© Scarborough October 2008 Math 141 Quiz 10 KEY

(10pts) NAME (printed neatly): ________________________________

(5pts) Section Number (circle correct section): 503 (10:20am) 521 (11:30am) 523 (1:50pm)

1. In a ballroom dance community, 4440 took rumba lessons (*R*), 3330 took samba lessons (*S*), and 5550 took waltz lessons (*W*). Assume everyone took only 1 type of dance lesson during each month. Of those who took rumba lessons one month, the next month 60% took samba and the rest took another rumba class. Of those who took samba lessons one month, the next month 24% took rumba, 52% took waltz, and the rest took another samba class. Of those who took waltz lessons one month, the next month 42% took samba, 36% took rumba, and the rest took another waltz class.

a) (10pts) What is the initial-state probability matrix X_0? Put all entries as exact fractions in lowest terms. Label the rows *R*, *S*, and *W*.

$$X_0 = \begin{matrix} R \\ S \\ W \end{matrix}\begin{bmatrix} \frac{4440}{13320} \\ \frac{3330}{13320} \\ \frac{5550}{13320} \end{bmatrix} = \begin{matrix} R \\ S \\ W \end{matrix}\begin{bmatrix} \frac{1}{3} \\ \frac{1}{4} \\ \frac{5}{12} \end{bmatrix}$$

b) (10pts) What is the transition matrix, *T*? Include the row and column headings for the matrix. Note the rows represent the next state and the columns represent the current state. The entries of the transition matrix are the probabilities associated with the transition from one state to the next.

$$\begin{matrix} & R & S & W \end{matrix}$$
$$T = \begin{matrix} R \\ S \\ W \end{matrix}\begin{bmatrix} 0.4 & 0.24 & 0.36 \\ 0.6 & 0.24 & 0.42 \\ 0 & 0.52 & 0.22 \end{bmatrix}$$

c) (10pts) How many, to the nearest whole number, would you expect to be taking rumba after 2 months?

$$X_2 = T^2 X_0 = \begin{matrix} R \\ S \\ W \end{matrix}\begin{bmatrix} 0.3215333333 \\ 0.4035 \\ 0.2749666667 \end{bmatrix}$$

0.3215333333(13320) = 4282.824

Therefore you would expect **4283** to be taking rumba after 2 months.

d) (5pts) Is the transition matrix, T, a regular stochastic matrix? **YES** or NO

(A stochastic matrix must be square, all entries must be non-negative, and the sum of each column must be one. A stochastic matrix is regular if and only if some power of the matrix has entries that are all positive.) If yes, name one power of T that has all positive entries. ________________
2 or 3 or 4 or any natural number greater than 1

e) (10pts) If T is a regular stochastic matrix, then the steady-state distribution vector X can be found by solving the matrix equation $TX = X$, along with the condition that the sum of the elements of vector X is one.

i) Find the equations represented by $TX = X$ by substituting in your transition matrix T, by substituting in $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$ for vector X, and then by getting all the variable terms to the left of the equal sign and all the constants to the right.

$$\begin{bmatrix} 0.4 & 0.24 & 0.36 \\ 0.6 & 0.24 & 0.42 \\ 0 & 0.52 & 0.22 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$$\begin{aligned} 0.4x + 0.24y + 0.36z &= x \\ 0.6x + 0.24y + 0.42z &= y \\ 0x + 0.52y + 0.22z &= z \end{aligned}$$

$$\begin{aligned} -0.6x + 0.24y + 0.36z &= 0 \\ 0.6x - 0.76y + 0.42z &= 0 \\ 0x + 0.52y - 0.78z &= 0 \end{aligned}$$

ii) To the system of equations found in the previous part i, add the equation representing the condition that the sum of the elements of vector X sum to one. Give matrix, M, which represents this system of equations.

$$\begin{aligned} -0.6x + 0.24y + 0.36z &= 0 \\ 0.6x - 0.76y + 0.42z &= 0 \\ 0x + 0.52y - 0.78z &= 0 \\ 1x + 1y + 1z &= 1 \end{aligned}$$

$$M = \begin{bmatrix} -0.6 & 0.24 & 0.36 & 0 \\ 0.6 & -0.76 & 0.42 & 0 \\ 0 & 0.52 & -0.78 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix}$$

f) (10pts) Find the steady-state vector, X, for the long-term distribution of the players by first putting matrix M into row-reduced echelon form with exact entries.

$$\text{rref } M = \begin{bmatrix} 1 & 0 & 0 & \frac{12}{37} \\ 0 & 1 & 0 & \frac{15}{37} \\ 0 & 0 & 1 & \frac{10}{37} \\ 0 & 0 & 0 & 0 \end{bmatrix} \text{ so } X = \begin{matrix} x \\ y \\ z \end{matrix}\begin{bmatrix} \frac{12}{37} \\ \frac{15}{37} \\ \frac{10}{37} \end{bmatrix} \text{ and thus } X = \begin{matrix} R \\ S \\ W \end{matrix}\begin{bmatrix} \frac{12}{37} \\ \frac{15}{37} \\ \frac{10}{37} \end{bmatrix}$$

g) (10pts) In the long run, what proportion of the dancers will be taking samba?

$$\frac{15}{37}$$

h) (10pts) In the long run, how many dancers, to the nearest whole number, will be taking waltz?

$$\left(\frac{10}{37}\right)(13320) = 3600 \text{ or } 13320\begin{bmatrix} \frac{12}{37} \\ \frac{15}{37} \\ \frac{10}{37} \end{bmatrix} = \begin{matrix} R \\ S \\ W \end{matrix}\begin{bmatrix} 4320 \\ 5400 \\ 3600 \end{bmatrix}$$

In the long run, you would expect **3600** dancers to take waltz.

2. (10pts) The probability that a person defaults on his/her auto loan is 5.4%. Out of 5184 people with auto loans, what is the probability, to 4 decimal places, that at least 300 but less than 326 default on their auto loans? Use the appropriate normal distribution to approximate this probability.

binomial n = 5184 p = 0.054

$$P(300 \le X < 326) = P(300 \le X \le 325) \approx P299.5 < Y < 325.5)$$
$$= normalcdf\left(299.5, 325.5, (5184)(0.054), \sqrt{(5184)(0.054)(1-0.054)}\right) \approx 0.1121$$