RESEARCH STATEMENT

Dmytro Savchuk
Department of Mathematics, Texas A&M University, College Station, TX 77843
`savchuk@math.tamu.edu` `http://www.math.tamu.edu/~savchuk`

My research interests concentrate mostly on combinatorial and geometric group theory. During my studies at Texas A&M University most of my work was related to various aspects of groups acting on trees and, in particular, generated by automata (or self-similar groups). The first appearance of these groups goes back to the beginning of 1960's. Self-similar groups have many interesting and important properties. The class of self-similar groups contains exotic examples, such as Burnside groups, groups of intermediate growth, as well as familiar examples, such as free groups or free products of finite groups that are well known and are regular objects of study in combinatorial group theory.

It was observed recently that the class of self-similar groups naturally appears in mathematics. The most recent examples come from combinatorics and are related to the famous combinatorial problem known as Hanoi Towers Game [9]. Slightly older examples, due first of all to Nekrashevych [11], are related to holomorphic dynamics and random walks. In the subsequent sections I briefly describe my major results, outline possible generalizations and my future plans.

1 Iterated monodromy groups

One of the most remarkable discoveries in the recent years is the observation, due to Nekrashevych, that the so-called iterated monodromy groups (IMG), which can be related to any self-covering map, belong to the class of self-similar groups and that, in the most natural situations, there is an explicit procedure representing them by finite automata [11].

In paper [8] (joint with R. Grigorchuk and Z. Šunić) we study the iterated monodromy group $\mathrm{IMG}(z^2+i)$ of the map $z \mapsto z^2+i$, which is generated by 3 nontrivial states of the 4-state automaton over a 2-letter alphabet. We show that this group is a regular branch group, thus presenting an example of a branch group which naturally appears in holomorphic dynamics. The main body of the article is devoted to the calculation of an L-presentation for $\mathrm{IMG}(z^2+i)$ (i.e., a presentation which involves a finite set of relators and their iterations by substitution). More precisely, we prove that

$$\begin{aligned}\mathrm{IMG}(z^2+i) \cong \big\langle a,b,c \;\big|\; &\phi^n(a^2), \phi^n\big((ac)^4\big), \phi^n([c,ab]^2), \phi^n([c,bab]^2),\\ &\phi^n([c,ababa]^2), \phi^n([c,ababab]^2), \phi^n([c,bababab]^2), n \geq 0\big\rangle,\end{aligned}$$

where ϕ is a substitution defined on words in the free monoid over the alphabet $\{a,b,c\}$ by $\phi(a)=b$, $\phi(b)=c$ and $\phi(c)=aba$.

Although it is known that L-presentations are quite common for groups of branch type the number of examples in which explicit computation is possible is rather small.

The presence of L-presentations is important from different points of view. For example, such presentations can be used to embed a group into a finitely presented group in a way that preserves many properties of the original group. We use the obtained L-presentation of $\mathrm{IMG}(z^2+i)$ to embed it into a finitely presented group with 4 generators and 10 relators, which is amenable but not elementary amenable.

It was shown by K.-U. Bux and R. Pérez that $\mathrm{IMG}(z^2+i)$ has intermediate growth and, hence, is amenable. We find a self-similar measure on $\mathrm{IMG}(z^2+i)$ providing a different proof that the group is amenable.

The self-similar measure is closely related to the problem of computation of the spectrum of a Hecke type operator that can be related to any group acting on a rooted tree and to the problem of computation of the spectrum of the discrete Laplace operator on the boundary Schreier graph of a group. Unfortunately, the spectral problem is not solved yet for $\mathrm{IMG}(z^2+i)$. What we were able to construct is a rational map on $\mathbb{R}^3$ whose proper invariant set gives the spectrum of the Markov operator after intersection by a corresponding line. Further efforts in the description of the shape of the attractor (and hence of the spectrum) should be a task for future research.

2 Groups of intermediate growth

Part of my dissertation is devoted to the study of Sushchansky groups introduced by V. Sushchansky in [18] in 1979 as one of the pioneering examples of Burnside groups. The results of this section are published in [4]. Sushchansky used a different language, namely the language of tableaux, introduced by L. Kaluzhnin to study properties of iterated wreath products. For each prime $p > 2$, V. Sushchansky constructed a finite family of infinite p-groups generated by two tableaux. Each such a tableau naturally defines an automorphism of a rooted tree and can be represented by a finite initial automaton. In paper [4] we describe these automata and study Sushchansky groups and their actions on rooted trees by means of this well-developed language.

The associated action of any Sushchansky group on a rooted tree is not level-transitive and we describe its orbit tree and show that there exists a faithful level-transitive action given by finite initial automata. Unlike the Grigorchuk group, Sushchansky groups are not self-similar. We consider a self-similar closure and prove that it is weakly regular branch, neither torsion nor torsion free and is generated by bounded automata, yielding contracting property and amenability. The question if the self-similar closure of any of Sushchansky groups is branch (or regular branch) is still open.

Our main result in the paper [4] is the fact that all Sushchansky groups have intermediate growth, which is a contribution to the Milnor question on the existence of such groups, which was solved in [6] by R.I. Grigorchuk. Also we give an upper bound on the period growth function.

3 Automata generating free products of groups of order 2

All tree automorphisms defined by states of finite invertible automata over a fixed alphabet form a group of automatic transformations over this alphabet. The structure of this large group is yet to be understood. An interesting question is the embedding of known groups into this group. For example, Brunner and Sidki proved in [5] that $GL_n(\mathbb{Z})$ can be generated by finite automata over the alphabet with 2^n letters. The first embedding of free products of groups C_2 of order 2 into the group of automatic transformations over the 2-letter alphabet was constructed by Olijnyk [12], but the group was not generated by all the states of corresponding automaton.

The first self-similar example is the 3-state Bellaterra automaton $\mathcal{B}_3$ over 2-letter alphabet. It was proved [1, 11] that it generates the group isomorphic to the free product of 3 copies of C_2. The Bellaterra automaton gives rise to a family of automata in which all states define involutive transformations. Namely, we modify the automaton $\mathcal{B}_3$ by inserting new states. More precisely, each automaton in the family is defined by wreath recursion

$$\begin{array}{l} a=(c,b),\quad b=(b,c),\quad c=(q_1,q_1)\sigma_0,\\ q_i=(q_{i+1},q_{i+1})\sigma_i,\ i=1,\dots,n-4,\\ q_{n-3}=(a,a)\sigma_{n-3}, \end{array} \tag{1}$$

where $\sigma_i\in\mathrm{Sym}(\{0,1\})$ is chosen arbitrarily.

Conjecturally, each nontrivial automaton in the family generates the free product of C_2. The first result supporting this conjecture was obtained by M. Vorobets and Y. Vorobets [19]. It was shown that if the number of states is odd and $\sigma_i=(12)$ for all i, then the conjecture holds. In the subsequent paper by the same authors and B. Steinberg [17], the conjecture was proved for the automata with even number of states and the additional condition that the number of nontrivial σ_i is odd.

In [14] (joint with Y. Vorobets) we prove that any n-state automaton from the family (1) with $n\geq 4$ satisfying $\sigma_0=(12)$ and $\sigma_{n-3}=(12)$ generates the free product of C_2. This covers the series constructed in [19] except the case $n=3$, and overlaps with a family constructed in [17].

4 Software development

Groups and semigroups generated by automata are particularly nice from the computational point of view. Major algorithmic problems are unsolved so far in the general case but have solutions in certain special cases. Certain problems have several solutions of different complexity depending on the type of the group. For example, the general algorithm solving the word problem has exponential complexity, but there is a polynomial time algorithm for contracting groups, whose complexity bound I obtained in [15] in terms of the size of the nucleus of the group. The computations related to these groups are often cumbersome to be performed by hand and the computers may be very helpful here.

There was a strong need in the implementation of the algorithms related to automata groups and semigroups in some computer algebra system. The package `AutomGrp` [10] for `GAP`-system was developed by myself and Yevgen Muntyan to satisfy this need. The package was successfully used in the project of the classification of groups generated by 3-state automata over 2-letter alphabet [1], as well as by several other authors. Currently the status of the package is *deposited* (it is available from `GAP` web page), but we are planning to submit it for refereeing in the nearest future. The package is constantly developing with new releases published regularly. I consider the support and developing of this package, as well as producing new pieces of software as an important part of my future research.

5 Classification of groups generated by automata

Similar to other classes of groups, the question of classification of groups generated by automata naturally arises. The first step in this direction was completed in [7], where it was proven that there are 6 nonisomorphic groups generated by 2-state automata over 2-letter alphabet.

During my studies at Texas A&M University I was involved in the project of classification of groups generated by 3-state automata over 2-letter alphabet. The results of our work were published in [1, 2, 3]. The situation here is much more complicated than in the case of 2-state automata. We have shown that there are no more then 122 nonisomorphic groups in the class. Substantial information about these groups was obtained: all finite groups and all abelian groups were detected, it was proved that there are no infinite torsion groups, and there is only one nonabelian free group is in the class, etc.

A natural continuation of this project would be the attempt to analyze the class of all groups generated by 4-state automata over 2-letter alphabet, or 2-state automata over 3-letter alphabet.

6 Thompson's group F

Another area of my research (although not part of my dissertation) is related to Thompson's group F. This group can be defined as a group of all piecewise linear orientation preserving homeomorphisms of the unit interval $[0,1]$ into itself, whose break points are dyadic rational numbers and the slopes are powers of 2. This group was first considered by R. Thompson in 1965. One of the most intriguing open questions about this group is whether F is amenable. Originally this question was asked by Geoghegan in 1979 and since then dozens of papers were in some extent devoted to it.

In paper [13] I completely described Schreier graphs of the action of F on the set of dyadic rational numbers on the interval $(0,1)$ and on the direct powers of these set. The studying of the Schreier graphs of F was partially inspired by the question of amenability of F. In particular, if any Schreier graph with respect to any subgroup is non-amenable the whole group F would be non-amenable. Unfortunately, all Schreier graphs described in [13] are amenable yielding no information about the amenability of F. But the knowledge about the structure of Schreier graphs provides some additional information about F itself.

It happens that the described Schreier graph of the action of F on the set of dyadic rational numbers on the interval $(0,1)$ is closely related to the unitary representation of F in the space $B(L_2([0,1]))$ of all bounded linear operators on $L_2([0,1])$. It reflects (modulo a finite part) the dynamics of F on the Haar wavelet basis in $L_2([0,1])$. In [13] I make this connection precise.

Grigorchuk and Stepin reduced the question of amenability of F to the right amenability of the positive monoid P of F. Moreover, the amenability of F is equivalent to the amenability of the induced subgraph Γ_P of the Cayley graph Γ_F of F containing the positive monoid P. We construct the induced subgraph Γ_S of Γ_F containing all the vertices of the form $x_n v$ for $n \geq 0, v \in \{x_0, x_1\}^*$ and prove that this graph is non-amenable. It is also worth noting that certain generalization of my results are related to the induced subgraphs of P obtained by Staley in [16]. Using Block and Weinberger uniformly finite homology and Ponzi flows he proved that all graphs from certain family which includes the one constructed in [13] are nonamenable. It is interesting to investigate this approach for further generalizations. Currently I am studying random walks on constructed Schreier graphs. There is a hope that this could lead to new perspectives in the question of amenability of F.

7 Future plans

My plans for the nearest future include generalizing my results and attempting to answer the questions posted in the previous sections. I will continue to develop the software for computations in group theory (undergraduate and graduate students can be incorporated in such projects).

Particular open problems I wish to attack are the algorithmic problems for self-similar groups, such as word problem and finding the order of an element. Many partial solutions are known for special classes of groups, but it is not clear yet whether these problems are decidable or not in general.

I am also interested in finding applications of groups generated by automata in coding theory and cryptography. There is certain potential and not much has been done in this direction yet. And, of course, I plan to widen my research interests and will be open for collaboration with mathematicians in other fields of algebra and mathematics in general.

References

[1] I. Bondarenko, R. Grigorchuk, R. Kravchenko, Y. Muntyan, V. Nekrashevych, D. Savchuk, and Z. Šunić. Classification of groups generated by 3-state automata over 2-letter alphabet. *Algebra Discrete Math.*, (1):1–163, 2008. (available at `http://arxiv.org/abs/0803.3555`).

[2] I. Bondarenko, R. Grigorchuk, R. Kravchenko, Y. Muntyan, V. Nekrashevych, D. Savchuk, and Z. Šunić. Groups generated by 3-state automata over a 2-letter alphabet, I. *São Paulo Journal of Mathematical Sciences*, 1(1):1–40, 2007. (available at `http://arxiv.org/abs/math.GR/0612178`).

[3] I. Bondarenko, R. Grigorchuk, R. Kravchenko, Y. Muntyan, V. Nekrashevych, D. Savchuk, and Z. Šunić. Groups generated by 3-state automata over 2-letter alphabet, II. J. Math. Sci. (New York), to appear (available at `http://arxiv.org/abs/0704.3876`), 2007.

[4] I. Bondarenko and D. Savchuk. On Sushchansky p-groups. *Algebra Discrete Math.*, (2):22–42, 2007. (available at `http://arxiv.org/abs/math/0612200`).

[5] A. Brunner and S. Sidki. The generation of $\mathrm{GL}(n, \mathbf{Z})$ by finite state automata. *Internat. J. Algebra Comput.*, 8(1):127–139, 1998.

[6] R. Grigorchuk. On the Milnor problem of group growth. *Dokl. Ak. Nauk SSSR*, 271(1):30–33, 1983.

[7] R. Grigorchuk, V. Nekrashevich, and V. Sushchansky. Automata, dynamical systems, and groups. *Tr. Mat. Inst. Steklova*, 231(Din. Sist., Avtom. i Beskon. Gruppy):134–214, 2000.

[8] R. Grigorchuk, D. Savchuk, and Z. Šunić. The spectral problem, substitutions and iterated monodromy. In *Probability and mathematical physics*, volume 42 of *CRM Proc. Lecture Notes*, pages 225–248. Amer. Math. Soc., Providence, RI, 2007.

[9] R. Grigorchuk and Z. Šuniḱ. Asymptotic aspects of Schreier graphs and Hanoi Towers groups. *C. R. Math. Acad. Sci. Paris*, 342(8):545–550, 2006.

[10] Y. Muntyan and D. Savchuk. *AutomGrp –* `GAP` *package for computations in self-similar groups and semigroups, Version 1.1.3*, 2008. (available at `http://finautom.sourceforge.net`).

[11] V. Nekrashevych. *Self-similar groups*, volume 117 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, 2005.

[12] A. Olĭĭnik. Free products of C_2 as groups of finitely automatic permutations. *Voprosy Algebry*, 14:158–165, 1999.

[13] D. Savchuk. Some graphs related to Thompson's group F. Trends in Mathematics (Birkhäuser), edited by O. Bogopolski, I. Bumagin, O. Kharlampovich and E. Ventura, to appear, 2008 (available at `http://arxiv.org/abs/0803.0043`).

[14] D. Savchuk and Y. Vorobets. Automata generating free products of groups of order 2. (available at `http://arxiv.org/abs/0806.4801`), 2008.

[15] D. Savchuk. On word problem in contracting automorphism groups of rooted trees. *Vīsn. Kiïv. Unīv. Ser. Fīz.-Mat. Nauki*, (1):51–56, 2003.

[16] D. Staley. Thompson's group f and uniformly finite homology. (available at `http://arxiv.org/abs/0806.2877`), 2008.

[17] B. Steinberg, M. Vorobets, and Y. Vorobets. Automata over a binary alphabet generating free groups of even rank, 2006. (available at `http://arxiv.org/abs/math/0610033`).

[18] V. Sushchansky. Periodic permutation p-groups and the unrestricted Burnside problem. *DAN SSSR.*, 247(3):557–562, 1979. (in Russian).

[19] M. Vorobets and Y. Vorobets. On a series of finite automata defining free transformation groups, 2006. (available at `http://arxiv.org/abs/math/0604328`).