrm{rank}(\Phi)$ its rank. Let $\ell > 1$ be an odd integer and assume that ℓ is not divisible by 3 if Φ is of type G_2. Let q be a primitive ℓ-th root of unity in $\mathbb{C}$. Let $U_q(\mathfrak{g})$ denote Lusztig's quantum group [22] and let $u_q(\mathfrak{g})$ denote the small quantum group [23].

Now fix a set of simple roots, and let Φ^+ and Φ^- denote the corresponding sets of positive and negative roots, respectively. Then $\mathfrak{g}$ has a standard Borel subalgebra corresponding to Φ^+ and an opposite standard Borel subalgebra corresponding to Φ^-. We will also use the notation $u_q^+(\mathfrak{g})$ to denote the Hopf subalgebra of $u_q(\mathfrak{g})$ corresponding to the standard Borel subalgebra of $\mathfrak{g}$ and $u_q^-(\mathfrak{g})$ to denote the Hopf subalgebra of $u_q(\mathfrak{g})$ corresponding to the opposite standard Borel subalgebra of $\mathfrak{g}$. Note that $u_q^-(\mathfrak{g}) \cong u_{q^{-1}}^+(\mathfrak{g})$ (cf. [12, (1.2.10)]).

We will need some knowledge about the complexity of the trivial $u_q^+(\mathfrak{g})$-module $\mathbb{C}$ (which according to Proposition 2.1(1) is the module of largest complexity). Let ρ be half the sum of the positive roots, let $\langle -, - \rangle$ denote the positive definite symmetric bilinear form on the real vector space spanned by Φ such that $\langle \alpha, \alpha \rangle = 2$ if α is a short root of Φ, and set

$$\Phi_0^+ := \{\alpha \in \Phi^+ \mid \langle \rho, \alpha^\vee \rangle \in \ell\mathbb{Z}\}.$$

Let $U_q^0(\mathfrak{g})$ denote the subalgebra of $U_q(\mathfrak{g})$ generated by the group-like elements $K_i^{\pm 1}$ ($1 \leq i \leq r$) and the elements $\begin{bmatrix} K_i; N \\ n \end{bmatrix}$ ($1 \leq i \leq r$; $N \in \mathbb{Z}$, $n \in \mathbb{Z}_{\geq 0}$) (see [22, (4.1)(b)]). Since Weyl's character formula holds for the quantized induced modules $H_q^0(\lambda)$ (see [1, Corollary 5.12]) and the rest of the argument in [32, §2.2–§2.5] depends only on the characters of the involved modules, one can prove the following result by replacing the category of B_1T-modules by the category of $u_q^+(\mathfrak{g})U_q^0(\mathfrak{g})$-modules, the induced module $H^0(\lambda)$ by the quantized induced module $H_q^0(\lambda)$, p by ℓ, and then specializing to $\lambda = 0$ (see [32, (2.5.2)] for the analogue for the first Frobenius kernel B_1 of the Borel subgroup B):

Lemma 4.1. *Let q be a primitive ℓ-th root of unity and assume that $\ell > 1$ is an odd integer not divisible by 3 if Φ is of type G_2. Then $\mathrm{cx}_{u_q^+(\mathfrak{g})}(\mathbb{C}) \geq |\Phi^+| - |\Phi_0^+|$.*

The following result, a consequence of [13, Theorem 5.6] or [24, Theorem 6.3], ensures that assumption (**fg**) holds for $A = u_q^+(\mathfrak{g})$. In these papers it is shown that the full cohomology ring is finitely generated. It follows that the even cohomology ring is finitely generated: Without loss of generality, the generators of the full cohomology ring are homogeneous. Take the set consisting of all generators in

even degree and all products of pairs of generators in odd degree. Since odd degree elements are nilpotent, this set generates the cohomology in even degrees. A similar argument applies to modules over the cohomology ring.

Theorem 4.2. *Let q be a primitive ℓ-th root of unity and assume that ℓ is an odd integer not divisible by 3 if Φ is of type G_2. Then the even cohomology ring $\mathrm{H}^{\mathrm{ev}}(u_q^+(\mathfrak{g}), \mathbb{C})$ is finitely generated. Moreover, if M is a finite dimensional $u_q^+(\mathfrak{g})$-module, then $\mathrm{H}^{\bullet}(u_q^+(\mathfrak{g}), M)$ is finitely generated as an $\mathrm{H}^{\mathrm{ev}}(u_q^+(\mathfrak{g}), \mathbb{C})$-module.*

The following theorem was proved by Cibils [10, Propositions 3.1 and 3.3] in the simply laced case for $\ell \geq 5$ by purely representation-theoretic methods using quivers and bimodule complements, and, in general, for $\ell > h$ by Gordon [21, Theorem 7.1(a)(i)] using geometric methods similar to ours (see also [8, Corollary 4.6(ii)] for replacing the restriction $\ell > h$ by ℓ good).

Theorem 4.3. *Let $r \geq 2$ and let q be a primitive ℓ-th root of unity. Assume that $\ell > 1$ is an odd integer not divisible by 3 if Φ is of type G_2. Then $u_q^+(\mathfrak{g})$ is wild.*

Note that $u_q^+(\mathfrak{g})$ has only one block. If $r = 1$, then $u_q^+(\mathfrak{g})$ is known to be representation-finite [10, Proposition 3.3].

Proof. First suppose $\ell \geq h$. Then $\Phi_0^+ = \emptyset$ and consequently Lemma 4.1 implies that $\mathrm{cx}_{u_q^+(\mathfrak{g})}(\mathbb{C}) \geq |\Phi^+| \geq 3$ since the rank r of Φ is at least 2 by hypothesis. In particular, this argument applies to A_2 for any ℓ, since in this case $h = 3$. By Theorem 3.1, $u_q^+(\mathfrak{g})$ is wild.

Now suppose $\ell < h$ and that $\mathfrak{g}$ is neither of type B_2 nor of type G_2. Then $u_q^+(\mathfrak{g})$ contains $u_q^+(\mathfrak{sl}_3)$ as a Hopf subalgebra. Applying Proposition 2.1(2), because $\ell \geq 3$ by hypothesis, it follows that $\mathrm{cx}_{u_q^+(\mathfrak{g})}(\mathbb{C}) \geq \mathrm{cx}_{u_q^+(\mathfrak{sl}_3)}(\mathbb{C}) \geq 3$, and as before, $u_q^+(\mathfrak{g})$ is wild by Theorem 3.1.

It remains to consider the cases when $\mathfrak{g}$ is of type B_2 or of type G_2, and $\ell < h$. If $\mathfrak{g}$ is of type B_2, then $h = 4$, and by hypothesis $\ell = 3$, which yields $|\Phi_0^+| = 1$. It follows from Lemma 4.1 that $\mathrm{cx}_{u_q^+(\mathfrak{g})}(\mathbb{C}) \geq 3$, implying again that $u_q^+(\mathfrak{g})$ is wild by Theorem 3.1. If $\mathfrak{g}$ is of type G_2, then $h = 6$ and by hypothesis $\ell = 5$, which yields $|\Phi_0^+| = 1$. It follows as before that $\mathrm{cx}_{u_q^+(\mathfrak{g})}(\mathbb{C}) \geq 5$, implying again that $u_q^+(\mathfrak{g})$ is wild by Theorem 3.1. $\square$

Remark 4.4. By carrying out the first part of the proof only for type A_2 and instead dealing with B_2 and G_2 also for $\ell \geq h$ one indeed needs to use Lemma 4.1 only in the rank two case.

The following result is a consequence of [24, Corollary 6.5], and it ensures that assumption (**fg**) holds for $A = u_q(\mathfrak{g})$.

Theorem 4.5. *Let q be a primitive ℓ-th root of unity and assume that $\ell > 1$ is an odd integer not divisible by 3 if Φ is of type G_2. Then the even cohomology*

ring $\mathrm{H}^{\mathrm{ev}}(u_q(\mathfrak{g}),\mathbb{C})$ *is finitely generated. Moreover, if M is a finite dimensional $u_q(\mathfrak{g})$-module, then $\mathrm{H}^{\bullet}(u_q(\mathfrak{g}), M)$ is finitely generated as an $\mathrm{H}^{\mathrm{ev}}(u_q(\mathfrak{g}),\mathbb{C})$-module.*

Note that under slightly stronger conditions on ℓ the above theorem is also a consequence of the explicit computation of the cohomology ring $\mathrm{H}^{\bullet}(u_q(\mathfrak{g}),\mathbb{C})$ which is due to Ginzburg and Kumar [19, Theorem 3] for $\ell > h$, and to Bendel, Nakano, Parshall, and Pillen [4, Theorem 1.3.4] for $\ell \leq h$ such that ℓ is neither a bad prime for Φ nor $\ell = 3$ for Φ of types B and C.

The *principal block* of a Hopf algebra is defined to be the block corresponding to the one-dimensional trivial module. The following theorem implies a conjecture of Cibils [10, p. 542].

Theorem 4.6. *Let $r \geq 2$ and let q be a primitive ℓ-th root of unity. Assume that $\ell > 1$ is an odd integer not divisible by 3 if Φ is of type G_2. Then the principal block of $u_q(\mathfrak{g})$ is wild.*

If $r = 1$, then $u_q(\mathfrak{g})$ is known to be tame [31, 33].

Proof. It follows from the proof of Theorem 4.3 that $\mathrm{cx}_{u_q^+(\mathfrak{g})}(\mathbb{C}) \geq 3$ if $r \geq 2$. Since $u_q^+(\mathfrak{g})$ is a Hopf subalgebra of $u_q(\mathfrak{g})$, we can apply Proposition 2.1(2) to obtain $\mathrm{cx}_{u_q(\mathfrak{g})}(\mathbb{C}) \geq \mathrm{cx}_{u_q^+(\mathfrak{g})}(\mathbb{C}) \geq 3$. Then Theorem 3.1 implies that the principal block of $u_q(\mathfrak{g})$ is wild. $\square$

Remark 4.7. It is also possible to give a direct proof of Theorem 4.6 along the lines of the proof of Theorem 4.3 by replacing Φ_0^+ by $\Phi_0 := \{\alpha \in \Phi \mid \langle \rho, \alpha^{\vee} \rangle \in \ell\mathbb{Z}\}$ and then using [4, Theorem 8.2.1(a)] instead of Lemma 4.1.

Note that Gordon also proved the wildness of certain truncated quantized function algebras of simply-connected connected complex semisimple algebraic groups at roots of unity of odd degree (not divisible by 3 if the group has a component of type G_2) [21, Remark (ii) after Theorem 7.1].

We conclude by mentioning that Cibils' proof also works for even ℓ to which we hope to return in a future paper. Note also that Cibils' proof uses the subalgebra of $u_q^+(\mathfrak{g})$ corresponding to the largest nilpotent ideal of the standard Borel subalgebra of $\mathfrak{g}$ (which is not a Hopf subalgebra) whereas our proof does not need this.

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