Chapter 7A - Systems of Linear Equations

Geometry of Solutions

In an earlier chapter we learned how to solve a single equation in one unknown. The general form of such an equation has the form

\[ ax = b, \]

where the constants \( a \neq 0 \) and \( b \) are assumed known, and we are looking for a value of \( x \) which satisfies the equation. Since \( a \neq 0 \), the equation is easy to solve. Multiply by \( a^{-1} \). Thus, we have the solution \( x = \frac{b}{a} \). The situation is not quite so simple when we have more than one equation and unknown. First we give the standard form for a system of two equations in two unknowns.

\[ ax + by = c \\
\[ dx + ey = f, \]

where the constants \( a, b, c, d, e, \) and \( f \) are assumed known.

By a solution of this system we mean a pair of numbers \( x_0 \) and \( y_0 \) which satisfy the system of equations. That is, when the substitutions \( x = x_0 \) and \( y = y_0 \) are made in the system both equations become identities.

This pair of numbers is commonly written as \((x_0, y_0)\) and interpreted as a point in the Euclidean plane, \( R^2 \). By the solution set of a system we mean the totality of all possible solutions to the system.

**Example 1:** Which of the pairs of numbers \((1, 2)\), \((11, 3)\), or \((-19, -9)\) are solutions to the given system?

\[
\begin{align*}
2x - 5y & = 7 \\
-x + 2y & = 1.
\end{align*}
\]

Solution: To check that a pair of numbers is a solution we substitute the values in for \( x \) and \( y \). In the first equation if we substitute \( x = 1 \) and \( y = 2 \) we have

\[
2x - 5y = 7 \rightarrow 2(1) - 5(2) = 7 \rightarrow -8 = 7
\]

Since \(-8 \neq 7\) the first pair does not satisfy the first equation let alone both equations.

We try the second pair \((11, 3)\) next:

\[
2x - 5y = 7 \rightarrow 2(11) - 5(3) = 7 \rightarrow 7 = 7
\]

Okay, the second pair satisfies the first equation, but that is not enough to guarantee that we have a solution to the system. The second equation still has to be checked, which we do below.

\[
-x + 2y = 1 \rightarrow -(11) + 2(3) = 1 \rightarrow -5 = 1
\]

Since \(-5 \neq 1\), this pair does not solve the system.
Now let’s check the last pair \((-19, -9)\) to see if it is a solution
\[
2x - 5y = 7 \rightarrow 2(-19) - 5(-9) = 7 \rightarrow 7 = 7
\]
\[
2x - 5y = 7 \rightarrow 2(-19) - 5(-9) = 7 \rightarrow 7 = 7
\]
Since the pair \((-19, -9)\) satisfies both equations, this pair is a solution to the system.

**Question:** Does the pair of numbers \((1, -1)\) satisfy the system?
\[
2x - y = 3
\]
\[
-3x + 6y = 3
\]
**Answer:** No, the first equation is satisfied, but the second is not.

**Note:** There is no reason why a system must consist of two equations. In the following pages we will have examples of systems which consist of a single equation with more than one unknown, systems which consist of two equations with three unknowns, and finally the general system which consists of \(n\) equations in \(m\) unknowns. In this later case there need be no a-priori relationship between the sizes of \(m\) and \(n\).

**Example 2:** Show that the pair of number \((-3, 2)\) satisfies the system
\[
2x + 7y = 8
\]
\[
-x + y = 5
\]
**Solution:** To verify that \((-3, 2)\) is a solution we just substitute \(-3\) for \(x\) and 2 for \(y\) in each equation and then check that we have an identity.
\[
2x + 7y = 8 \rightarrow 2(-3) + 7(2) = 8 \rightarrow 8 = 8
\]
\[
-x + y = 5 \rightarrow -(3) + (2) = 5 \rightarrow 5 = 5
\]
Since both equations are true the pair \((-3, 2)\) is a solution of the system.

**Example 3:** Consider the system
\[
-x + 3y = 2
\]
\[
x + y = 1
\]
Is there a number \(y_0\) so that the pair \((1, y_0)\) is a solution to the system?

**Solution:** If the pair \((1, y_0)\) is a solution to the system then \(y_0\) must satisfy the first equation and the second equation with 1 substituted for \(x\).
\[
-1 + 3y_0 = 2
\]
\[
1 + y_0 = 1
\]
Solving the first equation for \(y_0\) and then the second equation for \(y_0\) we have
\[
y_0 = \frac{2 + 1}{3} = 1 \quad \text{from the first equation}
\]
\[
y_0 = 1 - 1 = 0 \quad \text{from the second equation}
\]
Thus, there is no value of \(y_0\) which can satisfy both equations, and we conclude there is no pair of numbers of the form \((1, y_0)\) which satisfies this system.
Example 4: Is there a number \( x_0 \) such that the pair \((x_0, -2)\) satisfies the system

\[
\begin{align*}
3x + y &= 4 \\
5x - 3y &= 16
\end{align*}
\]

Solution: As in the previous example we substitute \(-2\) for \( y \) in both equations and see if there is a single value of \( x \) which will satisfy both equations simultaneously.

\[
\begin{align*}
3x - 2 &= 4 \Rightarrow 3x = 6 \Rightarrow x = 2 \\
5x - 3(-2) &= 16 \Rightarrow 5x = 16 - 6 = 10 \Rightarrow x = 2
\end{align*}
\]

The value \( x = 2 \) works. Thus, the pair \((2, -2)\) satisfies the system.

We now begin our study of the geometry of solutions to systems of equations. First we examine the solution set for the single equation in two unknowns

\[2x - 3y = 1.\]

As we have seen, the locus of points in the plane whose coordinates satisfy an equation of this sort is a straight line. The graph of the solution set of the equation \(2x - 3y = 1\) is shown below.

What can we say about the geometry of solutions to the system

\[
\begin{align*}
2x - 3y &= 1 \\
x + y &= 2
\end{align*}
\]

If we have a solution to this system then we know that it must lie on the straight line shown above, since the solution to the system is a solution to the first equation. However, it is also a solution to the second equation. Thus, the solution must also lie on the straight line \(x + y = 2\). Hence, any solution to the system must lie on the intersection of both lines. The two lines are shown below. Their point of intersection is \((7/5, 3/5)\).

Question: Verify that \((7/5, 3/5)\) is a solution to this system.

Solution: \(2x - 3y = 1 \Rightarrow 2(7/5) - 3(3/5) = 1 \Rightarrow 1 = 1.\) \(x + y = 2 \Rightarrow (7/5) + 3/5 = 2 \Rightarrow 2 = 2\)
There are three possibilities for the configuration of two lines in a plane. The lines intersect in a unique point, as they do above: the lines are parallel and not equal to each other, which means they have no points of intersection: the third possibility is that the two lines are the same line, in which case there are an infinite number of points of intersection. This leads us to believe that for a linear system of equations the solution set can have one of the following three characterizations

1. The solution set consists of a single solution.
2. The solution set is empty, that is there are no solutions.
3. There are an infinite number of solutions.

**Question:** How many solutions does the following system have, one, none, or an infinite number?

\[
\begin{align*}
5x - 3y &= 2 \\
10x - 6y &= 4
\end{align*}
\]

**Answer:** Since the second equation is a multiple of the first equation, every solution of the first equation will also be a solution of the second equation. Moreover, the first equation has an infinite number of solutions. Thus, the system has an infinite number of solutions.

The plots below exhibit the three possibilities for the number of solutions to a linear system of equations.

If we want to find the solution set to a system of equations in two unknowns, we can do so by graphing the straight line which corresponds to a particular equation in our system. However, this is not quickly done, nor is it accurate. In the next sections we will describe algebraic ways of solving a system of equations.
Example 5: Graph the solution set of the system, and from your plot describe the solution set to this system.

\[
\begin{align*}
    x + y &= 2 \\
    2x - 3y &= 1 \\
    x - y &= 4
\end{align*}
\]

Solution:

We note that each pair of lines has a single point of intersection, but there is no point which lies on all three lines. Hence, this system has no solution. Its solution set is empty.
Algebraic Methods of Solution

There are essentially two algebraic methods to solve a system of equations. The first, which is called the substitution method, is easy to use. However, its use is usually relegated to very simple systems. The second method, elimination, leads into a method of Gaussian-elimination that allows you to solve large systems. We will look at how to solve a system of two equations and two unknowns using both the method of substitution and the method of elimination.

The Method of Substitution

Pick one of the equations, solve this equation for one of the unknowns in terms of the other unknown(s), substitute the expression for the solved unknown in the remaining equations. Repeat this process until one of the equations has been reduced to an equation in only one unknown. Solve for this unknown, and then use this value to determine the values of the other unknowns.

The next few examples demonstrate how to solve a system of equations using the method of substitution.

Example 1: Solve the linear system of equations

\[ x - y = 4 \]
\[ 2x + 5y = 8. \]

Solution: Solve the first equation for \( y \) in terms of \( x \). (We could just as easily solve for \( x \) in terms of \( y \).)

\[ x - y = 4 \rightarrow y = x - 4. \]

Substitute \( x - 4 \) for \( y \) in the second equation, and then solve for \( x \).

\[ 2x + 5y = 8 \]
\[ 2x + 5(x - 4) = 8 \]
\[ 2x + 5x - 20 = 8 \]
\[ 7x - 20 = 8 \]
\[ 7x = 28 \]
\[ x = 4 \]

Take this value of \( x \) and calculate \( y \).

\[ y = x - 4 \]
\[ y = (4) - 4 \]
\[ y = 0 \]

Thus, the pair \((4, 0)\) is a solution to this system. You can easily verify that the pair \((4, 0)\) really is a solution to the system.
Example 2: Use the method of substitution to solve the system

\[
\begin{align*}
    x - 3y &= 6 \\
    2x + 5y &= 7
\end{align*}
\]

Solution: Solve the first equation for \( x \) (That seems easier than solving for \( y \)), and then substitute this expression for \( x \) into the second equation.

\[
\begin{align*}
    x - 3y &= 6 \rightarrow x = 6 + 3y \\
    2x + 5y &= 7
\end{align*}
\]

\[
\begin{align*}
    2(6 + 3y) + 5y &= 7 \\
    12 + 6y + 5y &= 7 \\
    12 + 11y &= 7 \\
    11y &= 12 - 7 \\
    y &= \frac{-5}{11} \\
    x &= 6 + 3y = 6 + 3\left(\frac{-5}{11}\right) \\
    &= 6 - \frac{15}{11} \\
    &= \frac{51}{11}
\end{align*}
\]

The solution is \( \left(\frac{51}{11}, \frac{-5}{11}\right) \).

Example 3: Solve the following system using the method of substitution. Then plot the equations and the solution.

\[
\begin{align*}
    6x + 17y &= 9 \\
    2x + y &= 6
\end{align*}
\]

Solution: First solve the second equation for \( y \) in terms of \( x \), to get \( y = 6 - 2x \). Substitute this expression into the first equation and solve for \( y \).

\[
\begin{align*}
    6x + 17y &= 9 \\
    6x + 17(6 - 2x) &= 9 \\
    6x + 102 - 34x &= 9 \\
    -28x &= 9 - 102 \\
    x &= \frac{-93}{-28} \\
    &= \frac{93}{28} \approx 3.32143
\end{align*}
\]

Thus, \( y = 6 - 2x = 6 - 2\left(\frac{93}{28}\right) = -\frac{9}{14} \approx -0.642857 \). Thus, the solution is \( \left(\frac{93}{28}, \frac{-9}{14}\right) \approx (3.321, -0.643) \). The plot of the solution set is given below.
Example 4: Solve the following system of equations using the method of substitution.
\[ \begin{align*}
2x - 6y &= 5 \\
-x + 3y &= 4 
\end{align*} \]

Solution: Solve the second equation for \( x \), and then substitute into the first equation.
\[ \begin{align*}
-x + 3y &= 4 \\
2x - 6y &= 5 \\
6y - 8 - 6y &= 5 \\
-8 &= 5 
\end{align*} \]

Something bad has happened. After substituting \( 3y - 4 \) for \( x \) in the first equation the terms involving \( y \) cancel, and we are left with a contradiction \(-8 = 5\). This means that there is no solution for this system. If we go back and examine the system, we soon see that the left side of the second equation is a multiple of the first equation.

\[ 2x - 6y = 5 \text{ multiply by } \frac{-1}{2} \rightarrow -x + 3y = \frac{-5}{2} \]

Thus, any solution to our system must satisfy the two equations
\[ \begin{align*}
-x + 3y &= \frac{-5}{2} \\
-x + 3y &= 4 
\end{align*} \]

Clearly this cannot happen. Geometrically these equations represent two parallel lines which are plotted below.

Example 5: Solve the following system using substitution.
\[ \begin{align*}
3x - 15y &= 18 \\
-6x + 30y &= -36 
\end{align*} \]

Solution: Solving the first equation for \( x \) we have
\[ \begin{align*}
3x - 15y &= 18 \\
3x &= 18 + 15y \\
x &= 6 + 5y 
\end{align*} \]

Substitute this expression for \( x \) into the second equation
\[ \begin{align*}
-6x + 30y &= -36 \\
-6(6 + 5y) + 30y &= -36 \quad \text{simplify the left hand side} \\
-36 &= -36 
\end{align*} \]

This last line implies that no matter what value \( y \) has as long as we set \( x = 6 + 5y \) we will have a solution. Thus, the solution set for this system consists of all pairs of numbers \((6 + 5y, y)\) for any value of \( y \).
We have seen how to solve a system using the process of substitution. However, this method is highly ineffective when the number of equations starts getting large. For the method of substitution large typically means more than 2. The second method (elimination), which we consider next, is somewhat more complicated, but it is a lot more efficient, and better suited to handle larger numbers of equations and unknowns.

Example 6: Solve the system below by using the method of elimination.

\[
\begin{align*}
2x + y &= 8 \\
x - y &= 6
\end{align*}
\]

Solution: We ask ourselves the following question: "What can we multiply one of the equations by so that when we add the equations together one of the variables is eliminated?" We see here that we don’t need to multiply one of the equations by a constant as the \( y \)'s will be eliminated if we add the equations together as is.

\[
\begin{align*}
2x + y &= 8 \\
x - y &= 6
\end{align*}
\]

\[3x + 0 = 14 \Rightarrow x = \frac{14}{3}\]

Once we have a value for one of the variables we can substitute this into either of the original equations to find the value for the other variable:

\[
\begin{align*}
2x + y &= 8 \\
2\left(\frac{14}{3}\right) + y &= 8 \Rightarrow y = 8 - \frac{28}{3} = -\frac{4}{3}
\end{align*}
\]

Thus, the solution to this system is the pair \(\left(\frac{14}{3}, -\frac{4}{3}\right)\).

Example 7: Use the method of elimination to solve the following system of equations

\[
\begin{align*}
2x + y &= 7 \\
-4x - y &= 6
\end{align*}
\]

Solution: Again, we see that the \( y \)'s will be eliminated if we add the equations together as is:

\[
\begin{align*}
2x + y &= 7 \\
-4x - y &= 6
\end{align*}
\]

\[-2x + 0 = 13 \Rightarrow x = -\frac{13}{2}\]

We now substitute this value for \( x \) into either of the equations:

\[
\begin{align*}
2x + y &= 7 \\
2\left(-\frac{13}{2}\right) + y &= 7 \Rightarrow y = 7 + 13 = 20
\end{align*}
\]

Thus, we have found the solution \(\left(-\frac{13}{2}, 20\right)\).
Example 8: Solve the following system of equations using the method of elimination.

\[
\begin{align*}
3x - 5y &= 16 \\
x + y &= 2
\end{align*}
\]

Solution: Here, we see that if we added the equations together as is, neither of the variables would be eliminated. If we wanted to eliminate the y’s, we would need to multiply both sides of the second equation by 5:

\[
\begin{align*}
3x - 5y &= 16 \\
5x + 5y &= 10
\end{align*}
\]

Adding the two equations together gives us:

\[
\begin{align*}
3x - 5y &= 16 \\
5x + 5y &= 10
\end{align*}
\]

\[
8x + 0 = 26 \Rightarrow x = \frac{13}{4}
\]

Substituting this value for x into either equation gives us:

\[
3 \left( \frac{13}{4} \right) - 5y = 16 \Rightarrow -5y = 16 - \frac{39}{4} = \frac{25}{4}
\]

\[
\Rightarrow y = \left( \frac{-1}{5} \right) \left( \frac{25}{4} \right) = \frac{-5}{4}
\]

Thus, we have found the solution \( \left( \frac{13}{4}, \frac{-5}{4} \right) \)

Example 9: Solve the following system by the process of elimination.

\[
\begin{align*}
5x + 2y &= 8 \\
3x + y &= 5
\end{align*}
\]

Solution: Let’s multiply both sides of the second equation by -2:

\[
\begin{align*}
5x + 2y &= 8 \\
-6x - 2y &= -10
\end{align*}
\]

Adding the two equations together gives us:

\[
\begin{align*}
5x + 2y &= 8 \\
-6x - 2y &= -10
\end{align*}
\]

\[
-x + 0 = -2 \Rightarrow x = 2
\]

Substituting this into either of the original equations gives us:

\[
5(2) + 2y = 8 \Rightarrow 2y = 8 - 10 = -2
\]

\[
\Rightarrow y = -1
\]

Thus, we have found the solution \((2, -2)\)

In the next few examples we will see how systems of equations can arise quite naturally. Pay particular attention to how the mathematical equations arise from the words which describe the problem.
Example 10: A plane flies a round trip between two cities. The flight from the first city is into a strong headwind and takes 1 hour and 30 minutes. The return flight is with the wind and takes 55 minutes. If the cities are 100 miles apart what is the aircraft’s speed, and what is the wind’s speed. Assume that both the aircraft’s and wind’s speeds are constant.

Solution: Let $p$ denote the aircraft’s speed in miles per hour and let $w$ denote the wind’s speed also in miles per hour. The basic fact which we need here is that distance equals rate times time. In that part of the trip against the wind, the speed of the plane is $p - w$ and the plane’s speed with the wind is $p + w$. This leads to the following system of equations.

\[
\frac{3}{2} (p - w) = 100 \quad \text{rate times time equals distance}
\]
\[
\frac{55}{60} (p + w) = 100
\]

Note that we’ve converted the flight times from minutes to hours, so that the speeds we solve for will be in miles per hour and not miles per minute.

The above system leads to the following system

\[
p - w = \frac{200}{3}
\]
\[
p + w = \frac{6000}{55}
\]

Adding the two equations together we get

\[
2p = \frac{200}{3} + \frac{6000}{55} = \frac{5800}{33}
\]
\[
p = \frac{5800}{66} = \frac{2900}{33} \approx 87.89 \text{ miles per hour}
\]

To determine the wind’s speed we have from the second equation

\[
w = \frac{6000}{55} - p
\]
\[
= \frac{6000}{55} - \frac{2900}{33}
\]
\[
= \frac{700}{33}
\]
\[
\approx 21.21 \text{ miles per hour}
\]

Example 11: A salad dressing manufacturer wants to make a new version of a honey mustard dressing by combining two other honey mustard dressings. Dressing number 1 contains 5% honey and dressing number 2 contains 4% honey. How many quarts of each of these dressings must the manufacture combine in order to produce 1000 quarts of a 4.75% honey dressing?

Solution: Let $d_1$ represent the number of quarts used of the first dressing and $d_2$ the number of quarts of the second dressing. Thus,

\[
d_1 + d_2 = 1000
\]
\[
0.05d_1 + 0.04d_2 = 0.0475(1000).
\]

Solving the first equation for $d_2$ and substituting this expression into the second equation gives

\[
0.05d_1 + 0.04(1000 - d_1) = 0.0475(1000)
\]
\[
5d_1 + 4(1000 - d_1) = 4750
\]
\[
d_1 + 4000 = 4750
\]
\[
d_1 = 750.
\]

Thus, the desired mixture can be made by combining 750 quarts of dressing number 1 with 250 quarts of dressing number 2.
Exercises for Chapter 7A - Systems of Linear Equations

1. Graph the solution set to the equation $2x - 3y = 1$

2. Graph the solution set to the system $2x - 3y = 1, x + y = -1$

3. How many solutions are there to the system $2x - 3y = 1, x + y = -1, x - y = 2$?

4. Graph the straight lines which make up the system
   
   $\begin{align*}
   x + y &= 5 \\
   2x - 3y &= 1 \\
   x - y &= 2
   \end{align*}$

   and from your graph determine the solution set.

5. Find a system of two equations in two unknowns for which $(2, -3)$ is a solution.

6. Use the method of substitution to solve the system
   
   $\begin{align*}
   x + y &= 6 \\
   2x - y &= 5
   \end{align*}$

7. Solve the system
   
   $\begin{align*}
   5x - 4y &= 9 \\
   4x + y &= 7
   \end{align*}$

8. Solve the system
   
   $\begin{align*}
   x + y &= 5 \\
   3x - 2y &= 12 \\
   2x - 3y &= 7
   \end{align*}$

9. Use the method of elimination to solve the system
   
   $\begin{align*}
   2x - y &= 5 \\
   x + 7y &= 4
   \end{align*}$

10. If $(3, -4)$ is a solution to the equation $ax - 5y = -2$, what must $a$ equal?

11. Suppose that $(-2, 5)$ and $(1, 1)$ solve the equation $ax + by = 1$. What do $a$ and $b$ equal?

12. Use the method of elimination to solve the system $3x - 4y = 5, x + 7y = 1$.

13. Tickets to see Star Wars 18 at a local movie theater cost $25 for an adult and $15 for a child. If the theater takes in $1300, and sold a total of 70 tickets, how many child and adult tickets were sold?
14. A research chemist has two different salt water solutions. Solution 1 is 5% salt and solution 2 is 15% salt. She wants to have 1 liter of a 7% solution. How much of each solution should she mix together to get the desired solution?

15. Suppose a boat has to make a round trip up and down a river. The trip up takes 1 hour and is with the current. The trip back takes 3 hours. Suppose the round trip is a total of 20 miles, and that the boat’s speed and the speed of the current are constant. What is the boat’s speed, and what is the current’s speed?

16. Solve the system

\[
\begin{align*}
3x - 5y &= 11 \\
3.9x - 6.5y &= 18.2
\end{align*}
\]

17. Solve the system

\[
\begin{align*}
-20x + 30y &= 50 \\
-40x + 60y &= 100
\end{align*}
\]
Answers to Exercises for Chapter 7A - Systems of Linear Equations

1. Every point on the line is a solution to the system.

\[ 2x - 3y = 1 \]

\[ y = -\frac{1}{3} \]

\[ x = \frac{1}{2} \]

2. \[ x = -\frac{2}{5} \]

\[ y = -\frac{3}{5} \]

3. We saw in the preceding problem that \( x = -\frac{2}{5} \) and \( y = -\frac{3}{5} \) are solutions to the first two equations. We also note that they do not solve the third equation \( x - y = 2 \).

Thus, this system has no solutions.

4. Since there is no point common to all three lines, the solution set of this system is empty.

5. There are an infinite number of such systems. Geometrically they are all possible pairs of straight lines which pass through the point \((2, -3)\). One such pair is

\[ 2x + 3y = -5 \]

\[ x - 6y = 20 \]

6. From the first equation we have

\[ y = 6 - x. \]

Substituting into the second equation we have

\[ 2x - y = 5 \]

\[ 2x - (6 - x) = 5 \]

\[ 3x = 11 \]

\[ x = \frac{11}{3} \]

For \( y \) we have \( y = 6 - x = 6 - \frac{11}{3} = \frac{7}{3} \). Thus, the solution is \( \left( \frac{11}{3}, \frac{7}{3} \right) \).

7. \( x = \frac{37}{21}, \ y = -\frac{1}{21} \)
8. \( x = \frac{22}{5}, \quad y = \frac{3}{5} \)

9. \[
2x - y = 5 \\
x + 7y = 4
\]

Multiply both sides of the first equation by 7:

\[
14x - 7y = 35 \\
x + 7y = 4
\]

Adding the two equations together gives us:

\[
14x - 7y = 35 \\
x + 7y = 4
\]

Substituting into either of the original equations gives us:

\[
x + 7y = 4 \\
\frac{13}{5} + 7y = 4 \Rightarrow 7y = 4 - \frac{13}{5} = \frac{7}{5}
\]

\[
\Rightarrow y = \frac{1}{5}
\]

Thus, our solutions is \( \left( \frac{13}{5}, \frac{1}{5} \right) \)

10. \( -\frac{22}{3} \)

11. Since the points (-2,5) and (1,1) satisfy the equation \( ax + by = 1 \), we have the following system of equations

\[
-2a + 5b = 1 \\
a + b = 1
\]

The solution to this system is

\[
a = \frac{4}{7}, \quad b = \frac{3}{7}.
\]

12. \( x = \frac{39}{25}, y = -\frac{2}{25} \)

13. Let \( c \) and \( a \) denote the number of child’s tickets and adult tickets sold respectively. They we know

\[
c + a = 70 \\
15c + 25a = 1300.
\]

Solving this system by substitution, we have

\[
15c + 25a = 1300 \\
15c + 25(70 - c) = 1300 \\
-10c + 1750 = 1300 \\
10c = 1750 - 1300 \\
10c = 450 \\
c = 45
\]

Thus, the number of children’s tickets sold is 45, and the number of adult tickets sold is \( 70 - 45 = 25 \).
14. Let \( x_1 \) denote the number of liters of solution 1 and \( x_2 \) the number of liters of solution 2 that are used. Then we have

\[
\begin{align*}
x_1 + x_2 &= 1 \\
0.05x_1 + 0.15x_2 &= 0.07(1)
\end{align*}
\]

Solving by substitution we have

\[
\begin{align*}
0.05x_1 + 0.15x_2 &= 0.07 \\
0.05x_1 + 0.15(1 - x_1) &= 0.07 \\
-0.1x_1 + 15 &= 0.07 \\
0.1x_1 &= 0.15 - 0.07 \\
x_1 &= 0.8 \\
\end{align*}
\]

So, she should use \( \frac{4}{5} \) of a liter of the first solution, and \( \frac{1}{5} \) of a liter of the second solution.

15. Let \( b \) represent the boat’s speed in miles per hour, and let \( c \) represent the current’s speed in miles per hour. Then we have

\[
\begin{align*}
1(b + c) &= 10 \\
3(b - c) &= 10
\end{align*}
\]

The solutions to this system are

\[
\begin{align*}
b &= \frac{20}{3} \text{ miles per hour} \\
c &= \frac{10}{3} \text{ miles per hour}
\end{align*}
\]

16.

\[
\begin{align*}
3x - 5y &= 11 \\
3.9x - 6.5y &= 18.2
\end{align*}
\]

Solving the first equation for \( y \) gives us:

\[
-5y = 11 - 3x \Rightarrow y = -\frac{1}{5}(11 - 3x) = -\frac{11}{5} + \frac{3}{5}x
\]

Substituting this into the second equation give us:

\[
\begin{align*}
3.9x - 6.5\left(-\frac{11}{5} + \frac{3}{5}x\right) &= 18.2 \\
3.9x + 14.3 - 3.9x &= 18.2 \\
14.3 &= 18.2
\end{align*}
\]

Recall that when we end up with an equation with no variables that is false, we can conclude that the system has no solution. Therefore, the above system has no solution.
17.

\[-20x + 30y = 50 \]
\[-40x + 60y = 100 \]

Using the method of elimination let’s multiply both sides of the first equation by \(-2\):

\[40x - 60y = -100 \]
\[-40x + 60y = 100 \]

Adding the two equations together gives us:

\[0 + 0 = 0 \]
\[0 = 0 \]

We recall that this leads us to conclude that the system has infinitely many solutions (both equations represent the exact same line). Thus, we can solve either equation for \(y\) to find the equation of the line:

\[-20x + 30y = 50 \implies 30y = 20x + 50 \]
\[\Rightarrow y = \frac{2}{3}x + \frac{5}{3} \]

Thus, all points on this line \((x, \frac{2}{3}x + \frac{5}{3})\) are solutions to the system.
Chapter 7B - Miscellaneous Exercises

1. Suppose that a population of bacteria satisfies an exponential growth law, \( p(t) = ae^{kt} \). If \( p(1) = 18 \) and \( p(5) = 72 \), determine \( a \) and \( k \).

2. Exactly solve the system of equations:
   \[
   \begin{align*}
   x^2 - y^2 &= 15 \\
   x - y &= 2
   \end{align*}
   \]

3. Exactly solve the system of equations:
   \[
   \begin{align*}
   y &= (x - 5)^2 + 7 \\
   (x - 5)^2 + (y - 7)^2 &= 9
   \end{align*}
   \]

4. Exactly solve the system of equations:
   \[
   \begin{align*}
   16x^2 - 64x + y^2 &= -39 \\
   x^2 - 4x + y^2 &= 21
   \end{align*}
   \]

5. Exactly solve the system of equations:
   \[
   \begin{align*}
   y^2 - 4x^2 &= 4 \\
   x &= y^2 - 4
   \end{align*}
   \]

6. Exactly solve the system of equations:
   \[
   \begin{align*}
   x^2 + 4y^2 &= 36 \\
   2y + x + 6 &= 0
   \end{align*}
   \]
Chapter 7B - Solutions to Miscellaneous Exercises

1. The equations we derive from the data are

\[ ae^k = 18 \]
\[ ae^{5k} = 72. \]

These are not linear equations. However, if the logarithm of both sides are taken, we get a linear system in \( \ln a \) and \( k \).

\[ \ln a + k = \ln 18 \]
\[ \ln a + 5k = \ln 72. \]

The solution to this system is

\[ k = \frac{\ln 4}{4} \]
\[ \ln a = \ln 18 - \frac{\ln 4}{4} \]

Which then gives

\[ a = e^{\ln a} \]
\[ = e^{\ln 18 - \frac{\ln 4}{4}} \]
\[ = 18e^{-\ln 4/4} \]
\[ = 18\left(\frac{1}{4}\right)^{1/4} \]

2.

\[ x^2 - y^2 = 15 \]
\[ x - y = 2 \]

Solving the second equation for \( x \) and substituting into the first equation yields:

\[ x = 2 + y \]

\[ (2 + y)^2 - y^2 = 15 \]
\[ 4 + 4y + y^2 - y^2 = 15 \]
\[ 4 + 4y = 15 \]
\[ 4y = 11 \]
\[ y = \frac{11}{4} \]

Substituting this back into the second equation yields:

\[ x = 2 + \frac{11}{4} = \frac{19}{4} \]

Thus, the solution to this system is \( \left(\frac{19}{4}, \frac{11}{4}\right) \).
3. \[ y = (x - 5)^2 + 7 \]
\[ (x - 5)^2 + (y - 7)^2 = 6 \]

Solving the first equation for \((x - 5)^2\) and substituting into the second equation yields:
\[ (x - 5)^2 = y - 7 \]
\[ y - 7 + (y - 7)^2 = 6 \]
We notice this is in quadratic form. Let \(u = y - 7\). This gives us:
\[ u + u^2 = 6 \]
\[ u^2 + u - 6 = 0 \]
\[ (u + 3)(u - 2) = 0 \Rightarrow u = -3 \text{ or } u = 2 \]

Thus,
\[ y - 7 = -3 \text{ or } y - 7 = 2 \]
\[ y = 4 \text{ or } y = 9 \]

We can now plug each of these back into the first equation to find the corresponding values of \(x\):
\[ y = 4: \]
\[ 4 = (x - 5)^2 + 7 \]
\[ 4 = x^2 - 10x + 25 + 7 \]
\[ 0 = x^2 - 10x + 23 \]

Using the quadratic formula to solve this quadratic yields:
\[ x = \frac{10 \pm \sqrt{(-10)^2 - 4(1)(28)}}{2(1)} = \frac{10 \pm \sqrt{-12}}{2} = \frac{10 \pm 2i\sqrt{3}}{2} = 5 \pm i\sqrt{3} \]
\[ y = 9: \]
\[ 9 = (x - 5)^2 + 7 \]
\[ 9 = x^2 - 10x + 25 + 7 \]
\[ 0 = x^2 - 10x + 23 \]

Using the quadratic formula to solve this quadratic yields:
\[ x = \frac{10 \pm \sqrt{(-10)^2 - 4(1)(23)}}{2(1)} = \frac{10 \pm \sqrt{8}}{2} = \frac{10 \pm 2\sqrt{2}}{2} = 5 \pm \sqrt{2} \]
Thus, our solutions to this system are \((5 + i\sqrt{3}, 4), (5 - i\sqrt{3}, 4), (5 + \sqrt{2}, 9), \) and \((5 - \sqrt{2}, 9)\).

4. \[ 16x^2 - 64x + y^2 = -39 \]
\[ x^2 - 4x + y^2 = 21 \]

Multiply the second equation by \(-1\) and then add the two equations together:
\[
16x^2 - 64x + y^2 = -39 \\
-x^2 + 4x - y^2 = -21 \\
15x^2 - 60x = -60
\]

We now have a quadratic that we can easily factor:
\[
15x^2 - 60x + 60 = 0 \\
15(x^2 - 4x + 4) = 0 \\
15(x - 2)(x - 2) = 0 \Rightarrow x = 2
\]

Substituting into either of the original equations gives us the corresponding value(s) for \( y \):
\[
16x^2 - 64x + y^2 = -39 \\
16(2)^2 - 64(2) + y^2 = -39 \\
-64 + y^2 = -39 \\
y^2 = 25 \\
y = \pm 5
\]

Thus, we have two solutions to this system: \((2, 5)\) and \((2, -5)\).

5.

\[
y^2 - 4x^2 = 4 \\
x = y^2 - 4
\]

Rearranging the first equation gives us:
\[
y^2 - 4 = 4x^2 \\
x = y^2 - 4
\]

Substitution yields:
\[
x = 4x^2 \\
x - 4x^2 = 0 \\
x(1 - 4x) = 0 \Rightarrow x = 0 \text{ or } x = \frac{1}{4}
\]

We can then get the corresponding value(s) of \( y \) by substituting into either of the original equations:
\[
x = 0:
\]
\[
y^2 - 4x^2 = 4 \\
y^2 - 4(0)^2 = 4 \\
y^2 = 4 \Rightarrow y = \pm 2
\]

\[
x = \frac{1}{4}:
\]
\[
y^2 - 4x^2 = 4 \\
y^2 - 4\left(\frac{1}{4}\right)^2 = 4 \\
y^2 - \frac{1}{4} = 4 \\
y^2 = \frac{17}{4} \Rightarrow y = \pm \frac{\sqrt{17}}{2}
\]

Thus, we have the following solutions for this system: \((0, -2)\), \((0, 2)\), \(\left(\frac{1}{4}, \frac{\sqrt{17}}{2}\right)\), and \(\left(\frac{1}{4}, -\frac{\sqrt{17}}{2}\right)\).
6. 

\[ x^2 + 4y^2 = 36 \]
\[ 2y + x + 6 = 0 \]

Solving the second equation for \( x \) and then substituing into the first equation yields:

\[ x = -2y - 6 \]
\[ (-2y - 6)^2 + 4y^2 = 36 \]
\[ 4y^2 + 24y + 36 + 4y^2 = 36 \]
\[ 8y^2 + 24y + 36 = 36 \]
\[ 8y^2 + 24y = 0 \]
\[ 8y(y + 3) = 0 \Rightarrow y = 0 \text{ or } y = -3 \]

Substituting back into the second equation yields:

\( y = 0: \)

\[ 2(0) + x + 6 = 0 \]
\[ x = -6 \]

\( y = -3: \)

\[ 2(-3) + x + 6 = 0 \]
\[ x = 0 \]

Thus, our solutions to the system are \((-6, 0)\) and \((0, -3)\).