Volumes of projection bodies of some classes of convex bodies

Christos Saroglou
University of Crete
Department of Mathematics

Abstract

Schneider posed the problem of determining the maximal value of the affine invariant $|\Pi K|/|K|^{d-1}$, where $\Pi K$ is the projection body of the $d$-dimensional convex body $K$. Some three-dimensional conjectures of Brannen, related to Schneider’s problem are confirmed. Namely, we determine the maximal value of $|\Pi K|/|K|^2$ in the class of three-dimensional zonoids, cones and double cones. Equality cases are, also, investigated. Moreover, results related to a conjecture of Petty, concerning the minimal value of the above quantity are obtained. In particular, we provide a negative answer to a question of Martini and Mustaev.

1. Introduction

Let $K$ be a convex body in $\mathbb{R}^d$, that contains 0 in its interior. Its support function is defined by:

$$h_K(x) = \max\{ < x, y > \mid y \in K \}, \; x \in \mathbb{R}^d,$$

where $< \cdot, \cdot >$ stands for the usual inner product in $\mathbb{R}^d$. Obviously, $h_K$ is convex and positively homogeneous. On the other hand, it is known that any convex and positively homogeneous function is the support function of a unique convex set. Moreover, support functions are additive under Minkowski sums (i.e. vector sums). To be more specific, if $L$ is another convex body, then $h_{K+L} = h_K + h_L$.

One may compute the support function of a line segment $[-y,y]$, centered at the origin:

$$h_{[-y,y]}(x) = | < x, y > |, \; x \in \mathbb{R}^d.$$
Thus, if $Z$ is the Minkowski sum of the line segments $[-y_i, y_i]$, $i = 1, ..., n$,

$$h_Z(x) = \sum_{i=1}^{n} |\langle x, y_i \rangle|, \quad x \in \mathbb{R}^d.$$ 

Translations of such bodies are called zonotopes and the members of the closure of the set of all zonotopes with respect to the Hausdorff metric are called zonoids.

The projection body $\Pi K$ of $K$ is defined by its support function:

$$h_{\Pi K}(x) = |K| x^+ = \frac{1}{2} \int_{S^{d-1}} |\langle x, y \rangle| dS(K, y), \quad x \in S^{d-1},$$ 

where $|\cdot| = |\cdot|_d$ denotes the volume functional in $\mathbb{R}^d$, $K | x^+$ is the orthogonal projection of $K$ onto the subspace $x^+$ orthogonal to $x$ and $dS(K, \cdot)$ is the surface area measure of $K$ on the unit sphere $S^{d-1}$.

Clearly, the projection body of a convex body is always a zonoid. The opposite is also true; any zonoid is the projection body of a convex body. In fact, the operator $\Pi$ is a continuous bijection between the class of centrally symmetric convex bodies and the class of zonoids. We refer to [6] or [20] for proofs, extensions and topics related to the mentioned properties, concerning support functions and projection bodies.

One of the outstanding problems in convex geometry is the determination of the extremal values of the affine invariant $|\Pi K|/|K|^{d-1}$ (a proof of the fact that this functional is indeed affine invariant can be found in [14]). Many variants of this problem are solved. For example, Petty [15] showed that the quantity $|\langle (\Pi K)^* \rangle| |K|^{d-1}$ is maximal if and only if $K$ is an ellipsoid and Zhang [23] proved that it attains its minimal value if and only if $K$ is a simplex. Here, $(\Pi K)^* = \{ y | \langle x, y \rangle \leq 1, \; x \in \Pi K \}$ is the polar body of $\Pi K$. Other types of modifications have been considered by Lutwak (see e.g. [8], [9]). Unfortunately, very little progress has taken place towards the direction of the initial problem.

Petty’s conjecture [15] states that the ratio $|\Pi K|/|K|^{d-1}$ is minimal if and only if $K$ is an ellipsoid (see [7] for applications). Schneider [19] conjectured that in the class of centrally symmetric convex bodies

$$\frac{|\Pi K|}{|K|^{d-1}} \leq 2^d,$$

with equality if and only if $K$ is the affine image of cartesian products of line segments or centrally symmetric planar convex figures (the class of these bodies is identical to the class of symmetric cylinders in the three-dimensional
Both Petty’s and Schneider’s conjectures make sense only if $d \geq 3$. In fact, if $K$ is a planar convex body, it is well known that:

$$4 \leq \frac{|\Pi K|}{|K|} \leq 6,$$

with equality in the left if and only if $K$ is centrally symmetric and in the right if and only if $K$ is a triangle. These facts follow immediately from (5), (7) below and the two-dimensional Rogers-Shephard inequality [18], respectively.

Counterexamples to Schneider’s conjecture were given by Brannen [2]. The same author [2] [3] conjectured that it would be true if we restricted ourselves in the class of zonoids. We prove this fact in three dimensions. We hope that the method described below can be modified to work in any dimension.

**Theorem 1.** If $Z$ is a three-dimensional zonoid, then

$$|\Pi Z| \leq 2^4 |Z|^2.$$

Equality holds if and only if $Z$ can be written as the Minkowski sum of five line segments or as the sum of a cylinder and a line segment.

Using an inequality for the volume of polar projection bodies due to Rei-
sner [16] [17], Theorem 1 shows that for all 3-dimensional zonoids $Z$ we have

$$|\Pi Z|^* \cdot |Z|^2 \geq \frac{4}{3},$$

with equality if and only if $Z$ is a parallelepiped. We note here that the problem of finding the minimum of the quantity $|\Pi K|^* \cdot |K|^{d-1}$ among centrally symmetric convex bodies still remains open. Makai and Martini [11] conjectured that this minimum is attained if and only if $K$ is a parallelepiped.

The proof of (2) and of the characterization of equality cases will be given in Sections 3 and 4 respectively.

We also deal with two other three-dimensional classes of convex bodies. To be more specific, we determine the extremal values of $|\Pi K|/|K|^2$ in the special case, in which $K$ is a cone or a centrally symmetric double cone.

**Theorem 2.** Let $K = conv(P \cup \{e_3\})$ be a three-dimensional cone, where $P$ is a convex body in $\mathbb{R}^2 \times \{0\}$ of area 1 and $e_3 = (0, 0, 1)$. Then,

$$|\Pi K| = \frac{1}{2} + \frac{1}{4}|\Pi P|.$$
Corollary 1. Let $K$ be a cone in $\mathbb{R}^3$. Then,
\[ 13.5 \leq \frac{\|\Pi K\|}{|K|^2} \leq 18. \]
Equality holds in the right if and only if $K$ is a simplex and in the left if and only if $K$ has centrally symmetric basis.

Corollary 1 follows immediately from (1) and Theorem 2.

Corollary 2. Let $K$ be a centrally symmetric double cone in $\mathbb{R}^3$. Then,
\[ \frac{\|\Pi K\|}{|K|^2} = 9. \]

Corollaries 1 and 2 are also conjectures of Brannen [3]. Theorem 2 and Corollary 2 will be treated in Section 5. We mention that Brannen conjectured that $\|\Pi K\|/|K|^2$ is maximal in the class of centrally symmetric convex bodies if and only if $K$ is a centrally symmetric double cone and in the class of general convex bodies if and only if $K$ is a simplex.

The Steiner symmetrization $St_\nu K$ of a $d$-dimensional convex body $K$ along the direction $\nu \in S^{d-1}$ is defined to be the unique convex body with the property that for any line $l$ parallel to $\nu$, the line segment $l \cap St_\nu K$ is symmetric with respect to the hyperplane $\nu^\perp$ and also $|l \cap St_\nu K|_1 = |l \cap K|_1$. Martini and Mustafaev [10] asked if the inequality
\[ |\Pi(St_\nu K)| \leq |\Pi K| \] (3)
holds for every direction $\nu \in S^{n-1}$. It is well known that the volume of $K$ remains unchanged under Steiner symmetrization and, furthermore, $K$ can always be transformed to a ball after applying an appropriate sequence of Steiner symmetrizations. Thus, it is clear that Petty’s conjectured inequality would follow from (3). We prove, however, that (3) is not true in general. This will be an easy application of Corollary 2.

Theorem 3. For any $d \geq 3$, there exists a convex body $K$ and a direction $\nu \in S^{d-1}$, such that
\[ |\Pi(St_\nu K)| > |\Pi K| . \]

Proof. We will make use of the following easy fact: If $K$ is a convex body of volume 1 in $\mathbb{R}^{d-1}$, then
\[ |\Pi(K \times [-1/2, 1/2])|_d = 2 \cdot |\Pi K|_{d-1} . \] (4)
Note that if $K$ is a convex body contained in the subspace orthogonal to some direction $\nu$ and $I$ is a line segment parallel to $\nu$, then for any unit vector $w$ in $\nu^\perp$, we have
\[ St_w(K + I) = (St_w K) + I. \]
Using this fact, (4) and an inductive argument, we conclude that we only have to construct a three-dimensional counterexample for (3).

Let $C$ be a centered three dimensional cube of volume 1. Choose some vertex $v$ of $C$ and set $\nu$ to be the direction parallel to $[-v, v]$. Then, one may check that the Steiner symmetrization of $C$ along the direction $\nu$ is a centrally symmetric double cone built on a regular hexagon, which is contained in the plane $\nu^\perp$. Then, by Corollary 2 we have:
\[ |\Pi C| = 8 < 9 = |\Pi (St_\nu C)|. \]

Some further results on Petty’s conjecture, involving centroid bodies and mean values of volumes of projection bodies, are included in Section 6.

2. Some basic formulas

Let $Z = \sum_{i=1}^n [-x_i, x_i]$ be a zonotope in $\mathbb{R}^d$. The volume of $Z$ is given by (see [22] for proof and extensions):
\[ |Z| = 2^d \sum_{\{i_1, \ldots, i_d\} \subseteq [n]} |\det(x_{i_1}, \ldots, x_{i_d})| = \frac{2^d}{d!} \sum_{i_1, \ldots, i_d \in [n]} |\det(x_{i_1}, \ldots, x_{i_d})|, \tag{5} \]
where $[n] := \{1, \ldots, n\}$.

Thus, if $F_1, \ldots, F_n$ are the facets of a polytope $K$ in $\mathbb{R}^d$ with corresponding outer normal unit vectors $x_1, \ldots, x_n$, by the definition of $\Pi K$ and (5), we have:
\[ |\Pi K| = \sum_{\{i_1, \ldots, i_d\} \subseteq [n]} |F_{i_1}| \cdots |F_{i_d}| \cdot |\det(x_{i_1}, \ldots, x_{i_d})| \tag{6} \]

Suppose, now, that $K = Z = \sum_{i=1}^n [-x_i, x_i]$. If we, in addition, assume that any $d$ vectors from $x_1, \ldots, x_n$ are linearly independent, it can be proven that its facets are exactly (up to translation) the $(d-1)$-dimensional parallelepipeds of the form $\sum_{i=1}^{d-1} [-x_{j_i}, x_{j_i}]$, where $1 \leq j_1 < \ldots < j_{d-1} \leq n$. In other words, the outer unit normals to the facets of $Z$, multiplied by the $(d-1)$-dimensional volume of the corresponding facet, are exactly the vectors:
\[ \pm 2^{d-1} x_{i_1} \wedge \ldots \wedge x_{i_{d-1}}, \ 1 \leq i_1 < \ldots < i_{d-1} \leq n, \]
where \( x_1 \wedge \ldots \wedge x_{d-1} \) stands for the vector product of \( x_1, \ldots, x_{d-1} \).

Applying formula (6), we immediately obtain:

\[
|\Pi Z| = \frac{2^{d^2}}{(d-1)!^d d!} \sum_{i_1, \ldots, i_d(d-1) \in [n]} |\det(x_{i_1} \wedge \ldots \wedge x_{i_{d-1}}, \ldots, x_{i(d-1)(d-1)+1} \wedge \ldots \wedge x_{i(d-1)})|.
\]

(7)

It is obvious that this identity holds even if we do not assume the \( x_i \)'s to be in general position.

### 3. Proof of the main inequality

Define the functions \( S, T : (\mathbb{R}^3)^6 \to \mathbb{R}_+ \) by:

\[
S(x_1, \ldots, x_6) = \sum_{i_1, \ldots, i_6 \in [6]} |\det(x_{i_1}, x_{i_2}, x_{i_3}) \cdot \det(x_{i_4}, x_{i_5}, x_{i_6})|,
\]

\[
T(x_1, \ldots, x_6) = \sum_{i_1, \ldots, i_6 \in [6]} |\det(x_{i_1} \wedge x_{i_2}, x_{i_3} \wedge x_{i_4}, x_{i_5} \wedge x_{i_6})|.
\]

Clearly, \( S \) and \( T \) are convex and positively homogeneous on each one of their variables. Also, it is easy to see that \( S(x_1, \ldots, x_6) = 0 \) if and only if \( T(x_1, \ldots, x_6) = 0 \). We will use the convexity property in the following form:

**Lemma 3.1.** Let \( f, g \) be real functions defined on an open interval \((a, b)\) of the real line. Suppose, also, that \( g \) is strictly positive in \((a, b)\), \( f \) is convex and \( g \) is affine. Then, the ratio \( f/g \) admits a maximum value in \((a, b)\) if and only if \( f \) is a constant multiple of \( g \).

Let us now rewrite (5) and (7) involving \( T \) and \( S \). If \( Z = \sum_{i=1}^n [-x_i, x_i] \) is a zonotope in \( \mathbb{R}^3 \), we first note that we may assume that \( n \geq 6 \) (by taking some of the \( x_i \)'s equal to each other, if necessary). We have:

\[
|\Pi Z| = \frac{2^9}{2^3 \cdot 3!} \sum_{i_1, \ldots, i_6 \in [n]} |\det(x_{i_1} \wedge x_{i_2}, x_{i_3} \wedge x_{i_4}, x_{i_5} \wedge x_{i_6})|
\]

\[
= \frac{2^6}{3!} \sum_{i_1, i_2, i_3 \in [n]} T(x_{i_1}, x_{i_2}, x_{i_3}, x_{i_4})
\]

6
of the following inequality:

Observing (8) and (9), we conclude that the proof of (2) reduces to the proof of

Similarly,

Similarly,

Similarly,

Similarly,

Similarly,

Similarly,

Similarly,

Similarly,

Similarly,

Similarly,

Similarly,

Similarly,

Similarly,

Similarly,

Similarly,

Similarly,

Similarly,

Similarly,

Similarly,

Similarly,

Similarly,

Similarly,

Similarly,

Similarly,

Similarly,

Similarly,

Similarly,

Similarly,

Similarly,

Similarly,

Similarly,

Similarly,

Similarly,

Similarly,

Similarly,

Similarly,

Similarly,

Similarly,

Similarly,

Similarly,

Similarly,

Similarly,

Similarly,

Similarly,

Similarly,

Similarly,

Similarly,

Similarly,

Similarly,

Similarly,

Similarly,

Similarly,

Similarly,

Similarly,

Similarly,

Similarly,

Similarly,

Similarly,

Similarly,

Similarly,

Similarly,

Similarly,

Similarly,

Similarly,

Similarly,

Similarly,

Similarly,

Similarly,

Similarly,

Similarly,

Similarly,

Similarly,

Similarly,

Similarly,

Similarly,
where

\[
\lambda_4 \cdot \det(x_1, x_2, x_4) \cdot \det(x_3, x_4, x_6) = \det(x_1, x_2, x_4) \cdot \det(x_3, \lambda_4 x_4 + \lambda_6 x_6, x_6)
\]

\[
= \det(x_1, x_2, x_4) \cdot \det(x_3, x_5, x_6) . \quad \Box
\]

**Lemma 3.3.** Let \(x_1, \ldots, x_6\) be vectors in \(\mathbb{R}^3\). If two of them are parallel, then

\[
T(x_1, \ldots, x_6) = \frac{4}{3} S(x_1, \ldots, x_6) .
\]

Proof. Obviously, we may assume that all \(x_1, \ldots, x_6\) are unit vectors. Also, by symmetry, one can take \(x_5 = x_6\). It follows by the previous lemma that

\[
| \det(x_5 \wedge x_{i_1}, x_5 \wedge x_{i_2}, x_{i_3} \wedge x_{i_4}) | = | \det(x_5, x_{i_1}, x_{i_2}) \cdot \det(x_5, x_{i_3}, x_{i_4}) |
\]

\[
= | \det(x_5 \wedge x_{i_2}, x_5 \wedge x_{i_4}, x_{i_1} \wedge x_{i_2}) | ,
\]

where \(\{i_1, i_2, i_3, i_4\} = \{1, 2, 3, 4\} \). Thus,

\[
T(x_1, \ldots, x_6) = 2 \cdot 2^3 \cdot 3! \sum_{\{i_1, i_2, i_3, i_4\} = \{1, 2, 3, 4\}} | \det(x_5 \wedge x_{i_1}, x_5 \wedge x_{i_2}, x_{i_3} \wedge x_{i_4}) |
\]

\[
= 2 \cdot 2^3 \cdot 3! \sum_{\{i_1, i_2, i_3, i_4\} = \{1, 2, 3, 4\}} | \det(x_5, x_{i_1}, x_{i_2}) \cdot \det(x_5, x_{i_3}, x_{i_4}) |
\]

\[
= \frac{2 \cdot 2^3 \cdot 3!}{2 \cdot 2} \sum_{\{i_1, i_2, i_3, i_4\} = \{1, 2, 3, 4\}} | \det(x_5, x_{i_1}, x_{i_2}) \cdot \det(x_5, x_{i_3}, x_{i_4}) |
\]

\[
= \frac{2 \cdot 2^3 \cdot 3!}{2^2 \cdot 2 \cdot 3^2} S(x_1, \ldots, x_6) = \frac{4}{3} S(x_1, \ldots, x_6) . \quad \Box
\]

Let \(Z\) be a zonotope in \(\mathbb{R}^3\), which is the sum of five line segments. As mentioned above, one can write \(Z = \sum_{i=1}^{6} [-x_i, x_i]\), for some \(x_1, \ldots, x_6\) in \(\mathbb{R}^3\) with \(x_5 = x_6\). The previous lemma combined with (8) and (9) ensures that \(|\Pi Z| = 8|Z|^2\).

Suppose, now, that \(x_1, \ldots, x_s\) are vectors in \(\mathbb{R}^3\). We set for simplicity \(\mathcal{E}(x_1, \ldots, x_s)\) to be the set of all planes through 0, spanned by pairs of vectors from \(x_1, \ldots, x_s\).

**Lemma 3.4.** Let \(x_1, \ldots, x_6\) be vectors that span \(\mathbb{R}^3\), such that any two of them are not parallel. Assume that for every \(i = 1, \ldots, 6\), there exist two different planes \(E_1, E_2\) from \(\mathcal{E}(x_1, \ldots, x_i-1, x_{i+1}, \ldots, x_6)\), that contain \(x_i\). Then, after a possible rearrangement of indices, the sets of coplanar vectors from \(x_1, \ldots, x_6\) are exactly the following:

\[
\{x_1, x_2, x_3\} , \ {x_2, x_4, x_5} , \ {x_1, x_5, x_6} , \ {x_3, x_4, x_6} .
\]
Proof. Clearly, any five vectors from \( x_1, ..., x_6 \) cannot lie in the same plane. We assume, without loss of generality that \( x_1, x_2, x_3 \) are linearly dependent. Since there exists a plane \( E \) in \( \mathcal{E}(x_1, x_3, ..., x_6) \) that contains \( x_2 \), different than the one spanned by \( x_1, x_3 \), we may assume that \( x_2, x_4, x_5 \) are linearly dependent, while each one of \( x_4, x_5 \) is not coplanar with \( x_1, x_3 \).

Similarly, either \( x_1 \) is linearly dependent with \( x_5, x_6 \) or it is linearly dependent with \( x_4, x_6 \). We may assume that the first case occurs. Now, \( x_6 \) cannot be contained in any of the planes spanned by \( x_4, x_5 \) and \( x_1, x_2 \) (in the opposite case, five vectors from the \( x_i \)'s would be coplanar). This forces \( x_3 \) to be coplanar with \( x_4, x_6 \).

We have shown that these sets are indeed linearly dependent. If there existed another subset of \( \{x_1, ..., x_6\} \) with this property, five vectors from the \( x_i \)'s would be coplanar, which is impossible. \( \square \)

The key to the proof of (10) will be the next lemma.

**Lemma 3.5.** Let \( x_1, ..., x_6 \) be vectors, for which the conclusion of Lemma 3.4 holds. Then

\[
T(x_1, ..., x_6) < \frac{4}{3}S(x_1, ..., x_6).
\]

Proof. Consider the following subsets of the set \( U \) of summands in \( T(x_1, ..., x_6) \):

\[
U_{ij} = \left\{ \left| \det(x_i \wedge x_j, x_i \wedge x_{i2}, x_i \wedge x_{i4}) \right| \neq 0 \mid \{i_1, i_2, i_3, i_4\} = \{1, ..., 6\} \setminus \{i, j\} \right\},
\]

\( i, j = 1, ..., 6, \ i \neq j \). It is clear, that the sets \( U_{12}, U_{23}, U_{13}, U_{24}, U_{25}, U_{45}, U_{26} \) cover \( U \). It follows from Lemma 3.2 that the elements of \( U_{12} \) are exactly the terms of the form:

\[
\left| \det(x_1, x_2, x_{i1}) \cdot \det(x_{i2}, x_{i3}, x_{i4}) \right| \neq 0, \ \{i_1, i_2, i_3, i_4\} = \{3, 4, 5, 6\}.
\]

Similar expressions can be derived for the elements of \( U_{23} \) and \( U_{13} \). Hence, since \( |\det(x_1, x_2, x_3)| = 0 \) and \( U_{12}, U_{23}, U_{13} \) are disjoint the sum of all terms that belong to \( V_1 := U_{12} \cup U_{23} \cup U_{13} \), is a constant multiple of \( S(x_1, ..., x_6) \).

One may easily compute this constant to be \( 2/3 \).

Similarly, the sum of all terms contained in \( V_2 := U_{21} \cup U_{25} \cup U_{45} \) also equals \( 2/3 \cdot S(x_1, ..., x_6) \).

Clearly, terms of the form (each one counted \( 2^3 \cdot 3\)-times)

\[
\left| \det(x_4 \wedge x_5, x_1 \wedge x_6, x_2 \wedge x_3) \right|, \ \left| \det(x_4 \wedge x_5, x_1 \wedge x_2, x_3 \wedge x_6) \right|, \ \left| \det(x_2 \wedge x_5, x_1 \wedge x_3, x_4 \wedge x_6) \right| \text{ belong both to } V_1 \text{ and } V_2.
\]

Thus, if \( A \) is the sum of terms from \( V_1 \cup V_2 \), we have:

\[
A \leq \frac{4}{3}S(x_1, ..., x_6) - 2^3 \cdot 3! \left| \det(x_4 \wedge x_5, x_1 \wedge x_6, x_2 \wedge x_3) \right|
\]

9
Consequently, such that: For all $i$ the assertion is obvious. In any other case, there exists some 

Proof of (10): completing the proof.

where we used once again Lemma 3.2.

Next, we observe that $x_1, x_5, x_6$ are coplanar, so by Lemma 3.2 we have: 

Consequently, 

$$U_{26 \setminus (V_1 \cup V_2) = \{ | det(x_2 \wedge x_6 , x_1 \wedge x_4 , x_3 \wedge x_5) | \}}$$

and since $| det(x_4, x_5, x_3) \cdot det(x_1, x_2, x_6) | > 0$, we conclude: 

$$T(x_1, ..., x_6) < \frac{4}{3} S(x_1, ..., x_6) + 2^3 \cdot 3! \left[ | det(x_2 \wedge x_6 , x_1 \wedge x_4 , x_3 \wedge x_5) | 

- | det(x_4, x_5, x_1) \cdot det(x_2, x_3, x_6) | - | det(x_1, x_3, x_5) \cdot det(x_2, x_4, x_6) | \right].$$

Finally, by assumption, there exist numbers $\lambda_4, \lambda_6$ such that $x_3 = \lambda_4 x_4 + \lambda_6 x_6$. Using the fact that $x_1, x_5, x_6$ are coplanar, we have: 

$$| det(x_2 \wedge x_6 , x_1 \wedge x_4 , x_3 \wedge x_5) | \leq | \lambda_4 \cdot det(x_2 \wedge x_6 , x_1 \wedge x_4 , x_4 \wedge x_5) | 

+ | \lambda_6 \cdot det(x_2 \wedge x_6 , x_1 \wedge x_4 , x_6 \wedge x_5) | 

= | \lambda_4 \cdot det(x_4, x_5, x_1) \cdot det(x_4, x_2, x_6) | + | \lambda_6 \cdot det(x_4, x_5, x_6) \cdot det(x_1, x_2, x_6) | 

= | det(x_3, x_5, x_1) \cdot det(x_4, x_2, x_6) | + | det(x_4, x_5, x_1) \cdot det(x_2, x_3, x_6) |,$$ 

completing the proof. □

Proof of (10):

If the assumptions of Lemma 3.3 or Lemma 3.4 are true or $S(x_1, ..., x_6) = 0$, the assertion is obvious. In any other case, there exists some $i$ in $\{1, ..., 6\}$, such that $x_i$ belongs to at most one plane from $E(x_1, ..., x_{i-1}, x_{i+1}, ..., x_6)$. It is easy, then, to see that there exist real numbers $t_1 < 0 < t_2$ and a vector $\nu$ such that: For all $E \in E(x_1, ..., x_{i-1}, x_{i+1}, ..., x_6)$, we have:

$$x_i \in E \iff x_i + t \nu \in E \text{, for all } t \in (t_1, t_2)$$
and also
\[ \#E(x_1, \ldots, x_{i-1}, x_i + t_j \nu, x_{i+1}, \ldots, x_6) < \#E(x_1, \ldots, x_6), \ j = 1, 2. \]  \tag{11} 

We note here that this fact will be also used in the next section.

Consequently, the function
\[ [t_1, t_2] \ni t \mapsto S(x_1, \ldots, x_{i-1}, x_i + t \nu, x_{i+1}, \ldots, x_6) \]

is affine. Thus, by the convexity of \( T(x_1, \ldots, x_{i-1}, x_i + t \nu, x_{i+1}, \ldots, x_6) \) and by Lemma 3.1, we conclude that
\[ \frac{T}{S}(x_1, \ldots, x_6) \leq \frac{T}{S}(x_1, \ldots, x_{i-1}, x_i + t_j \nu, x_{i+1}, \ldots, x_6), \]

for \( j = 1 \) or 2 (clearly, \( S(x_1, \ldots, x_i + t_j \nu, \ldots, x_6) \) cannot be zero for both \( j = 1, 2 \); in the opposite case, \( S(x_1, \ldots, x_6) \) would be zero).

We may repeat this procedure as many times as needed. Nevertheless, as (11) shows, after only a finite number of steps we will have found vectors \( z_1, \ldots, z_6 \) for which the conditions of Lemma 3.3 or Lemma 3.4 are true and also \((T/S)(x_1, \ldots, x_6) \leq (T/S)(z_1, \ldots, z_6)\).  \( \square \)

4. Characterization of extremal zonoids

**Lemma 4.1.** Let \( x_1, \ldots, x_6 \) be vectors in \( \mathbb{R}^3 \). If four of them are coplanar, then
\[ T(x_1, \ldots, x_6) = \frac{4}{3} S(x_1, \ldots, x_6). \]

Proof. Suppose e.g. that \( x_1, x_2, x_3, x_4 \) are coplanar. If \((i_1, i_2, i_3, i_4)\) is a permutation on \(\{1, 2, 3, 4\}\), it is clear that
\[ |\det(x_{i_1} \wedge x_{i_2}, x_{i_3} \wedge x_{i_4}, x_{i_5} \wedge x_{i_6})| = 0 = |\det(x_{i_1}, x_{i_2}, x_{i_3}) \cdot \det(x_{i_4}, x_{i_5}, x_{i_6})|. \]

Also, by Lemma 3.2, it follows that
\[ |\det(x_{i_1} \wedge x_{i_2}, x_{i_3} \wedge x_{i_4}, x_{i_5} \wedge x_{i_6})| = |\det(x_{i_1}, x_{i_2}, x_{i_3}) \cdot \det(x_{i_4}, x_{i_5}, x_{i_6})| \]
\[ = |\det(x_{i_1}, x_{i_2}, x_{i_3} \wedge x_{i_4}, x_{i_5} \wedge x_{i_6})|. \]

Hence,
\[ T(x_1, \ldots, x_6) = 2^3 \cdot 3! \sum_{\substack{(i_1, i_2, i_3, i_4) \
i i_1 < i_2 \\ni \{i_3, i_4\} \in S_4}} |\det(x_{i_1} \wedge x_{i_2}, x_{i_3} \wedge x_{i_4}, x_{i_5} \wedge x_{i_6})|. \]
\[ = 2 \cdot 2^3 \cdot 3! \sum_{(i_1, i_2, i_3, i_4) \in S_4} | \det(x_{i_1}, x_{i_2}, x_3) \cdot \det(x_{i_3}, x_{i_4}, x_6) | = \frac{4}{3} S(x_1, \ldots, x_6), \]

where \( S_4 \) is the set of permutations on \( \{1, 2, 3, 4\} \).

Suppose, now, that \( Z \) is the sum of at least six line segments \([-x_i, x_i], i = 1, \ldots, n\) (as mentioned above, this is no loss of generality). If \( Z \) is the sum of a cylinder and a line segment, then for any six vectors from \( x_1, \ldots, x_n \), at least two are collinear or at least four are coplanar. The previous Lemma, Lemma 3.3, (8) and (9) show that \( |\Pi Z| = 2^3|Z|^2 \). Since the same is true when \( Z \) is the sum of five line segments, we only have to prove the ”only if” part in Theorem 1.

It is now clear that the problem of characterization of zonoids, for which equality in (2) is attained, reduces to the determination of the 6-tuples \( (x_1, \ldots, x_6) \) such that \( T(x_1, \ldots, x_6) = (4/3) \cdot S(x_1, \ldots, x_6) \). If the conditions of Lemma 3.3 or Lemma 4.1 hold, the last equality is true. In what follows, we will show that these are the only possible equality cases.

To accomplish this, we need a series of geometric lemmas. The proof of the following is obvious.

**Lemma 4.2.** Let \( E \) be a plane in \( \mathbb{R}^3 \), that does not contain 0 and \( y_1, \ldots, y_5 \) be points in \( E \), with \( y_1 \neq y_5 \). Suppose, also, that \( y_i \) lies in the line segment \([y_1, y_2]\) and \( y_i \) is not collinear with \( y_1, y_2 \), \( i = 4, 5 \). If \( x_i \) is the position vector of \( y_i \), \( i = 1, \ldots, 5 \), then there exists a vector \( \nu \) in \( \mathbb{R}^3 \) and real numbers \( t_1 < 0 < t_2 \), such that \( x_3 + t_i \nu \) is parallel to \( x_i \), \( i = 1, 2 \) and \( \det(x_3 + t_i \nu, x_4, x_5) \neq 0 \) for all \( t \) in \( (t_1, t_2) \), if and only if \( y_3 \) is an interior point of \([y_1, y_2]\), while at the same time the line \( \text{aff} \{y_4, y_5\} \) and the interior of the segment \([y_1, y_2]\) are disjoint.

In order to make use of the previous lemma, we observe that the ratio \( T/S(x_1, \ldots, x_6) \) is independent of the length and the orientation of the \( x_i \)’s. Thus, we may assume that the endpoints of \( x_1, \ldots, x_6 \) all lie in the same plane in \( \mathbb{R}^3 \), not containing the origin.

Let \( A = \{x_1, \ldots, x_6\} \) be a set of six vectors in \( \mathbb{R}^3 \). We say that \( A \) has the (N)-property, if no two of \( x_1, \ldots, x_6 \) are parallel and no four of them are coplanar.

**Lemma 4.3.** Let \( \{x_1, \ldots, x_6\} \) be a set of vectors in \( \mathbb{R}^3 \) having the (N)-property, where \( x_i \) is the position vector of some point \( y_i \), \( i = 1, \ldots, 6 \). We assume the following:

i) \( y_1, \ldots, y_6 \) are coplanar and \( y_6 \) is an interior point of the segment \([y_1, y_2]\).

ii) For every \( i, j, i \neq j \), in \( \{3, 4, 5\} \) the line spanned by the points \( y_i, y_j \), and
the interior of \([y_1, y_2]\) are disjoint.

iii) There exists some permutation \((k_1, k_2, k_3)\) of \(\{3, 4, 5\}\) and some interior point \(y\) of \([y_1, y_2]\), such that the lines \(\text{aff}\{y_1, y_{k_1}\}\), \(\text{aff}\{y_2, y_{k_2}\}\), \(\text{aff}\{y, y_{k_3}\}\) are either parallel or they have a common point, which is different from \(y_3, y_4, y_5\).

Then,

\[
T(x_1, ..., x_6) < \frac{4}{3}S(x_1, ..., x_6) .
\]

Proof. Let us assume that \(T(x_1, ..., x_6) = (4/3) \cdot S(x_1, ..., x_6)\). According to the previous lemma, there exist numbers \(t_1 < 0 < t_2\) and a vector \(\nu\), such that \(x_6 + t\nu\) is parallel to \(x_i\), \(i = 1, 2\), and the quantity \(S(x_1, ..., x_5, x_6 + t\nu)\) is affine (and positive) in \([t_1, t_2]\), as a function of \(t\). This, combined with (10), Lemma 3.1 and the fact that \(T(x_1, ..., x_5, x_6 + t\nu)\) is convex on \(t\) in \([t_1, t_2]\), shows immediately that the function

\[
[t_1, t_2] \ni t \mapsto \frac{T}{S}(x_1, ..., x_5, x_6 + t\nu)
\]

is constant. In particular, \(T(x_1, ..., x_5, x_6 + t\nu)\) must be affine in \([t_1, t_2]\).

Since it is clear that there does not exist a \(j \in \{3, 4, 5\}\) such that \(y_j\) is collinear with \(y_1, y_2\) (in the opposite case, \(x_1, x_2, x_6, x_j\) would be coplanar), the point \(y\) of assumption (iii) is unique. In other words, for some permutation \((k_1, k_2, k_3)\) of \(\{3, 4, 5\}\), the lines \(\varepsilon_1 := \text{aff}\{y_1, y_{k_1}\}\), \(\varepsilon_2 := \text{aff}\{y_2, y_{k_2}\}\), \(\text{aff}\{y, y_{k_3}\}\) are parallel or have a common point, while for each \(y'\) in the interior of \([y_1, y_2]\), different from \(y\), the lines \(\varepsilon_1, \varepsilon_2, \text{aff}\{y', y_{k_3}\}\) neither are parallel nor contain a common point.

Consequently, for \(t_1 < t < t_2\), the intersection of the planes \(\text{span}\{x_1, x_{k_1}\}\), \(\text{span}\{x_2, x_{k_2}\}\), \(\text{span}\{x_6 + t\nu, x_{k_3}\}\) is non-trivial if and only if \(x_6 + t\nu\) is parallel to the position vector of \(y\). This shows that the quantity

\[
| \det(x_1 \wedge x_{k_1} \uplus x_2 \wedge x_{k_2} \uplus (x_6 + t\nu) \wedge x_{k_3}) |
\]

is zero for a unique interior point \(t\) of \([t_1, t_2]\).

It follows that the function \(T(x_1, ..., x_5, x_6 + t\nu)\) cannot be affine in \([t_1, t_2]\), proving our claim. \(\Box\)

**Lemma 4.4.** Let \(y_1, ..., y_5\) be points lying in the same plane, such that \(y_1, y_3, y_4\) are collinear, \(y_2, y_3, y_5\) are collinear and there are no other sets of three collinear points among them. Exactly one of the following are true:

i) \(\text{aff}\{y_4, y_5\} \cap \text{int}[y_1, y_2] \neq \emptyset\).

ii) The assumption (iii) in Lemma 4.3 holds true.
Proof. Up to a possible rearrangement of indices, there exist exactly the following cases:

Lemma 4.5. Let \( \{y_1, ..., y_5\} \) be discrete points, lying in the same plane, such that:

i) \( y_5 \) is an interior point of the line segment \([y_3, y_4]\).

ii) Each one of the segments \([y_1, y_2], [y_3, y_4]\) is contained in one of the two open half-planes defined by the other one.

Then, for some choice of \( k_1, k_2 \), there exists an interior point \( y \) of \([y_1, y_2]\), such that the lines \( \text{aff} \{y, y_5\}, \text{aff} \{y_1, y_{k_1}\}, \text{aff} \{y_2, y_{k_2}\} \) have a common point, where \( \{k_1, k_2\} = \{3, 4\} \).

Proof. The vertices of the polygon \( P = \text{conv}\{y_1, y_2, y_3, y_4\} \) are exactly the points \( y_1, y_2, y_3, y_4 \). Therefore, the line defined by \( y_5 \) and the intersection point of the diagonals of \( P \), crosses \([y_1, y_2]\) at one of its interior points.

Lemma 4.6. Suppose that the set \( \{x_1, ..., x_6\} \) satisfies the \((N)\)-property. Assume, furthermore, that there is no plane in \( E(x_1, ..., x_5) \), that contains \( x_6 \). Then, there exist a vector \( \nu \) and real numbers \( t_1 < 0 < t_2 \) with the following properties:

i) There exists a plane \( E_i \) from \( E(x_1, ..., x_5) \) that contains \( x_6 + t_i \nu \), \( i = 1, 2 \).

ii) For all \( t \) in \((t_1, t_2)\), there is no plane from \( E(x_1, ..., x_5) \), that contains \( x_6 + t \nu \).

iii) The set \( \{x_1, ..., x_5, x_6 + t_i \nu\} \) satisfies the \((N)\)-property, for \( i = 1 \) or \( 2 \).

Proof. If there exists at most one 3-tuple of coplanar vectors from \( x_1, ..., x_6 \), our claim follows easily. If there exist at least two such 3-tuples, let \( G \) be the open convex angle defined by the corresponding planes, so that \( G \) contains \( x_6 \). Clearly, there exists some plane \( E \), that contains exactly two vectors from \( x_1, ..., x_5 \) but not \( x_6 \), the intersection of \( E \) with \( G \) is not empty and for some point \( x \) in \( E \) the interior of the segment \([x, x_6]\) and any plane from \( E(x_1, ..., x_3) \) are disjoint. The result follows.

The proof of the following fact is easy and will be omitted.

Lemma 4.7. Suppose that the set \( \{x_1, ..., x_6\} \) has the \((N)\)-property and for every \( i = 1, ..., 6 \), there exists a plane \( E_i \) from \( E(x_1, ..., x_{i-1}, x_{i+1}, ..., x_6) \) that contains \( x_i \). Up to a possible rearrangement of indices, one of the following is true:

i) \( x_1, x_2, x_3 \) are coplanar and \( x_4, x_5, x_6 \) are coplanar.

ii) \( x_1, x_2, x_3 \) are coplanar, \( x_3, x_4, x_5 \) are coplanar and \( x_1, x_5, x_6 \) are coplanar.

Now we are ready to prove the key fact mentioned at the beginning of this
section.

**Lemma 4.8.** The set \( \{x_1, \ldots, x_6\} \) satisfies the (N)-property, if and only if

\[
T(x_1, \ldots, x_6) < \frac{4}{3} S(x_1, \ldots, x_6)
\]

Proof. It suffices to prove the ”only if” part. Suppose that the set \( \{x_1, \ldots, x_6\} \) satisfies the (N)-property. If the assumptions of Lemma 3.4 are true, the assertion is true by Lemma 3.5.

Case I: Assume that the following are true:

a) For every \( i = 1, \ldots, 6 \), there exists some plane \( E_i \) from \( E(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_6) \) that contains \( x_i \).

b) There exists an \( i \), \( i = 1, \ldots, 6 \), a vector \( \nu \) and real numbers \( t_1 < 0 < t_2 \), so that for all \( k, l = 1, \ldots, 6 \) and for all \( t \) in \( (t_1, t_2) \), \( \det(x_i + t\nu, x_k, x_l) = 0 \), if and only if \( \det(x_i, x_k, x_l) = 0 \) and, furthermore, the sets \( \{x_1, \ldots, x_{i-1}, x_i + t\nu, x_{i+1}, \ldots, x_6\} \) do not satisfy the (N)-property, \( j = 1, 2 \).

By assumption (a) and Lemma 4.7, there are two possibilities (rearranging the indices, if necessary):

i) \( x_1, x_2, x_3 \) are contained in some plane \( E_1 \) and \( x_4, x_5, x_6 \) are contained in some other plane \( E_2 \). Since the (N)-property holds, replacing \( x_i \) with \(-x_i \) if necessary, we may assume that \( x_1, x_2, x_3 \) are contained in the same open half-space of \( E_2 \) and \( x_4, x_5, x_6 \) are contained in the same open half-space of \( E_1 \). One can check that we can, simultaneously, take \( x_i \) to be the position vector of some point \( y_i, i = 1, \ldots, 6 \), where \( y_1, \ldots, y_6 \) are coplanar. Then, it is clear that five of the points \( y_1, \ldots, y_6 \) satisfy the assumptions of Lemma 4.5, thus by Lemma 4.3 we obtain \( T(x_1, \ldots, x_6) < (4/3) \cdot S(x_1, \ldots, x_6) \).

ii) \( \{x_1, x_2, x_3\}, \{x_2, x_4, x_3\}, \{x_1, x_5, x_6\} \) are sets of linearly dependent vectors. Then, we may assume that there exists some vector \( \nu \) and real numbers \( t_1 < 0 < t_2 \), so that for all \( t \) in \( (t_1, t_2) \), we have \( \det(x_3 + t\nu, x_k, x_l) \neq 0 \) if and only if \( \det(x_3, x_k, x_l) \neq 0 \), while the sets \( \{x_1, x_2, x_3 + t_j\nu, x_4, x_5, x_6\} \), \( j = 1, 2 \), do not satisfy the (N)-property. If for \( j = 1 \) or 2, four of the vectors \( x_1, x_2, x_3 + t_j\nu, x_4, x_5, x_6 \) were coplanar, then \( x_4, x_5, x_6 \) would also be coplanar and \( x_1, \ldots, x_6 \) would not span \( \mathbb{R}^3 \). This forces \( x_3 + t_j\nu \) to be parallel to \( x_j \), \( j = 1, 2 \).

As before, we may assume that the \( x_i \)'s are the position vectors of some coplanar points \( y_1, \ldots, y_6 \) respectively. It is clear that we can apply Lemma 4.4 for the points \( y_1, y_2, y_4, y_5, y_6 \). If (i) of Lemma 4.4 is satisfied, then Lemma 4.2 contradicts to our assumption in the present Lemma. Therefore, the assertion (ii) of Lemma 4.4 holds, hence \( T(x_1, \ldots, x_6) < (4/3) \cdot S(x_1, \ldots, x_6) \).

By Lemma 4.6, the only remaining case is the following:

Case II: There exist an index \( i \) from \( \{1, \ldots, 6\} \), a vector \( \nu \) and an interval \( [t_1, t_2] \) that contains 0 in its interior, which is maximal under the assumption
that the following are true:

a) For all \( t \in (t_1, t_2) \) and for all \( k, l = 1, \ldots, 6 \), \( k, l \neq i \), \( x_i + t \nu, x_k, x_l \) are coplanar, if and only if \( x_i, x_k, x_l \) are coplanar.

b) The set \( \{ x_1, \ldots, x_{i-1}, x_i + t_j \nu, x_{i+1}, \ldots, x_6 \} \) satisfies the \((N)\)-property, for \( j = 1 \) or \( 2 \).

Let us assume that equality holds in (10). By assumption (a), it follows that the function

\[
[t_1, t_2] \ni t \mapsto S(x_1, \ldots x_i-1, x_i + t \nu, x_{i+1}, \ldots, x_6),
\]

is affine, thus Lemma 3.1, combined with (10), implies that

\[
\frac{T}{S}(x_1, \ldots x_i-1, x_i + t \nu, x_{i+1}, \ldots, x_6) = \frac{4}{3}, \quad t \in [t_1, t_2].
\]

Hence, by assumption, it is clear that there exists a set of vectors \( \{ z_1, \ldots, z_6 \} \) with the \((N)\)-property, such that

\[
T(z_1, \ldots z_6) = \frac{4}{3} S(z_1, \ldots z_6) \quad \text{and} \quad \#E(z_1, \ldots, z_6) < \#E(x_1, \ldots, x_6).
\]

Clearly, after a finite number of repetitions of the same procedure, we will have constructed a set of six vectors with the \((N)\)-property, that satisfies the assumptions of Lemma 3.4 or falls into Case I. This is impossible and the conclusion follows. \( \Box \)

The proof of the remaining part of Theorem 1 follows easily from Lemma 4.8. Indeed, let \( Z \) be a zonotope in \( \mathbb{R}^3 \) with support function

\[
h_Z(x) = \int_{S^2} | < x, y > | \, d\mu(y),
\]

where \( \mu(\cdot) = \sum_{i=1}^{n} \alpha_i \delta_{x_i}(\cdot) \) and \( \delta_{x_i}(\cdot) \) is the Dirac measure in \( x_i \), for some unit vectors \( x_i \) and some positive numbers \( \alpha_i \) \( (n \geq 6) \). By Lemma 3.3, (8) and (9), we have

\[
6! \left( \frac{2^6}{3!} \right)^{-1} \left[ 2^3 |Z|^2 - |\Pi Z| \right] = 6! \sum_{i_1 < \ldots < i_6 \in [n]} \alpha_{i_1} \ldots \alpha_{i_6} \left[ \frac{4}{3} S(x_{i_1}, \ldots, x_{i_6}) - T(x_{i_1}, \ldots, x_{i_6}) \right]
\]

or

\[
6! \left( \frac{2^6}{3!} \right)^{-1} \left[ 2^3 |Z|^2 - |\Pi Z| \right] = \int_{x_1 \in S^2} \ldots \int_{x_6 \in S^2} \varphi(x_1, \ldots, x_6) \, d\mu(x_1) \ldots d\mu(x_6),
\]

where we set \( \varphi := (4/3) \cdot S - T \). By approximation, the last identity holds for any measure on \( S^2 \), thus for every three-dimensional zonoid.
Now, if \( Z \) is not the sum of five line segments or the sum of a cylinder and a line segment, there clearly exists a set of vectors \( \{ y_1, \ldots, y_6 \} \), contained in the support of \( \mu \), that satisfies the \((N)\)-property. By the continuity of \( \varphi \) and the fact that \( \varphi(y_1, \ldots, y_6) > 0 \), we have

\[
\int_{x_1 \in S^2} \ldots \int_{x_6 \in S^2} \varphi(x_1, \ldots, x_6) \, d\mu(x_1) \ldots d\mu(x_6) > 0,
\]

which completes the proof. □

5. Cones and double cones

Suppose that \( K = \text{conv}(P \cup \{ e_3 \}) \) is a cone in \( \mathbb{R}^3 \), where \( P \) is a convex body in \( \mathbb{R}^2 \times \{0\} \). For our purpose, we may assume that \( P \) is a polygon that contains the origin. Let \( A_1, \ldots, A_n \) be the edges of \( P \). Set, also, \( h_i \) to be the outer normal vector to \( A_i \), of length equal to the distance of \( A_i \) from the origin and take vectors \( a_i \), parallel to \( A_i \), which have length equal to the length of \( A_i \), \( i = 1, \ldots, n \). We may choose the orientations of the \( a_i \)'s, so that \( \det(a_i, h_i) > 0 \), \( i = 1, \ldots, n \).

Now the facets of \( K \) are exactly the sets

\[ P_i := \text{conv}(A_i \cup \{ e_3 \}) , \quad i = 1, \ldots, n. \]

Let \( x_i \) be the outer unit normals to \( F_i \), \( i = 1, \ldots, n \). Then, since \( -e_3 \) is the outer unit normal vector to \( P \), by (6), we have:

\[
|\Pi K| = \sum_{\{i_1,i_2,i_3\} \subseteq [n]} |F_{i_1}| \cdot |F_{i_2}| \cdot |F_{i_3}| \cdot |\det(x_{i_1}, x_{i_2}, x_{i_3})| \\
+ \sum_{\{i_1,i_2\} \subseteq [n]} |P| \cdot |F_{i_1}| \cdot |F_{i_2}| \cdot |\det(x_{i_1}, x_{i_2}, e_3)| \quad (12)
\]

A crucial observation for what follows is the fact that all terms of the form

\[
|F_{i_1}| \cdot |F_{i_2}| \cdot |F_{i_3}| \cdot |\det(x_{i_1}, x_{i_2}, x_{i_3})| \quad \text{are non-zero.}
\]

One can easily see this by taking a suitable affine transformation that maps \( F_{i_1}, F_{i_2} \) to facets that are parallel to the vector \( e_3 \). Then, for any \( i_3 \) different than \( i_1, i_2 \), the image of \( F_{i_3} \) through this transformation is necessarily not parallel to \( e_3 \).

Clearly, the vector \( h_i - e_3 \) is parallel to the facet \( F_i \). Thus, the vector \( a_i \wedge (h_i - e_3) \) is orthogonal to \( F_i \), hence a multiple of \( x_i \). Moreover, since \( h_i - e_3 \) is orthogonal to \( a_i \),

\[
|a_i \wedge (h_i - e_3)| = |a_i| \cdot |h_i - e_3| = 2|F_i| , \quad i = 1, \ldots, n.
\]
Consequently, we have shown that
\[ |F_i| \cdot x_i = \pm \frac{1}{2} a_i \wedge (h_i - e_3) \]  
\hspace{1cm} (13)

We may now use (13), Lemma 3.2 (i) and the fact that the vectors \( a_1 \wedge h_1, a_2 \wedge h_2 \) are parallel, to deal with every term of (12) separately:
\[ |P| \cdot |F_1| \cdot |F_2| \cdot |det(x_1, x_2, e_3)| = |P| \cdot \left| \frac{1}{4} \right| det(a_1 \wedge (h_1 - e_3), a_2 \wedge (h_2 - e_3), e_3) | \]
\[ = \frac{1}{4} |P| \cdot |det(a_1 \wedge e_3, a_2 \wedge e_3, e_3)| = \frac{1}{4} |P| \cdot |det_{2 \times 2}(a_1, a_2)| . \]

Thus,
\[ \sum_{\{i_1, i_2\} \subseteq [n]} |P| \cdot |F_{i_1}| \cdot |F_{i_2}| \cdot |det(x_{i_1}, x_{i_2}, e_3)| = \frac{1}{4} |P| \cdot \sum_{\{i_1, i_2\} \subseteq [n]} |det(a_{i_1}, a_{i_2})| \]
\[ = \frac{1}{4} |P| \cdot |\Pi P| . \]  
\hspace{1cm} (14)

Moreover,
\[ |det\left(a_1 \wedge (h_1 - e_3), a_2 \wedge (h_2 - e_3), a_3 \wedge (h_3 - e_3)\right)| \]
\[ = \left| det(-a_1 \wedge e_3, -a_2 \wedge e_3, a_3 \wedge h_3) + det(-a_1 \wedge e_3, a_2 \wedge h_2, -a_3 \wedge e_3) \right. \]
\[ + det(a_1 \wedge h_1, -a_2 \wedge e_3, -a_3 \wedge e_3) \right| \]
\[ = |(-a_1 \wedge e_3) \wedge (e_3 \wedge a_2), a_3 \wedge h_3 > + \left< (-a_1 \wedge e_3) \wedge (e_3 \wedge -(a_3)), a_2 \wedge h_2 > \right. \]
\[ + \left< a_1 \wedge h_1, (-a_2 \wedge e_3) \wedge (e_3 \wedge a_3) > \right| \]
\[ = \left| det(-a_1, e_3, a_2) \cdot det(e_3, a_3, h_3) + det(-a_1, e_3, -a_3) \cdot det(e_3, a_2, h_2) \right. \]
\[ + \left. det(-a_2, e_3, a_3) \cdot det(a_1, h_1, e_3) \right| \]
\[ = \left| det_{2 \times 2}(a_1, a_2) \cdot det_{2 \times 2}(a_3, h_3) + det_{2 \times 2}(a_3, a_1) \cdot det_{2 \times 2}(a_2, h_2) \right. \]
\[ + \left. det_{2 \times 2}(a_2, a_3) \cdot det_{2 \times 2}(a_1, h_1) \right| . \]

Since \( |conv\{\{0\} \cup A_i\}| = det(a_i, h_i)/2 \), it follows that
\[ |F_1| \cdot |F_2| \cdot |F_3| \cdot |det(x_1, x_2, x_3)| = \frac{1}{4} \left| det(a_2, a_3) \cdot |conv\{\{0\} \cup A_i\}| \right. \]
\[ + det(a_3, a_1) \cdot |conv\{\{0\} \cup A_2\}| + det(a_1, a_2) \cdot |conv\{\{0\} \cup A_3\}| \right| . \]  
\hspace{1cm} (15)
If we assume, in addition, that $A_1, A_2$ are adjacent edges, then
\[ |\text{conv}\{0\} \cup A_1 \cup A_2| = |\text{conv}\{0\} \cup A_1| + |\text{conv}\{0\} \cup A_2| . \]

In this case,
\[ \text{det}(a_3, a_1) \cdot |\text{conv}\{0\} \cup A_2| + \text{det}(a_2, a_3) \cdot |\text{conv}\{0\} \cup A_1| \]
\[ = \text{det}(a_3, a_1) \cdot |\text{conv}\{0\} \cup A_1 \cup A_2| + \text{det}(a_1 + a_2, a_3) \cdot |\text{conv}\{0\} \cup A_1| . \quad (16) \]

Set $R(K) := |\Pi K| - (1/4) \cdot |P| \cdot |\Pi P|$. To prove Theorem 2, it suffices to prove that $R(K)$ depends only on the area of $P$ and that equality in Theorem 2 holds for some $P$.

Let $v_1, ..., v_n$ be the vertices of $P$. Suppose that $P$ is not a triangle. We may assume that the line segments $[v_1, v_2]$, $[v_2, v_3]$ are the edges $A_1, A_2$ respectively and that 0 is not contained in the triangle $\text{conv}(A_1 \cup A_2)$. The fact that $\text{det}(a_i, b_i) > 0$ easily implies that the vector $a_1 + a_2$ and the segment $[v_1, v_3]$ have equal lengths and parallel directions.

We employ here a method often used by Campi, Colesanti and Gronchi (see e.g. [4] or [5]). Consider the family of polygons
\[ P_t = \text{conv}\{v_1, v_2 + tv, v_3, ..., v_n\} , \quad t \in [t_1, t_2] , \]
where $\nu$ is any vector parallel to $a_1 + a_2$ and $[t_1, t_2]$ is the largest interval, in which $v_1, v_3$ are vertices of $P_t$ for all $t$ in $(t_1, t_2)$. Clearly, $(t_1, t_2)$ contains 0, $P_0 = P$ and, furthermore, the volume of $P_t$ is constant in $[t_1, t_2]$. Also, $P_t$ contains 0 in its interior for all $t$ in $[t_1, t_2]$.

If $A_1, A_2, t, A_3, ..., A_n$ are the edges of $P_t$, the corresponding parallel vectors are $a_1, t = a_1 \pm tv$, $a_2, t = a_2 \mp tv$, $a_3, ..., a_n$. Substituting $\nu$ by $-\nu$ if necessary, we may assume that $a_1, t = a_1 - tv$ and $a_2, t = a_2 + tv$.

By (12), (14), (15), (16) we have:
\[ R\left(\text{conv}\{e_3\} \cup P_t\right) = \sum_{\{i_1, i_2, i_3\} \subseteq [n] \setminus \{1, 2\}} |F_{i_1}| \cdot |F_{i_2}| \cdot |F_{i_3}| \cdot |\text{det}(x_{i_1}, x_{i_2}, x_{i_3})| \]
\[ + \frac{1}{4} \sum_{\{i_1, i_2, i_3\} \subseteq [n] \setminus \{1, 2\}} \left| \text{det}(a_i + \epsilon tv, a_{i_2}) \cdot \text{conv}\{0\} \cup A_{i_3} \right| \]
\[ + \text{det}(a_{i_3}, a_i + \epsilon tv) \cdot \text{conv}\{0\} \cup A_{i_2} \right| + \text{det}(a_{i_2}, a_{i_3}) \cdot \text{conv}\{0\} \cup A_{i_1} \right| \]
\[ + \frac{1}{4} \sum_{i \in [n] \setminus \{1, 2\}} |\text{det}(a_i, a_1 - tv) \cdot \text{conv}\{0\} \cup A_{1,t} \cup A_{2,t}| \]

19
Thus, \( |\text{conv}(\{0\} \cup A_{1,t})| + det(a_1 - tv, a_2 + tv) \cdot |\text{conv}(\{0\}) \cup A_t| \),

where \( \varepsilon_1 = -1 \), \( \varepsilon_2 = 1 \).

Then, \( |\text{conv}(\{0\} \cup A_{1,t} \cup A_{2,t})| \) and \( det(a_1 - tv, a_2 + tv) = 2|\text{conv}(v_1, v_2 + tv, v_3)| \) are clearly constant in \([t_1, t_2]\). Also, \( det(a_1, a_1 - tv) \) and \( |\text{conv}(\{0\} \cup A_{i,t})| \) are affine in \([t_1, t_2], i = 1, 2\). As observed previously, each term of the sum above is strictly positive in \((t_1, t_2)\). All these facts imply that the quantity \( R(\text{conv}(P_t \cup \{e_3\})) \) is affine in \([t_1, t_2]\).

We conclude that for some \( i, j \), with \( \{i, j\} = \{1, 2\}\),

\[
R \left( \text{conv}(P_i \cup \{e_3\}) \right) \leq R(K) = R \left( \text{conv}(P_0 \cup \{e_3\}) \right) \leq R \left( \text{conv}(P_j \cup \{e_3\}) \right). 
\]

It is true that the number of vertices of \( P_{t_1} \) and \( P_{t_2} \) is strictly less than the number of vertices of \( P \). Thus, by an inductive argument, there exist triangles \( T_1, T_2 \) in \( \mathbb{R}^2 \times \{0\} \) of the same area as \( P \), with:

\[
R \left( \text{conv}(T_1 \cup \{e_3\}) \right) \leq R(K) \leq R \left( \text{conv}(T_2 \cup \{e_3\}) \right). 
\]

However, by definition, \( R(K) \) is invariant under maps of the form

\[
\mathbb{R}^3 \ni (s_1, s_2, s_3) \mapsto (\Phi(s_1, s_2), s_3) \in \mathbb{R}^3,
\]

where \( \Phi \) is an area-preserving, affine transformation on \( \mathbb{R}^2 \). This shows that \( R(K) = R(\text{conv}(T \cup \{e_3\})) \), where \( T \) is any triangle of the same area as \( P \). Thus, \( R(K) \) depends only on the area of \( P \).

In the particular case in which \( |P| = 1 \) and \( K \) is the simplex, it is clear that \( |P| \cdot |\Pi P|/4 = 1.5 \) and one easily calculates (see e.g. \([3]\)) \( |\Pi K| = 2 \). Thus, \( R(K) = 1/2 \). \( \square \)

To prove Corollary 2, take \( K \) to be the double cone \( \text{conv}(P \cup \{\pm e_3\}) \) and \( K' \) to be the cone \( \text{conv}(P \cup \{e_3\}) \), where \( P \) is a centrally symmetric polygon in \( \mathbb{R} \times \{0\} \) of area 1. If \( F_1, \ldots, F_n \) are the facets of \( K' \), that are different from \( P \), by (6), (12) and (14) it follows that:

\[
|\Pi K'| = \frac{1}{4}|P| \cdot |\Pi P| + \sum_{\{i_1, i_2, i_3\} \subseteq [n]} |F_{i_1}| \cdot |F_{i_2}| \cdot |F_{i_3}| \cdot |\text{det}(x_{i_1}, x_{i_2}, x_{i_3})|,
\]

\[
|\Pi K| = 2^3 \cdot \sum_{\{i_1, i_2, i_3\} \subseteq [n]} |F_{i_1}| \cdot |F_{i_2}| \cdot |F_{i_3}| \cdot |\text{det}(x_{i_1}, x_{i_2}, x_{i_3})|.
\]

Thus,

\[
\frac{|\Pi K|}{|K|^2} = 2^3 \cdot \left( |\Pi K'| - \frac{1}{4}|P| \cdot |\Pi P| \right) \cdot \frac{9}{4} = 9,
\]

20
where we used Theorem 2 and the fact that $|K| = 2/3$.  

6. Projection bodies and centroid bodies

Let $K$ be a star body in $\mathbb{R}^d$. The centroid body $\Gamma K$ of $K$ is defined by its support function

$$h_{\Gamma K}(x) = \int_K |<x, y>| \, dy = \frac{1}{d+1} \int_{S^{d-1}} |<x, y>| \rho_K^{d+1}(y) \, dy, \quad x \in S^{d-1},$$

where $\rho_K$ is the radial function of $K$ and the last equality follows by integration in polar coordinates. Obviously, $\Gamma K$ is a zonoid. It can be easily shown that the functional $|\Gamma K|/|K|^{d+1}$ is invariant under non-singular linear transformations. A basic inequality for volumes of centroid bodies is due to Busemann and Petty [1] [13]:

$$\frac{|\Gamma B_1|}{|B_1|^{d+1}} \leq \frac{|\Gamma K|}{|K|^{d+1}}$$  \hspace{1cm} (17)

where $B_1$ is the unit ball. Here, equality holds if and only if $K$ is an origin symmetric ellipsoid. We prove the following:

**Proposition 6.1.** If $K$ is a star body in $\mathbb{R}^d$, then

$$\frac{|\Pi(\Gamma B_1)|}{|B_1|^{(d+1)(d-1)}} \leq \frac{|\Pi(\Gamma K)|}{|K|^{(d+1)(d-1)}}$$  \hspace{1cm} (18)

with equality if and only if $K$ is an ellipsoid with center at the origin.

Note here that the quantity $|\Pi(\Gamma K)|/|K|^{(d+1)(d-1)}$ is also invariant under linear maps. Also, as (17) shows, (18) would follow by Petty’s conjectured inequality.

In what follows, $\alpha_d$, $\beta_d$ etc. will be positive constants that depend only on the dimension $d$. For $p = 1$ and $p = 2$, define the quantity

$$S_p(K) = \int_{x_1 \in K} \ldots \int_{x_d \in K} |\det(x_1, ..., x_d)|^p \, dx_1 \ldots dx_d =$$

$$\int_{x_1 \in S^{d-1}} \ldots \int_{x_d \in S^{d-1}} |\det(x_1, ..., x_d)|^p \rho_K(x_1)^{d+p} \ldots \rho_K(x_d)^{d+p} \, dx_1 \ldots dx_d.$$  \hspace{1cm} (19)

It is clear that $S_p(K)$ is invariant under volume preserving linear transformations. Also, it follows from (5) that the volume of $\Gamma K$ is given by

$$|\Gamma K| = \alpha_d S_1(K).$$
We say that $K$ is in isotropic position, if the function

$$S^{d-1} \ni x \mapsto \int_{K} <x, y>^2 \, dy$$

constant. In this case, the quantity

$$L_K := \left( \int_{K} <x, y>^2 \, dy \right)^{\frac{1}{2}}$$

is called the isotropic constant of $K$. By definition, if $K'$ is an affine image of $K$ then $L_{K'} = L_K$. An obvious fact is that if $K$ is isotropic of volume 1 and $\{e_1, ..., e_d\}$ is an orthonormal basis, then

$$\int_{K} <x, e_i> <x, e_j> \, dx = L_K^2 \delta_{ij}, \ i, j = 1, ..., d .$$

It is well known (see e.g. [12]) that there is always a linear transformation $T$, such that $TK$ is isotropic and of the same volume as $K$. Thus, it is clear by the above discussion that

$$S_2(K) = \beta_d L_K^2 |K|^{d+2} .$$

**Lemma 6.1.** There exists some constant $\delta_d$, such that if $K$ is a star body in $\mathbb{R}^d$ then

$$S_1(K)^{\frac{d+2}{d+1}} \leq \delta_d S_2(K) ,$$

with equality if and only if $K$ is an origin symmetric ellipsoid.

Proof.

$$S_2(K) = \beta_d L_K^2 |K|^{d+2} = \beta_d \left( \frac{1}{d} \int_{TK} |x|^2 \, dx \right)^d$$

$$= \bar{\beta}_d \left( \int_{S^{d-1}} \rho_{TK}^{d+2}(x) \, dx \right)^d \geq \bar{\beta}_d \left( \int_{S^{d-1}} \rho_{TK}^{d+1}(x) \, dx \right)^{\frac{d+2}{d+1}} ,$$

where $T$ is a transformation, such that $TK$ is isotropic of the same volume as $K$ and we used Hölder’s inequality in the last part. Equality holds if and only if $\rho_{TK}$ is constant i.e. $TK$ is a ball centered at the origin or, equivalently, $K$ is an ellipsoid with center at 0.

On the other hand, it is clear that there exist some constants $\gamma_d, \gamma_d'$, such that

$$\gamma_d \int_{S^{d-1}} \rho_{TK}^{d+2}(x) \, dx = \gamma_d \int_{S^{d-1}} \int_{TK} |<x, y>| \, dx \, dy = V(\Gamma(TK), B_1, ..., B_1) ,$$
where \( \omega_d \) is the volume of the \( d \)-dimensional unit ball and \( V(\Gamma(TK), B_1, \ldots, B_1) \) denotes the mixed volume of \( \Gamma(TK) \) and \( B_1 \) (see [20] for definitions and related inequalities concerning mixed volumes). Now, the Minkowski inequality gives:

\[
V(\Gamma(TK), B_1, \ldots, B_1) \geq |\Gamma(TK)| \frac{1}{d} \cdot |B_1|^{\frac{d-1}{d}},
\]

with equality if and only if \( \Gamma(TK) \) is a ball. If \( TK \) is a ball centered at the origin, then \( \Gamma(TK) \) is a ball. Hence if \( K \) is an ellipsoid of center at 0, then equality holds in the last inequality. Combining both inequalities together with the equality cases, we conclude the desired result. □

Proposition 6.1, follows from Lemma 6.1. First we need some additional well known facts. The Busemann formula [1] states:

\[
|K|^{d-1} = \zeta_d \int_{S^{d-1}} S_1(K \cap x^\perp) \, dx.
\]

(19)

Using a generalization of Busemann’s formula Weil [22] showed that if

\[
h_Z(x) = \frac{1}{2} \int_{S^{d-1}} |<x, y>| f(y) \, dy \, , \, x \in \mathbb{R}^d,
\]

is the support function of a \( d \)-dimensional zonoid \( Z \), for some measurable function \( f : S^{d-1} \rightarrow \mathbb{R} \), then its surface area measure is absolutely continuous with respect to the Lebesque measure and its density function \( f_Z \) is given by:

\[
f_Z(x) = \theta_d \int_{S^{d-1} \cap x^\perp} \ldots \int_{S^{d-1} \cap x^\perp} \det(x_1, \ldots, x_{d-1})^2 \, f(x_1) \ldots f(x_{d-1}) \, dx_1 \ldots dx_{d-1}.
\]

Let \( P \) be a convex body, having absolutely continuous surface area measure with density \( f \). Petty [14] proved the following inequality

\[
|\Pi P| \geq c_d \left( \int_{S^{d-1}} f^{\frac{d}{d+1}}(x) \, dx \right)^{d+1},
\]

(20)

with equality if and only if \( P \) is an ellipsoid. The quantity \( \Omega(P) := \int_{S^{d-1}} f^{d/d+1} \, dx \) is called the affine surface area of \( P \).

Proof of Proposition 6.1.

Note that (according to Weil’s result) the surface area measure of \( \Gamma K \) is absolutely continuous with respect to the Lebesque measure and its density is given by:

\[
f(x) = \ell_d \cdot S_2(K \cap x^\perp) \, , \, x \in S^{d-1}.
\]
Thus, by (20), Lemma 6.1 and (19), we have

\[ |\Pi(\Gamma K)| \geq \lambda_d \left( \int_{S^{d-1}} S_2(K \cap x^\perp) \frac{d}{\pi} \, dx \right)^{d+1} \]

\[ \geq \lambda'_d \left( \int_{S^{d-1}} S_1(K \cap x^\perp) \, dx \right)^{d+1} = \lambda''_d |K|^{(d-1)(d+1)}. \]

Equality holds in both inequalities if \( K \) is an ellipsoid with center at 0 and in the second one only if \( K \cap x^\perp \) is an ellipsoid centered at the origin, for every \( x \in S^{d-1} \).

It follows then from [6], Theorem 7.1.5 that \( K \) is necessarily an ellipsoid centered at the origin. \( \square \)

Next we state an application of Proposition 6.1 in three dimensions.

**Theorem 4.** Let \( n \geq 3 \) be an integer. Among all 3-dimensional convex bodies of volume 1, ellipsoids with center at the origin are exactly the bodies that minimize the functional

\[ Q_n(K) := \int_{x_1 \in K} \cdots \int_{x_n \in K} \left| \Pi \left( \sum_{i=1}^{n} [-x_i, x_i] \right) \right| \, dx_1 \cdots dx_n. \]

In other words, the mean value of the volume of the projection body of the sum of \( n \) line segments picked uniformly and independently from a convex body \( K \) of prescribed volume, is minimal if and only if \( K \) is an origin symmetric ellipsoid. Theorem 4 is formally related to Petty’s conjecture as follows: If one could replace the Minkowski sum with the convex hull in Theorem 4, then Petty’s conjecture would be correct in three dimensions. Note, also, that the functional \( Q_n(K)/|K|^3 \) is invariant under non-singular linear transformations.

**Proof of Theorem 4.**

Let us use (8) and Lemma 3.3 to derive a simple expression for \( Q_n(K) \):

\[ Q_n(K) = \alpha_1 \left( \frac{n}{3} \right) \int_{(x_1, x_2, x_3) \in K^3} \det(x_1, x_2, x_3)^2 \, dx_1 dx_2 dx_3 \]

\[ + \alpha_2 \left( \frac{n}{4} \right) \int_{(x_1, x_2, x_3, x_4) \in K^4} |\det(x_1, x_2, x_3) \cdot \det(x_1, x_2, x_4)| \, dx_1 dx_2 dx_3 dx_4 \]

\[ + \alpha_3 \left( \frac{n}{5} \right) \int_{(x_1, x_2, x_3, x_4, x_5) \in K^5} |\det(x_1, x_2, x_3) \cdot \det(x_1, x_4, x_5)| \, dx_1 dx_2 dx_3 dx_4 dx_5 \]

24
\[ +\alpha_4 \left( \frac{n}{6} \right) \int_{(x_1, \ldots, x_6) \in K^6} |\text{det}(x_1 \wedge x_2, x_3 \wedge x_4, x_5 \wedge x_6)| \, dx_1 \ldots dx_6 \]

\[ =: \alpha_1 \left( \frac{n}{3} \right) A_1(K) + \alpha_2 \left( \frac{n}{4} \right) A_2(K) + \alpha_3 \left( \frac{n}{5} \right) A_3(K) + \alpha_4 \left( \frac{n}{6} \right) A_4(K) , \]

where \( \alpha_1, \ldots, \alpha_4 \) are absolute positive constants.

Since \( A_1(K) = S_2(K) = \beta_3(L_K^2)^3 \) and since it is well known that the isotropic constant is minimal if and only if \( K \) is an ellipsoid centered at 0, it follows that

\[ A_1(K) \geq A_1(B_1) , \]

where \( B \) is the ball of center 0 and volume 1, with equality if and only if \( K \) is an ellipsoid centered at 0.

Also, by (7) and Proposition 6.1, we have:

\[ A_4(K) = \mu|\Pi(\Gamma K)| \geq \mu|\Pi(\Gamma B_1)| = A_4(B_1) , \]

where \( \mu \) is an absolute constant.

To complete the proof of Theorem 4, observe that \( A_2(K) = \)

\[ \int_{x_1 \in K} \int_{x_2 \in K} \left( \int_{x_3 \in K} |\text{det}(x_1, x_2, x_3)| \, dx_3 \cdot \int_{x_4 \in K} |\text{det}(x_1, x_2, x_4)| \, dx_4 \right) \, dx_1 \, dx_2 \]

\[ = \int_{x_1 \in K} \int_{x_2 \in K} \left( \int_{x_3 \in K} |\text{det}(x_1, x_2, x_3)| \, dx_3 \right)^2 \, dx_1 \, dx_2 \]

and, similarly,

\[ A_3(K) = \int_{x_1 \in K} \left( \int_{x_2 \in K} \int_{x_3 \in K} |\text{det}(x_1, x_2, x_3)| \, dx_2 \, dx_3 \right)^2 \, dx_1 . \]

One may, now, use the process of Steiner symmetrization in a standard way (taking, also in advantage the obvious fact that \( A_2(K), A_3(K) \) are invariant under volume-preserving linear maps) to show that

\[ A_i(K) \geq A_i(B_1) , \quad i = 2, 3 . \]

We omit the details. \( \Box \)
7. Some concluding remarks

§1) The fact that the class of three-dimensional zonoids, in which equality holds in (2), strictly contains the class of cylinders is of some interest. For instance, one can easily derive an infinite family of counterexamples to Schneider’s conjecture in three dimensions, also extended to any dimension using (4).

Indeed, suppose that $Z$ is a zonotope in $\mathbb{R}^3$, which is the sum of five line segments in general position or the sum of a cylinder and a line segment, but not a cylinder. Then,

$$|\Pi(\Pi^{-1} Z)| > 2^3 |\Pi^{-1} Z|^2 .$$

To see this, note that as shown in Schneider [20] (p. 417),

$$\frac{|\Pi(\Pi Z)|}{|\Pi Z|^2} \leq \frac{|\Pi Z|}{|Z|^2} ,$$

with equality if and only if $\Pi(\Pi Z)$ and $Z$ are homothetic. However, Weil [21] showed that cylinders are the only three-dimensional polytopes having this property. Since $Z$ is a polytope, but not a cylinder, so is $\Pi^{-1} Z$. Thus, replacing in (21) $Z$ by $\Pi^{-1} Z$ and using the equality characterization in Theorem 1, the conclusion follows. As an example, one may conclude that for every centrally symmetric three-dimensional polytope $K$ of volume 1, with at most ten facets, it is true that $|\Pi K| \geq 8$, with equality if and only if $K$ is a cylinder.

If $E$ is a three-dimensional ellipsoid, one may easily compute that $|\Pi E|/|E|^2 < 8$. Thus, if $K$ is a cone, a centrally symmetric double cone or of the form $\Pi^{-1} Z$, where $Z$ is a zonoid for which equality in (2) is attained, then $|\Pi E|/|E|^2 < |\Pi K|/|K|^2$. As far as we know, there are no other natural three-dimensional classes of convex bodies, beside cylinders, in which Petty’s conjecture has been confirmed.

§2) The problem of proving the analogous to (2) in all dimensions seems to be much more complicated than the three-dimensional case. One of the main reasons, in our opinion, is that it is not very clear what someone would expect as equality cases. An inequality analogous to (10) would be the key to this problem. One may consider the functions $S$ and $T$ in $d$ dimensions, defined by (9) and (8) respectively, to conclude that the inequality $|\Pi Z|/|Z|^{d-1} \leq 2^d$ holds for any zonoid $Z$, if and only if the same inequality holds for zonotopes being the sum of at most $d(d - 1)$-line segments.
§3) If $K$ is a 3-dimensional star body, set

$$P_n(K) := \int_{x_1 \in K} \ldots \int_{x_n \in K} \left| \sum_{i=1}^n [-x_i, x_i] \right|^2 \, dx_1 \ldots dx_n.$$ 

To conclude this paper, we would like to remark that the value of the ratio $Q_n(B)/P_n(B)$ agrees with Petty’s conjecture for all integers $n \geq 3$, where $B$ is a three-dimensional euclidian ball with center at 0. Indeed, one computes

$$P_n(K) = \beta_1 \left( \frac{n}{3} \right) A_1(K) + \beta_2 \left( \frac{n}{4} \right) A_2(K) + \beta_3 \left( \frac{n}{5} \right) A_3(K) + \beta_4 \left( \frac{n}{6} \right) B_4(K),$$

where

$$B_4(K) := \left( \int_{x_1 \in K} \int_{x_2 \in K} \int_{x_3 \in K} |\text{det}(x_1, x_2, x_3)| \, dx_1 \, dx_2 \, dx_3 \right)^2$$

and $\beta_1, \ldots, \beta_4$ are absolute constants. Taking $n$ to be 3, 4, 5 successively, it follows by Theorem 1 that $\alpha_i/\beta_i = 8 > |\Pi B|/|B|^2$, $i = 1, 2, 3$. Also, the law of large numbers implies easily that the random zonotope $\frac{1}{n} \sum_{i=1}^n [-x_i, x_i]$ converges almost surely to a multiple of the centroid body of $K$, as $n$ tends to infinity, thus $\lim_{n \to \infty} Q_n(K)/P_n(K) = |\Pi (\Gamma K)|/|\Gamma K|^2$. On the other hand, it is clear that $\lim_{n \to \infty} Q_n(K)/P_n(K) = \alpha_4 A_4(K)/\beta_4 B_4(K)$. Taking $K = B$, it follows that $\alpha_4 A_4(B)/\beta_4 B_4(B) = |\Pi B|/|B|^2$, so $Q_n(B)/P_n(B) > |\Pi B|/|B|^2$, $n \geq 3$. The last inequality would follow by Petty’s conjectured inequality.

**Acknowledgement.** Special thanks to Prof. Souzana Papadopoulou for significant contribution in this paper.

**References**


