Chapter 8A - Angles and Circles

Many applications of calculus use trigonometry, which is the study of angles and functions of angles and their application to circles, polygons, and science. We start with the definition of angles and their measures.

Angles
Roughly, an angle is the space between two rays or line segments with a common endpoint. The rays or line segments are called the sides and the common endpoint is called the vertex. More precisely, if a ray or line segment rotates about an endpoint from some initial position, called the initial side, to some final position, called the final side, then the angle between the sides is the space swept out. Within a plane, we say the angle is positive if the rotation is counterclockwise, and is negative if the rotation is clockwise. An angle is in standard position if the vertex is at the origin and the initial side is along the positive x-axis.

Measures of Angle
The size of an angle may be measured in revolutions (rev), in degrees (°) or in radians (rad).

An angle is called a full rotation if the ray rotates from the initial side all the way around so that the final side coincides with the initial side.
A full rotation is measured as $1\text{rev} = 360° = 2\pi\text{rad}$.

An angle is called a half rotation or a straight angle if the ray rotates from the initial side to a final side which is directly opposite to the initial side.
A half rotation is measured as $\frac{1}{2}\text{rev} = 180° = \pi\text{rad}$.

An angle is called a quarter rotation or a right angle if the ray rotates from the initial side to a final side which is perpendicular to the initial side.
A quarter rotation is measured as $\frac{1}{4}\text{rev} = 90° = \frac{\pi}{2}\text{rad}$.

An angle is called a null rotation if the ray never rotates so that the final side coincides with the initial side.
A null rotation is measured as $0\text{rev} = 0° = 0\text{rad}$. 
Several other important angles are:

$$\frac{3}{4}\text{ rev} = 270^\circ = \frac{3\pi}{2}\text{ rad}$$
$$\frac{1}{6}\text{ rev} = 60^\circ = \frac{\pi}{3}\text{ rad}$$
$$\frac{1}{8}\text{ rev} = 45^\circ = \frac{\pi}{4}\text{ rad}$$
$$\frac{1}{12}\text{ rev} = 30^\circ = \frac{\pi}{6}\text{ rad}$$

**Example 1:** Find the angular measure of one time zone on the surface of the earth.

Solution: The earth rotates once a day or by 1 rev in 24 hours. Thus a 1 hour time zone has an angular measure of

$$\frac{1}{24}\text{ rev} = 15^\circ = \frac{\pi}{12}\text{ rad}.$$ 

An angle will be bigger than a full rotation if the ray rotates from the initial side all the way around and past the initial side again. There is no limit to the size of an angle either positive or negative. Two angles with the same initial and final sides are called **coterminal** and their measures must differ by an integral multiple of

$$1\text{ rev} = 360^\circ = 2\pi\text{ rad}. $$

Below are some pictures of various angles. In each figure, the initial sides of the angles coincide.

Below are some pictures of various angles in standard position.
You must learn to identify angles in standard position in radians and degrees. These are the angular measurements most often used in trigonometry and calculus, with radians predominating in calculus. In the following either degrees or radians will be randomly used.

**Definition:** An angle is called **acute** if its measure is between 0° and 90°, and an angle is called **obtuse** if its measure is between \( \frac{\pi}{2} \) rad and \( \pi \) rad.

**Definition:** The **complement** of an angle whose measure is \( \theta \) is any angle whose measure is \( \frac{\pi}{2} - \theta \) (in radians), and the **supplement** of an angle \( \psi \) is an angle whose measure is \( 180 - \psi \) (in degrees).

### The Parts of a Circle

**Definition:** The circle with **center** \( P \) and **radius** \( r \) is the set of all points \( X \) in the plane whose distance from \( P \) is \( r \). If the center is \( P = (a,b) \) and the general point on the circle is \( X = (x,y) \) then the equation of the circle is

\[
(x - a)^2 + (y - b)^2 = r^2
\]

We often take the center to be the origin \( O = (0,0) \). Then its equation is

\[
x^2 + y^2 = r^2
\]

The region inside of a circle is called a **disk**.

**Definition:** A **radial line** (or a **radius**) is any straight line segment from the center of the circle to a point on the circle. The word radius can refer to either a radial line or its length \( r \).

**Definition:** A **diameter line** (or a **diameter**) is any line segment between two points on the circle which passes through the center. The word diameter can refer to either a diameter line or its length \( d = 2r \).

**Definition:** The **circumference** of the circle is the distance around the circle. By the definition of \( \pi \) the circumference is \( C = \pi d = 2\pi r \).

**Definition:** The **area** of the circle (actually of the disk) is \( A = \pi r^2 \).

**Definition:** An **arc** is any piece of the circle between two points on the circle. A **chord** is any line segment between two points on the circle. A **sector** is any piece of the disk between two radial lines.
Two radial lines are shown in black.
An arc is shown in blue.
A chord is shown in green.
A sector is shaded in cyan.

**Definition:** An angle whose vertex is at the center of a circle is called a central angle. The sides of a central angle are radial lines which intersect the circle at two points. The arc between these two points, the chord between these two points and the sector between the two sides of the angle are called the arc, chord and sector subtended by the central angle. We use the following notations:

- \( C \) = Circumference of the circle
- \( L \) = Length of an arc of the circle
- \( A_\bigcirc \) = Area of the whole circle
- \( A_\preceq \) = Area of a sector of the circle

In the previous figure the arc, chord and sector are subtended by the central angle \( \theta \).

The fraction of the circle or disk subtended by a central angle \( \theta \) is \( \frac{\theta}{360^\circ} \) for \( \theta \) in degrees, or \( \frac{\theta}{2\pi \text{rad}} \) for \( \theta \) in radians. So the length of an arc (the arc length) is this fraction of the circumference and the area of a sector (the sector area) is this fraction of the area of disk. Hence the arc length is

\[
L = \begin{cases} 
\frac{\theta}{360^\circ} C = \frac{2\pi \theta}{360} & \text{for } \theta \text{ in degrees} \\
\frac{\theta}{2\pi \text{rad}} C = \frac{2\pi \theta}{2\pi} = r\theta & \text{for } \theta \text{ in radians}
\end{cases}
\]

and the sector area is

\[
A_\preceq = \begin{cases} 
\frac{\theta}{360^\circ} A_\bigcirc = \frac{\pi r^2 \theta}{360} & \text{for } \theta \text{ in degrees} \\
\frac{\theta}{2\pi \text{rad}} A_\bigcirc = \frac{\theta}{2\pi} \pi r^2 = \frac{\theta}{2} r^2 & \text{for } \theta \text{ in radians}
\end{cases}
\]

Notice that the formulas for arc length and the area of a sector are much simpler when written in terms of radians than in terms of degrees. The formulas for angular velocity and angular acceleration used in physics are also simpler using radians. This is one of the reasons for using radians rather than degrees, and when you learn about derivatives of the trig functions in calculus, you will appreciate radians even more.

**Example 2:** In the figure above, suppose the radius is \( r = 6\text{cm} \) and the central angle is \( \theta = 105^\circ = \frac{7\pi}{12} \text{ rad} \). Find the circumference and area of the circle. Find the fraction subtended as well as the arc length and sector area of the arc and sector subtended by the central angle \( \theta \).

Solution: The circumference is \( C = 2\pi(6\text{cm}) = 12\pi \text{cm} \). The area is \( A = \pi(6\text{cm})^2 = 36\pi \text{cm}^2 \).

Using degrees: The fraction subtended is \( \frac{105^\circ}{360^\circ} = \frac{7}{24} \). The arc length is \( L = \frac{2\pi (6\text{cm})105}{360} = \frac{7\pi}{2} \text{cm} \).

And the sector area is \( A_\preceq = \frac{\pi (6\text{cm})^2105}{360} = \frac{21\pi}{2} \text{cm}^2 \).

Using radians: The fraction subtended is \( \frac{7\pi}{12 \cdot 2\text{rad}} = \frac{7}{24} \). The arc length is \( L = (6\text{cm}) \frac{7\pi}{12} = \frac{7\pi}{2} \text{cm} \). And the sector area is \( A_\preceq = \frac{1}{2} (6\text{cm})^2 \frac{7\pi}{12} = \frac{21\pi}{2} \text{cm}^2 \). Notice how much simpler the computations are in terms of radians.
**Secant line, Tangent line**

**Definition:** A secant line is a line which intersects the circle twice. The part inside the circle is a chord.

**Definition:** A tangent line is a line which intersects the circle at exactly one point called the point of tangency.

A secant line is shown in purple.
Its chord is shown in green.
A tangent line is shown in blue.
Its point of tangency is in yellow.
Exercises for Chapter 8A - Angles and Circles

1. If a point on a disk rotates about the disk’s axis 5 times, how many degrees will it rotate, how many radians?

2. An angle of $5^\circ$ equals how many radians?

3. An angle of 22 radians is how many degrees?

4. If a disk is rotating at 45 rotations per minute, how many radians will the disk rotate in 2 minutes?

5. A circular race track is being built. The outside diameter of the track is 200 ft. and the inside diameter is 180 feet. If two runners run around the track once with one of them running on the outside edge of the track, while the other is running on the inside edge, how much further will the outside runner have to run?

6. Referring to the runners in the preceding problem, if the runners both run at a speed of 6 miles per hour, how much time will elapse between the first runner finishing and the second runner finishing. Assume they start at the same time.

7. A child runs half way around a circle. If the radius of the circle is 20 meters, how far did the child run?

8. A dog is tied to a 10 centimeter diameter pole, with a 3 meter long leash. If the dog runs around the pole until his collar is tight to the pole, how many degrees has the dog run around the pole?

9. The apparent diameter of the planet Venus is 12,104 kilometers. Assuming that Venus is a sphere, what is its circumference?

10. The average distance of the earth to the sun is approximately 150 million kilometers. Assuming it takes the earth 365 days to orbit the sun, that there are 24 hours in a day, that its orbit is a circle, and that its speed is constant, what is the speed of the earth in kilometers per hour about the sun?

11. If a circle has a 20 meter diameter, find the exact arc length and exact area of the sector subtended by a central angle of 93 degrees.

12. If a circle has a radius of 50 centimeters, find the exact arc length and exact area of the sector subtended by a central angle of $\frac{7\pi}{12}$ radians.
Answers to Exercises for Chapter 8A - Angles and Circles

1. Since one rotation is equivalent to 360 degrees or $2\pi$ radians, 5 rotations equals $5(360) = 1800$ degrees or $10\pi$ radians.

2. If $x$ is the measure of the angle in radians, then we have the following ratio:

$$\frac{x}{2\pi} = \frac{5}{360}$$

Thus, $x = \frac{10\pi}{360} = \frac{x}{36}$.

3. If $x$ is the measure of the angle in degrees, then

$$\frac{x}{360} = \frac{22}{2\pi}$$

Thus, $x = \frac{22(360)}{2\pi} = \frac{7920}{2\pi} = \frac{3960}{\pi}$.

4. In two minutes the disk will have rotated 90 times, or $90(2\pi) = 180\pi$ radians.

5. The outer circumference is $200\pi$ feet while the inner circumference is $180\pi$ feet. The difference between these two distances, $20\pi$ feet, is how much further the first runner has to travel.

6. Since both runners are running at the same speed, we just need to determine how much time it will take to run $20\pi$ feet at a speed of 6 miles per hour. First we convert the speed from miles per hour into feet per second:

$$6 \text{ miles/hour} = 6(5280)\left(\frac{1}{3600}\right) = 8.8 \text{ ft/sec}$$

Since, see preceding problem, one of the runners has to run an additional $20\pi$ feet, it will take him

$$\frac{20\pi}{8.8} \approx 7.14 \text{ seconds}$$

longer to run once around the track.

7. distance = $r\theta = 20\pi$ meters.

8. Each turn of the dog around the pole uses $10\pi$ cm or 0.1$\pi$ meters of the dogs leash. Thus, number of revolutions = $\frac{3}{0.1\pi} \approx 9.5493$ revolutions or $(9.5493)(360) \approx 3437.7468$ degrees.

9. It’s circumference = $\pi (12104) \approx 38,026$ kilometers.

10. The distance the earth travels in one orbit about the sun is approximately $2\pi (150 \times 10^6) = 300\pi \times 10^6$ kilometers. The amount of time it takes for one orbit is $365 \times 24 = 8760$ hours. Thus, the speed of the earth in its orbit is
\[
\frac{300 \times 10^6}{8760} \approx 0.107589 \times 10^3 \approx 107589 \text{ kilometers/hour}.
\]

11. Since the diameter is 20 meters, here we have \( r = 10 \) and we are given that \( \theta = 93^\circ \). Thus, the arc length is

\[
\left( \frac{2\pi r}{360} \right) \theta = \left( \frac{2\pi (10)}{360} \right) (93) = \frac{31\pi}{6} \text{ meters},
\]

and the area of the sector is:

\[
\left( \frac{\pi r^2}{360} \right) \theta = \left( \frac{\pi (10)^2}{360} \right) (93) = \frac{155\pi}{6} \text{ square meters}
\]

12. Here \( r = 50 \) and \( \theta = \frac{7\pi}{12} \) radians. Thus, the arc length is

\[
\left( \frac{2\pi r}{2\pi} \right) \theta = r\theta = (50)\left( \frac{7\pi}{12} \right) = \frac{175\pi}{6} \text{ centimeters}
\]

and the area of the sector is:

\[
\left( \frac{\pi r^2}{2\pi} \right) \theta = \left( \frac{r^2}{2} \right) \theta = \left( \frac{(50)^2}{2} \right) \left( \frac{7\pi}{12} \right) = \frac{4375\pi}{6} \text{ square centimeters}
\]
Chapter 8B - Trig Functions

Triangle Definition
We are now ready to introduce the basic trig functions. There are two consistent ways to define them. The first uses a right triangle and is valid only for angles strictly between 0° and 90°, while the second uses a circle and is valid for all angles.

Consider a right triangle with one angle θ.

The sides are:
- the leg adjacent to θ: adj
- the leg opposite to θ: opp
- and the hypotenuse: hyp

In terms of these sides, the trig functions, sine, cosine, tangent, cotangent, secant and cosecant, of the angle θ are given by:

\[
\begin{align*}
\sin \theta &= \frac{\text{opp}}{\text{hyp}} & \tan \theta &= \frac{\text{opp}}{\text{adj}} & \sec \theta &= \frac{\text{hyp}}{\text{adj}} \\
\cos \theta &= \frac{\text{adj}}{\text{hyp}} & \cot \theta &= \frac{\text{adj}}{\text{opp}} & \csc \theta &= \frac{\text{hyp}}{\text{opp}}
\end{align*}
\]

In terms of \( \sin \theta \) and \( \cos \theta \), the other trig functions are:

\[
\begin{align*}
\tan \theta &= \frac{\sin \theta}{\cos \theta} & \cot \theta &= \frac{\cos \theta}{\sin \theta} = \frac{1}{\tan \theta} & \sec \theta &= \frac{1}{\cos \theta} & \csc \theta &= \frac{1}{\sin \theta}
\end{align*}
\]

By using similar triangles it is easy to see that these definitions do not depend upon any particular right triangle. Once again we note that this definition of the trig functions is limited to angles θ which are acute, that is, less than a right angle and greater than 0°. Below we give a definition which works for any angle and show that if we have an acute angle then both definitions give the same values.

Circle Definition

Consider a circle of radius \( r \) centered at the origin in which a radial line has been drawn at an angle \( \theta \) measured counterclockwise from the positive x-axis. The radial line intersects the circle at a point \( (x,y) \).

Note: \( x \) and/or \( y \) may be negative.

In terms of \( x, y \) and \( r \), the trig functions, sine, cosine, tangent, cotangent, secant and cosecant, of the angle \( \theta \) are given by: 
\[ \sin \theta = \frac{y}{r}, \quad \tan \theta = \frac{y}{x}, \quad x \neq 0 \]
\[ \cos \theta = \frac{x}{r}, \quad \cot \theta = \frac{x}{y}, \quad y \neq 0 \]

Note that \( \sin \theta \) and \( \cos \theta \) are defined for all values of \( \theta \), but this is not true for the other 4 trig functions. \( \tan \theta \) and \( \sec \theta \) are not defined when \( x = 0 \), that is, when \( \theta \) is an odd integer multiple of \( \pi/2 \). \( \cot \theta \) and \( \csc \theta \) are not defined when \( y = 0 \). That is, they are not defined when \( \theta \) is an integer multiple of \( \pi \).

Notice that this says the coordinates of a point on the circle are \((x, y) = (r \cos \theta, r \sin \theta)\).

In terms of \( \sin \theta \) and \( \cos \theta \), the other trig functions are
\[ \tan \theta = \frac{\sin \theta}{\cos \theta}, \quad \cot \theta = \frac{\cos \theta}{\sin \theta} = \frac{1}{\tan \theta}, \quad \sec \theta = \frac{1}{\cos \theta}, \quad \csc \theta = \frac{1}{\sin \theta} \]

Using similar triangles it is clear that the definition of these trig functions does not depend upon the radius of the circle. That is, no matter what the circle’s radius is, the values of these functions do not depend upon the value of \( r \).

Consider a small red circle of radius \( r \) and a big magenta circle of radius \( R \).
The radial line at an angle \( \theta \) intersects the red circle at \((x,y)\) and intersects the magenta circle at a point \((X,Y)\).

Since the blue and green triangles are similar, the ratios of corresponding sides are equal. Hence:
\[ \frac{y}{r} = \frac{Y}{R}, \quad \frac{x}{r} = \frac{X}{R} \]
But these equations say the sine and cosine are the same for both triangles:
\[ \sin \theta = \frac{y}{r} = \frac{Y}{R}, \quad \cos \theta = \frac{x}{r} = \frac{X}{R} \]
From the equality of the sine and cosine functions, the equalities of the rest of the trig functions follow.

Notice that this circle definition of the trig functions works for any angle \( \theta \) not just acute angles. However, to make sense, the definitions in terms of circles and triangles must agree for acute angles and they do, as one sees by comparing the definitions when the angle is acute.

**Trig Functions on a Unit Circle**

It is often convenient to use a unit circle. Then the trig functions are given by
\[ \sin \theta = y, \quad \tan \theta = \frac{y}{x}, \quad \sec \theta = \frac{1}{x} \]
\[ \cos \theta = x, \quad \cot \theta = \frac{x}{y}, \quad \csc \theta = \frac{1}{y} \]
and the coordinates of a point on the circle are \((x, y) = (\cos \theta, \sin \theta)\).
Special Angles

You need to know (or be able to figure out) the values of the trig functions for specific angles. That is those angles which are multiples of $30^\circ = \frac{\pi}{6}$ rad or $45^\circ = \frac{\pi}{4}$ rad.

Below is a table of the values of the trig functions for some special angles. You should absolutely not memorize this table. Rather, in each case, you should figure out the values of the trig functions using the circle definition and your knowledge of $30^\circ$ – $60^\circ$ – $90^\circ$ and $45^\circ$ right triangles.

<table>
<thead>
<tr>
<th>θ Deg</th>
<th>θ Rad</th>
<th>sin θ</th>
<th>cos θ</th>
<th>tan θ</th>
<th>cot θ</th>
<th>sec θ</th>
<th>csc θ</th>
</tr>
</thead>
<tbody>
<tr>
<td>0°</td>
<td>0 rad</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>±∞</td>
<td>1</td>
<td>±∞</td>
</tr>
<tr>
<td>30°</td>
<td>$\frac{\pi}{6}$ rad</td>
<td>$\frac{1}{2}$</td>
<td>$\frac{\sqrt{3}}{2}$</td>
<td>$\frac{1}{\sqrt{3}}$</td>
<td>$\sqrt{3}$</td>
<td>$\frac{2}{\sqrt{3}}$</td>
<td>2</td>
</tr>
<tr>
<td>45°</td>
<td>$\frac{\pi}{4}$ rad</td>
<td>$\frac{1}{\sqrt{2}}$</td>
<td>$\frac{1}{\sqrt{2}}$</td>
<td>1</td>
<td>1</td>
<td>$\sqrt{2}$</td>
<td>$\sqrt{2}$</td>
</tr>
<tr>
<td>60°</td>
<td>$\frac{\pi}{3}$ rad</td>
<td>$\frac{\sqrt{3}}{2}$</td>
<td>$\frac{1}{2}$</td>
<td>$\sqrt{3}$</td>
<td>$\frac{1}{\sqrt{3}}$</td>
<td>2</td>
<td>$\frac{2}{\sqrt{3}}$</td>
</tr>
<tr>
<td>90°</td>
<td>$\frac{\pi}{2}$ rad</td>
<td>1</td>
<td>0</td>
<td>±∞</td>
<td>0</td>
<td>±∞</td>
<td>1</td>
</tr>
</tbody>
</table>

$30^\circ$ – $60^\circ$ – $90^\circ$ triangle

To determine the values of the trig functions for either $30^\circ$ or $60^\circ$, draw an equilateral triangle with each side of length 2. Remember that an equilateral triangle is also equiangular, which means each of its angles is $1/3$ of $180^\circ$ or $60^\circ$. Now draw a perpendicular from any vertex to its opposite side. This gives you the triangle show below, and the Pythagorean theorem tells us that the vertical side must have length $\sqrt{2^2 - 1^2} = \sqrt{3}$.

Notice that the shortest side 1 is opposite the smallest angle $30^\circ = \frac{\pi}{6}$ rad, the middle length side $\sqrt{3} \approx 1.7$ is opposite the middle angle $60^\circ = \frac{\pi}{3}$ rad and the longest side 2 is opposite the biggest angle $90^\circ = \frac{\pi}{2}$ rad.

From this triangle we get

\[
\begin{align*}
\sin 60^\circ &= \frac{\sqrt{3}}{2}, \\ 
\cos 60^\circ &= \frac{1}{2}, \\ 
\tan 60^\circ &= \sqrt{3}, \\ 
\csc 60^\circ &= \frac{2}{\sqrt{3}}, \\ 
\sec 60^\circ &= 2, \\ 
\cot 60^\circ &= \frac{1}{\sqrt{3}}.
\end{align*}
\]

\[
\begin{align*}
\sin 30^\circ &= \frac{1}{2}, \\ 
\cos 30^\circ &= \frac{\sqrt{3}}{2}, \\ 
\tan 30^\circ &= \frac{1}{\sqrt{3}}, \\ 
\csc 30^\circ &= 2, \\ 
\sec 30^\circ &= \frac{2}{\sqrt{3}}, \\ 
\cot 30^\circ &= \sqrt{3}.
\end{align*}
\]
A $45^\circ$ right triangle must be an isosceles right triangle, which means the legs of such a triangle have the same length. In the picture to the left they are of length 1.

Computing the values of the trig functions we get

$$\sin 45^\circ = \frac{1}{\sqrt{2}}, \cos 45^\circ = \frac{1}{\sqrt{2}}, \tan 45^\circ = 1, \csc 45^\circ = \sqrt{2}, \sec 45^\circ = \sqrt{2}, \cot 45^\circ = 1.$$ 

**Example 1:** Find the trig functions at $\frac{5\pi}{3}$ rad.

Solution: Draw a circle of radius 2, draw the radial line at $\frac{5\pi}{3}$ rad and drop (raise) a perpendicular to the $x$-axis. The hypotenuse of the resulting right triangle is 2 since this is the radius of the circle. Moreover $2\pi - \frac{5\pi}{3} = \frac{\pi}{3} = 60^\circ$. Since this is the IVth quadrant, the $x$ coordinate is positive and the $y$ coordinate is negative and we label the two legs of the triangle with $x = 1$ and $y = -\sqrt{3}$.

Then the trig functions are

$$\sin \frac{5\pi}{3} = \frac{y}{r} = -\frac{\sqrt{3}}{2} = -\frac{\sqrt{3}}{2}, \cos \frac{5\pi}{3} = \frac{x}{r} = \frac{1}{2}, \tan \frac{5\pi}{3} = \frac{y}{x} = -\sqrt{3}, \cot \frac{5\pi}{3} = \frac{x}{y} = \frac{1}{-\sqrt{3}} = -\frac{1}{\sqrt{3}}, \sec \frac{5\pi}{3} = \frac{r}{x} = 2, \csc \frac{5\pi}{3} = \frac{r}{y} = \frac{2}{-\sqrt{3}} = -\frac{2}{\sqrt{3}}.$$
Example 2:  Find the trig functions at $\frac{3\pi}{4}$ rad.

Solution:  We draw a circle of radius $\sqrt{2}$, draw the radial line at $\frac{3\pi}{4}$ rad and drop a perpendicular to the x-axis. The hypotenuse of the resulting right triangle is $\sqrt{2}$ since this is the radius of the circle, and we note that this must be an isosceles right triangle since $\pi - \frac{3\pi}{4} = \frac{\pi}{4} = 45^\circ$. Since we are in the II$^{nd}$ quadrant, we label the two legs of the triangle with $x = -1$ and $y = 1$.

Then the trig functions are

\[
\begin{align*}
\sin \frac{3\pi}{4} &= \frac{y}{r} = \frac{1}{\sqrt{2}} \\
\cos \frac{3\pi}{4} &= \frac{x}{r} = \frac{-1}{\sqrt{2}} \\
\tan \frac{3\pi}{4} &= \frac{y}{x} = \frac{-1}{-1} = 1 \\
\cot \frac{3\pi}{4} &= \frac{x}{y} = \frac{-1}{1} = -1 \\
\sec \frac{3\pi}{4} &= \frac{r}{x} = \frac{\sqrt{2}}{-1} = -\sqrt{2} \\
\csc \frac{3\pi}{4} &= \frac{r}{y} = \frac{\sqrt{2}}{-1} = -\sqrt{2}
\end{align*}
\]

Example 3:  Find the trig functions at $\frac{3\pi}{2}$ rad.

Solution:  We draw a circle of radius 1 and draw the radial line at $\frac{3\pi}{2}$ rad. The coordinates at the endpoint of the radial line are $x = 0$ and $y = -1$.

So the values of the trig functions are:

\[
\begin{align*}
\sin \frac{3\pi}{2} &= \frac{y}{r} = \frac{-1}{1} = -1 \\
\cos \frac{3\pi}{2} &= \frac{x}{r} = \frac{0}{1} = 0 \\
\tan \frac{3\pi}{2} &= \frac{y}{x} = \frac{-1}{0} = \pm \infty \\
\cot \frac{3\pi}{2} &= \frac{x}{y} = \frac{0}{-1} = 0 \\
\sec \frac{3\pi}{2} &= \frac{r}{x} = \frac{1}{0} = \pm \infty \\
\csc \frac{3\pi}{2} &= \frac{r}{y} = \frac{1}{-1} = -1
\end{align*}
\]
Exercises for Chapter 8B - Trig Functions

1. If \( \sin \theta = 0.55 \) and the angle is acute what are the values of \( \cos \theta \) and \( \tan \theta \)?
2. If \( \sin \theta = 0.55 \) and the angle is obtuse what are the values of \( \cos \theta \) and \( \tan \theta \)?
3. If \( \cos \theta = 0.25 \) and the angle is acute, what are the values of \( \sin \theta \) and \( \tan \theta \)?
4. If \( \cos \theta = 0.25 \) and the angle is not acute, what are the values of \( \sin \theta \) and \( \tan \theta \)?
5. If \( \tan \theta = 0.5 \) and \( \theta \) is acute what are the values of \( \sin \theta \) and \( \cos \theta \)?
6. If \( \tan \theta = 0.5 \) and \( \theta \) is not acute what are the values of \( \sin \theta \) and \( \cos \theta \)?
7. Using the Pythagorean theorem deduce that \( \sin^2 \theta + \cos^2 \theta = 1 \) for all values of \( \theta \).
8. If \( \cos x = \frac{20}{21} \) and \( x \) is in Quadrant IV, exactly find the value of \( \sin x \), \( \tan x \), \( \csc x \), \( \sec x \), and \( \cot x \).
9. Give exact answers to each of these:
   a. \( \cos \frac{\pi}{3} \)
   b. \( \tan \frac{5\pi}{6} \)
   c. \( \csc \frac{\pi}{4} \)
   d. \( \sin \frac{13\pi}{6} \)
   e. \( \sin \frac{9\pi}{4} \)
   f. \( \tan \frac{\pi}{2} \)
10. Find the exact value of all the trig functions of the angle \( \frac{7\pi}{6} \).
11. Find the exact value of all the trig functions of the angle \( \frac{3\pi}{2} \).
Answers to Exercises for Chapter 8B - Trig Functions

1. Since the angle \( \theta \) is acute, the value of its cosine must be positive. If \((x, y)\) is that point on the unit circle, which corresponds to the angle \( \theta \), then we have

\[
\sin \theta = \frac{y}{1} = y = 0.55.
\]

Moreover we have \( x = \sqrt{1 - y^2} = \sqrt{0.695} \approx 0.835 \). Thus, \( \cos \theta \approx 0.835 \) and \( \tan \theta = \frac{y}{x} \approx \frac{0.55}{0.835} \approx 0.658 \)

2. Since the value of the sine function is positive, if \((x, y)\) is that point on the unit circle, which corresponds to the angle \( \theta \), then \( y > 0 \), which means that the point \((x, y)\) must lie in the second quadrant. That is, \( x < 0 \). Thus, using the \( x \) value from the preceding problem, we have

\[
\cos \theta = \frac{x}{1} = x \approx -0.835 \quad \text{and} \quad \tan \theta = \frac{y}{x} \approx \frac{0.55}{0.835} \approx -0.658
\]

3. If \((x, y)\) is that point on the unit circle, which corresponds to the angle \( \theta \), then \( x = 0.25 \) and \( y = \sqrt{1 - 0.0625} = \sqrt{0.9375} \approx 0.968 \). Thus,

\[
\sin \theta = \frac{y}{1} \approx 0.968 \quad \text{and} \quad \tan \theta = \frac{y}{x} \approx \frac{0.968}{0.25} \approx 3.873
\]

4. Since the angle is not acute and \( \cos \theta \) is positive then the point \((x, y)\) must lie in the fourth quadrant. That is \( y < 0 \) with \( x > 0 \). Using the \( y \) value computed in the previous problem, we have:

\[
\sin \theta = \frac{y}{1} \approx -0.968 \quad \text{and} \quad \tan \theta = \frac{y}{x} \approx \frac{-0.968}{0.25} \approx -3.873
\]

5. Since \( \theta \) is acute, the point \((x, y)\) must lie in the first quadrant. That is both \( x \) and \( y \) are positive, and we know that \( \tan \theta = \frac{y}{x} = 0.5 \) and \( x^2 + y^2 = 1 \). Thus, \( y = x/2 \), which implies that \( x^2 + x^2/4 = 1 \). Thus, \( x = 2/\sqrt{5} \), and \( y = 1/\sqrt{5} \). Thus,

\[
\sin \theta = y = \frac{1}{\sqrt{5}} \approx 0.447 \quad \text{and} \quad \cos \theta = x = \frac{2}{\sqrt{5}} \approx 0.894
\]

6. Since \( \tan \theta \) is positive, and if \((x, y)\) is that point on the unit circle determined by the angle \( \theta \), then \( x \) and \( y \) must have the same sign. Since \( \theta \) is not acute, this means that both \( x \) and \( y \) are negative. Solving the same system of equations as in the preceding problem, we now have \( x = -2/\sqrt{5} \) and \( y = -1/\sqrt{5} \). Thus,

\[
\sin \theta = y = \frac{-1}{\sqrt{5}} \approx -0.447 \quad \text{and} \quad \cos \theta = x = \frac{-2}{\sqrt{5}} \approx -0.894
\]
7. Let \((x, y)\) be that point on the unit circle determined by the angle \(\theta\), then, as \(|x|\) and \(|y|\) are the lengths of the legs of a right triangle whose hypotenuse has length 1, we have
\[
\cos^2 \theta + \sin^2 \theta = x^2 + y^2 = 1.
\]

8. We recall that if we are looking at a right triangle,
\[
\cos \theta = \frac{\text{adj}}{\text{hyp}} = \frac{20}{21}
\]
Thus, in our right triangle, the length of the side adjacent to the angle is 20, the length of the hypotenuse is 21, and we can use the pythagorean theorem to find the length of the side opposite the angle:
\[
\text{opp} = \sqrt{21^2 - 20^2} = \sqrt{41}
\]
But since we are in Quadrant IV, we know that \(\text{opp} = -\sqrt{41}\). Thus, we can use this triangle to find the values of our other trig functions:
\[
\sin \theta = \frac{-\sqrt{41}}{21}, \quad \tan \theta = \frac{-\sqrt{41}}{20}, \quad \csc \theta = \frac{1}{\sin \theta} = \frac{-21}{\sqrt{41}}, \quad \sec \theta = \frac{1}{\cos \theta} = \frac{21}{20}, \quad \cot \theta = \frac{1}{\tan \theta} = \frac{-20}{\sqrt{41}}
\]

9. 
   a. \(\cos \frac{\pi}{3} = \frac{1}{2}\)
   b. \(\tan \frac{5\pi}{6} = \tan(\pi - \frac{\pi}{6}) = -\tan \frac{\pi}{6} = -\frac{1}{\sqrt{3}}\)
   c. \(\csc \frac{5\pi}{3} = \csc(2\pi - \frac{\pi}{3}) = -\csc \frac{\pi}{3} = -\frac{1}{\sin \frac{\pi}{3}} = -\frac{2}{\sqrt{3}}\)
   d. \(\sin \frac{13\pi}{3} = \sin(4\pi + \frac{\pi}{3}) = \sin \frac{\pi}{3} = \frac{\sqrt{3}}{2}\)
   e. \(\sin \frac{9\pi}{4} = \sin(2\pi + \frac{\pi}{4}) = \sin \frac{\pi}{4} = \frac{\sqrt{2}}{2}\)
   f. \(\tan \frac{\pi}{2} = \frac{\sin(\pi/2)}{\cos(\pi/2)} = \frac{1}{0} = \pm\infty\)

10. We see here that \(\frac{7\pi}{6} = \pi + \frac{\pi}{6}\). Thus,
   - \(\cos \frac{7\pi}{6} = -\cos \frac{\pi}{6} = -\frac{\sqrt{3}}{2}\)
   - \(\sin \frac{7\pi}{6} = -\sin \frac{\pi}{6} = -\frac{1}{2}\)
   - \(\tan \frac{7\pi}{6} = \tan \frac{\pi}{6} = \frac{1}{\sqrt{3}}\)
8B: Pre-Calculus - Chapter 8B

- $\sec \frac{7\pi}{6} = \frac{1}{\cos \frac{\pi}{4}} = \frac{2}{-\sqrt{3}}$
- $\csc \frac{7\pi}{6} = \frac{1}{\sin \frac{\pi}{4}} = -2$
- $\cot \frac{7\pi}{6} = \frac{1}{\tan \frac{\pi}{4}} = \sqrt{3}$

11. We see here that $\frac{3\pi}{2} = \pi + \frac{\pi}{2}$. Thus,
- $\cos \frac{3\pi}{2} = \cos \frac{\pi}{2} = 0$
- $\sin \frac{3\pi}{2} = -\sin \frac{\pi}{2} = -1$
- $\tan \frac{3\pi}{2} = \tan \frac{\pi}{2} = \pm \infty$
- $\sec \frac{3\pi}{2} = \frac{1}{\cos \frac{\pi}{2}} = \frac{1}{0} = \pm \infty$
- $\csc \frac{3\pi}{2} = \frac{1}{\sin \frac{\pi}{2}} = -1$
- $\cot \frac{3\pi}{2} = \frac{\cos \frac{\pi}{2}}{\sin \frac{\pi}{2}} = 0$
Chapter 8C - Graphs of the Trig Functions

If we compute and plot the values of the six trig functions, for those values of $\theta$ in radians for which the computations are easy, we get the following six plots. Remember that $\theta$ is in radians.

If we add more points, or use a graphing calculator, we get the following six plots.
General Graphs

There are three ways we need to clarify our graphs of the trig functions.

- First, although calculators will compute trig functions using either degrees or radians, mathematicians almost always use radians when working with the trig functions. The reasons for this will become clear after you learn about the derivatives of the trig functions.

- Second, in the circle definition of the trig functions, there is no reason that the angle is restricted to 1 revolution. So the trig functions are also defined for angles less than 0 rad or greater than 2π rad ≈ 6.28 rad. Since the coordinates of the point on the circle repeat with each revolution, the graphs of sine, cosine, cosecant, and secant repeat every 2π rad. We denote this by saying that these four trig functions are periodic with period 2π.

- Third, mathematicians like to use the letter x for the independent variable and the letter y for the dependent variable of a function. So we often write y = sin x, or y = cos x, or y = tan x, or etc. Here x is the angle (in radians) and y is the value of the particular trig function.

**Caution:** The x and y in a formula such as y = sin x have absolutely nothing to do with the x and y in the circle definition of the trig functions.

With these modifications, the graphs of the six trig functions become:

\[ y = \sin x \quad \text{and} \quad y = \cos x \]

Notice that sin and cos have period 2π, while tan and cot (below) have period π. In the next four trig functions note the vertical asymptotes. Those for \( \tan \theta \) and \( \sec \theta \) are at odd multiples of \( \frac{\pi}{2} \); those for \( \cot \theta \) and \( \csc \theta \) are at multiples of \( \pi \).
Notice that sin, cos, sec and csc all have period $2\pi$, while tan and cot have period $\pi$. Also recall that tan and sec have vertical asymptotes at the odd multiples of $\frac{\pi}{2}$ while cot and csc have vertical asymptotes at the multiples of $\pi$.

**Shifting and Rescaling Graphs**

The trig functions may also be shifted and/or rescaled. We demonstrate using the sin function. We start with the graph of $y = \sin x$:

![Graph of $y = \sin x$]

**Shifting**

As an example, if the sine function is shifted to the right by $\frac{\pi}{4} \approx 0.785$ and up by $\frac{1}{2}$ then the shifted graph is

![Shifted sine graph]

which has the equation $y = \sin \left(x - \frac{\pi}{4}\right) + \frac{1}{2}$.

In general, if the sin function is shifted to the right by $x_0$ and up by $y_0$ then the shifted graph is

![General shifted sine graph]

which has the equation $y = \sin(x - x_0) + y_0$.

If the horizontal shift $x_0$ is positive, the shift is to the right and we say that the graph of $y = \sin(x - x_0)$ leads $y = \sin x$ by $x_0$. If $x_0$ is negative, the shift is to the left and we say that the graph of $y = \sin(x - x_0)$ lags behind $y = \sin x$ by $x_0$.

If the vertical shift $y_0$ is positive, the shift is up. If $y_0$ is negative, the shift is down.
Rescaling
As an example, if the sine function is expanded vertically by 2 and contracted horizontally by 3 then the rescaled graph is

![Rescaled graph](image)

which has the equation $y = 2 \sin(3x)$.

In general, if the sine function is expanded vertically by $A$ and contracted horizontally by $c$ then the rescaled graph is

![Rescaled graph](image)

which has the equation $y = A \sin(cx)$.

If the horizontal scale factor $c > 1$, the graph is contracted horizontally. If $0 < c < 1$, the graph is expanded horizontally. If $c < 0$, the graph is reflected horizontally and contracted or expanded by a factor of $|c|$.

If the vertical scale factor $A > 1$, the graph is expanded vertically. If $0 < A < 1$, the graph is contracted vertically. If $A < 0$, the graph is reflected vertically and contracted or expanded by a factor of $|A|$.

Shifting and Rescaling
The function $y = \sin x$ may be shifted and rescaled at the same time. For example, the graph of $y = 2 \sin\left(3\left(x - \frac{\pi}{4}\right)\right) + \frac{1}{2}$ is:

![Rescaled graph](image)

which is expanded vertically by 2, contracted horizontally by 3, shifted to the right by $\frac{\pi}{4}$ and shifted up by $\frac{1}{2}$.

In general, the graph of $y = A \sin(c(x - x_0)) + y_0$ is:

![Rescaled graph](image)

which is expanded vertically by $A$, contracted horizontally by $c$, shifted to the right by $x_0$ and shifted up by $y_0$. 

© WeBAdG : Pre-Calculus - Chapter 8C
Interpretation

- The vertical shift $y_0$ gives the average value of the function $y = A \sin(c(x - x_0)) + y_0$ and the line $y = y_0$ is called the center line of this function.
- The absolute value of the vertical scale factor $|A|$ is called the amplitude and gives the maximum distance the function goes above and below the centerline.
- The horizontal shift $x_0$ gives the distance the function $y = A \sin(c(x - x_0)) + y_0$ leads or lags behind $y = \sin(cx)$.
- The absolute value of the horizontal scale factor $|c|$ gives the angular frequency $\omega$ which is related to the period $P$ and the frequency $f$ by the equations $\omega = 2\pi f = \frac{2\pi}{P} = |c|$.
- Finally, the quantity $\theta = c(x-x_0)$ is called the phase of the wave and the product $\varphi = cx_0$ is called the phase shift of the wave. So the phase shift $\varphi = cx_0$ gives the amount the phase of $y = A \sin(c(x - x_0)) + y_0$ leads or lags behind the phase of $y = \sin(cx)$.

Using these various definitions, if $c > 0$ (for simplicity), the general shifted and rescaled sin curve can be rewritten in several other ways:

$$y = A \sin(\omega(x - x_0)) + y_0 = A \sin(\omega x - \varphi) + y_0 = A \sin\left(\frac{2\pi}{P} (x - x_0)\right) + y_0 = A \sin\left(\frac{2\pi x}{P} - \varphi\right) + y_0$$
Exercises for Chapter 8C - Graphs of the Trig Functions

1. Graph $\sin(x + 1)$ and $\sin x$ on the same axes. If you could pick up the plot of $\sin(x + 1)$ and move it around, how should you move this plot so that it lays right on top of the plot of $\sin x$?

2. Compute the value of $\sin 2x$ for $x = 0, \pi/12, \pi/6, \pi/3, \pi/8, \pi/4,$ and $\pi/2$. Then, knowing what the graph of $\sin x$ looks like, graph the function $\sin 2x$ on the interval $[0, \pi]$.

3. The graph of a function $f(x)$ looks like the graph of the function $\sin x$ except that it takes on the value 1 when $x = 2$. What could $f(x)$ equal?

4. The graphs of $\sin x$ and $\cos x$ look like each other except that they are out of phase. Conjecture a relationship between the sine and cosine functions.

5. Sketch the graphs of the functions $\sin x$, $\sin(x - \pi/2)$, and $\sin(x + \pi/2)$ on the same plot. Pay attention to the correspondence between the plots and the sign of the shift $\pm \pi/2$.

6. Sketch the graphs of $\cos x$ and $\cos(x - \pi)$. Conjecture a relationship between these two functions.

7. A scientist recorded data from an experiment. She noticed that the data appeared to be periodic with period $4\pi$ and had a maximum value of 5, which occurred at $x = 3$. After staring at the data for some time, she decided to use a cosine function to model the data. What was the form of the function she used? What would the form be if she had decided to use a sine function?

8. If $f(x) = -8 \sin(4x + 3) - 6$, what is the amplitude, period, phase shift, and vertical shift?

9. If $f(x) = 2 \cos(3x - 4\pi) + 9$, what is the amplitude, period, phase shift, and vertical shift?

10. Write a function of the form $f(x) = a \sin(k(x - b)) + c$ whose graph is given, where $a$, $k$, and $b$ are positive and $b$ is as small as possible.
11. Match each function with its graph below:
   a. \( y = \sin x \)
   b. \( y = \sin(x - \pi/2) \)
   c. \( y = \sin(x + \pi/2) \)
   d. \( y = 2 \sin x \)
   e. \( y = \sin(2(x + \pi/2)) \)
Answers to Exercises for Chapter 8C - Graphs of the Trig Functions

1. Moving the plot of $\sin(x + 1)$ one unit to the right or $2\pi - 1$ units to the left will place it on top of the plot of $\sin x$.

2. $\sin(2 \cdot 0) = 0$, $\sin(2 \cdot \frac{\pi}{12}) = \frac{1}{2}$, $\sin(2 \cdot \frac{\pi}{6}) = \frac{\sqrt{3}}{2}$, $\sin(2 \cdot \frac{\pi}{3}) = \frac{\sqrt{3}}{2}$, $\sin(2 \cdot \frac{\pi}{4}) = 1$, $\sin(2 \cdot \frac{\pi}{2}) = 0$

3. $f(x) = \sin(x + \frac{\pi}{2} - 2)$

4. $\cos x = \sin(\frac{\pi}{2} - x)$

5. $\sin x$ black, $\sin(x - \pi/2)$ red, $\sin(x + \pi/2)$ green

6. $\cos(x - \pi) = -\cos x$
7. \( f(x) = 5 \cos\left(\frac{x - 3}{2}\right) \) or \( f(x) = 5 \sin\left(\frac{x + \pi}{2}\right) \)

8. The amplitude is 8, the period is \( \frac{2\pi}{\frac{\pi}{2}} = \frac{\pi}{2} \), the phase shift is −3, and the vertical shift is −6.

9. The amplitude is 2, the period is \( \frac{2\pi}{\frac{\pi}{3}} = \frac{\pi}{2} \), the phase shift is 4π, and the vertical shift is 9.

10. \( 5 \sin(2(x - 3)) + 8 \)

11.
   a. (2)
   b. (3)
   c. (1)
   d. (5)
   e. (4)
Chapter 8D - Trigonometric Identities

There are many identities of an algebraic nature between the trigonometric functions. Some of them involve relationships between the different functions and some involve the same function but with sums or differences of their arguments. Two examples of this are:

\[
\sin^2 \theta + \cos^2 \theta = 1 \quad \text{and} \quad \sin(a + \beta) = \sin a \cos \beta + \sin \beta \cos a,
\]

respectively. In the following pages we will list some of the more useful trig identities and give their proofs. Many more identities can be found in the exercises.

The first series of identities we will list involve negative angles and complementary angles.

**Negative Angle Identities:**

\[
\begin{align*}
sin(-\theta) &= -\sin \theta \\
tan(-\theta) &= -\tan \theta \\
sec(-\theta) &= \sec \theta \\
cos(-\theta) &= \cos \theta \\
cot(-\theta) &= -\cot \theta \\
csc(-\theta) &= -\csc \theta
\end{align*}
\]

These identities follow from the fact that if \((x, y)\) is that point on the unit circle, which corresponds to the angle \(\theta\), then the point which corresponds to the angle \(-\theta\) is \((x, -y)\). So, for example, we have

\[
\begin{align*}
\sin(-\theta) &= -y = -\sin \theta \\
tan(-\theta) &= \frac{-y}{x} = -\tan \theta
\end{align*}
\]

Note that sin, tan, cot, and csc are odd functions, while cos and sec are even functions.

**Complementary Angle Identities:**

\[
\begin{align*}
sin\left(\frac{\pi}{2} - \theta\right) &= \cos \theta \\
tan\left(\frac{\pi}{2} - \theta\right) &= \cot \theta \\
sec\left(\frac{\pi}{2} - \theta\right) &= \csc \theta \\
cos\left(\frac{\pi}{2} - \theta\right) &= \sin \theta \\
cot\left(\frac{\pi}{2} - \theta\right) &= \tan \theta \\
csc\left(\frac{\pi}{2} - \theta\right) &= \sec \theta
\end{align*}
\]

Thus, the trig function of the complementary angle is equal to the complementary trig function of the original angle.

To prove these complementary angle identities it suffices to verify that \(\cos\left(\frac{\pi}{2} - \theta\right) = \sin \theta\) is true. The corresponding identity for \(\sin\left(\frac{\pi}{2} - \theta\right)\) follows from the one for cosine, and the other four follow from these two.

To see that \(\cos\left(\frac{\pi}{2} - \theta\right) = \sin \theta\) is valid, we first observe that the identity is valid if \(\theta = 0\) or \(\pi/2\). So let’s assume that \(0 < \theta < \pi/2\). Upon examining the figure below

we see that \(\sin \theta = y\) and that this is the value of \(\cos(\pi/2 - \theta)\). Similar arguments verify this identity for
values of $\theta$ larger than $\pi/2$. The corresponding identity for sine follows from:

$$\sin(\pi/2 - \theta) = \cos(\pi/2 - (\pi/2 - \theta)) = \cos \theta.$$  

**Pythagorean Identities:**

The trig identity $\sin^2 \theta + \cos^2 \theta = 1$ is called a Pythagorean identity as it follows from the Pythagorean theorem. To see this, let $(x, y)$ denote that point on the unit circle determined by the angle $\theta$. Then $\sin \theta = y$ and $\cos \theta = x$. Moreover, $|x|$ and $|y|$ are the lengths of the legs of a right triangle whose hypotenuse has length 1. Thus,

$$\sin^2 \theta + \cos^2 \theta = y^2 + x^2 = 1.$$ 

There are two other such identities, and they are obtained from this one by dividing this equation by $\cos^2 \theta$ or by $\sin^2 \theta$.

Thus,

$$\frac{\sin^2 \theta + \cos^2 \theta}{\cos^2 \theta} = 1$$

$$\frac{\sin^2 \theta}{\cos^2 \theta} + 1 = \frac{1}{\cos^2 \theta} = \sec^2 \theta$$

$$\tan^2 \theta + 1 = \sec^2 \theta.$$ 

Now divide by $\sin^2 \theta$.

$$\frac{1 + \frac{\cos^2 \theta}{\sin^2 \theta}}{\sin^2 \theta} = \frac{1}{\sin^2 \theta} = \csc^2 \theta$$

$$1 + \cot^2 \theta = \csc^2 \theta.$$

**Supplementary Angle Identities:**

The following identities are easy to prove as they follow from the corresponding complementary angle identities, and this is demonstrated below for one of the identities.

$$\sin(\pi - \theta) = \sin \theta$$

$$\tan(\pi - \theta) = -\tan \theta$$

$$\sec(\pi - \theta) = -\sec \theta$$

$$\cos(\pi - \theta) = -\cos \theta$$

$$\cot(\pi - \theta) = -\cot \theta$$

$$\csc(\pi - \theta) = \csc \theta$$

$$\sin(\pi - \theta) = \sin \left( \frac{\pi}{2} - \left( \theta - \frac{\pi}{2} \right) \right) = \cos \left( \theta - \frac{\pi}{2} \right)$$

$$= \cos \left( \frac{\pi}{2} - \theta \right) = \sin \theta.$$ 

Be sure you can explain why each of the equalities in the above lines is correct.

**Sum of Two Angles Formulas:**

$$\sin(\alpha + \beta) = \sin \alpha \cos \beta + \cos \alpha \sin \beta$$

$$\cos(\alpha + \beta) = \cos \alpha \cos \beta - \sin \alpha \sin \beta$$

$$\tan(\alpha + \beta) = \frac{\tan \alpha + \tan \beta}{1 - \tan \alpha \tan \beta}$$

The proofs of these identities are more complicated, but as usual once we have the identity for sine and cosine, then the two angle sum formula for the tangent function follows.
We give a proof for these formulas for the special case where \( \alpha \) and \( \beta \) are acute and so is their sum. That is, we assume

\[
0 < \alpha, 0 < \beta, \text{ and } 0 < \alpha + \beta < \pi/2.
\]

The figure below was constructed as follows:

Lines, which are here denoted by \( BP, BQ, \) and \( BR \) were drawn so that angle \( PBQ = \alpha \) and angle \( QBR = \beta \), which means that angle \( PBR = \alpha + \beta \). Point \( A \) on line \( BR \) was chosen arbitrarily, and a line from \( A \) perpendicular to line \( BP \) was drawn with point \( G \) being its intersection with line \( BP \). A similar line was drawn from \( D \) to line \( BP \) with \( C \) its point of intersection. Line \( DE \) was then drawn perpendicular to \( DC \) with \( E \) being its point of intersection with \( AG \).

Note: triangles \( DAE \) and \( DBC \) are similar, \( EG = DC \), and \( GC = DE \). We now have

\[
\sin(\alpha + \beta) = \frac{AG}{AB} = \frac{AE + EG}{AB} = \frac{AE}{AB} + \frac{EG}{AB} = \frac{AE}{AB} \frac{AD}{AB} + \frac{DC}{BD} \frac{BD}{AB}
\]

\[
= \frac{AE}{AD} \frac{AD}{AB} + \frac{DC}{BD} \frac{BD}{AB} = \sin \left( \frac{\pi}{2} - \alpha \right) \sin \beta + \sin \alpha \cos \beta
\]

\[
= \sin \alpha \cos \beta + \sin \beta \cos \alpha.
\]

\[
\cos(\alpha + \beta) = \frac{BG}{BA} = \frac{BC - CG}{BA} = \frac{BC}{BA} - \frac{DE}{BA}
\]

\[
= \frac{BC}{BA} \frac{BD}{AD} - \frac{DE}{BA} \frac{BD}{AD} = \frac{BC}{BD} \frac{BD}{BA} - \frac{DE}{AD} \frac{AD}{BA}
\]

\[
= \cos \alpha \cos \beta - \sin \alpha \sin \beta.
\]

The other cases for various values of \( \alpha \) and \( \beta \) can be shown by using the above identities for these restricted values of \( \alpha \) and \( \beta \) and the complementary trig identities we’ve already verified. See the exercises.

**Difference of Two Angles Formulas:**

These identities follow from the Sum of Two Angles Formulas as we show for the sine function.

\[
\sin(\alpha - \beta) = \sin \alpha \cos \beta - \cos \alpha \sin \beta
\]

\[
\cos(\alpha - \beta) = \cos \alpha \cos \beta + \sin \alpha \sin \beta
\]

\[
\tan(\alpha - \beta) = \frac{\tan \alpha - \tan \beta}{1 + \tan \alpha \tan \beta}
\]

\[
\sin(\alpha - \beta) = \sin(\alpha + (-\beta))
\]

\[
= \sin \alpha \cos(-\beta) + \sin(-\beta) \cos \alpha
\]

\[
= \sin \alpha \cos \beta - \sin \beta \cos \alpha.
\]
Double Angle Formulas:
The proofs of these formulas, as are all remaining trig identities in this section, are assigned as exercises.

\[ \sin(2\alpha) = 2\sin\alpha \cos\alpha \]
\[ \cos(2\alpha) = \cos^2\alpha - \sin^2\alpha = 2\cos^2\alpha - 1 = 1 - 2\sin^2\alpha \]
\[ \tan(2\alpha) = \frac{2\tan\alpha}{1 - \tan^2\alpha} \]

Square Formulas:
The proofs of these formulas, as are all remaining trig identities in this section, are assigned as exercises.

\[ \sin^2\alpha = \frac{1 - \cos(2\alpha)}{2} \]
\[ \cos^2\alpha = \frac{1 + \cos(2\alpha)}{2} \]

Half Angle Formulas:
The proofs of these formulas, as are all remaining trig identities in this section, are assigned as exercises.

\[ \sin\left(\frac{\alpha}{2}\right) = \pm \sqrt{\frac{1 - \cos\alpha}{2}} \quad \text{with} \quad \begin{cases} + \text{ if } \frac{\alpha}{2} \text{ is in Quadrants I or II} \\ - \text{ if } \frac{\alpha}{2} \text{ is in Quadrants III or IV} \end{cases} \]
\[ \cos\left(\frac{\alpha}{2}\right) = \pm \sqrt{\frac{1 + \cos\alpha}{2}} \quad \text{with} \quad \begin{cases} + \text{ if } \frac{\alpha}{2} \text{ is in Quadrants I or IV} \\ - \text{ if } \frac{\alpha}{2} \text{ is in Quadrants II or III} \end{cases} \]
\[ \tan\left(\frac{\alpha}{2}\right) = \pm \sqrt{\frac{1 - \cos\alpha}{1 + \cos\alpha}} \quad \text{with} \quad \begin{cases} + \text{ if } \frac{\alpha}{2} \text{ is in Quadrants I or III} \\ - \text{ if } \frac{\alpha}{2} \text{ is in Quadrants II or IV} \end{cases} \]
\[ \tan\left(\frac{\alpha}{2}\right) = \frac{1 - \cos\alpha}{\sin\alpha} = \frac{\sin\alpha}{1 + \cos\alpha} \]

Product to Sum Formulas:
The proofs of these formulas, as are all remaining trig identities in this section, are assigned as exercises.

\[ 2\cos\alpha \cos\beta = \cos(\alpha - \beta) + \cos(\alpha + \beta) \]
\[ 2\sin\alpha \sin\beta = \cos(\alpha - \beta) - \cos(\alpha + \beta) \]
\[ 2\sin\alpha \cos\beta = \sin(\alpha - \beta) + \sin(\alpha + \beta) \]
\[ 2\cos\alpha \sin\beta = -\sin(\alpha - \beta) + \sin(\alpha + \beta) \]

Sum to Product Formulas:
The proofs of these formulas are assigned as exercises.

\[ \cos\alpha + \cos\beta = 2\cos\left(\frac{\alpha + \beta}{2}\right) \cos\left(\frac{\alpha - \beta}{2}\right) \]
\[ \cos\alpha - \cos\beta = -2\sin\left(\frac{\alpha + \beta}{2}\right) \sin\left(\frac{\alpha - \beta}{2}\right) \]
\[ \sin\alpha + \sin\beta = 2\sin\left(\frac{\alpha + \beta}{2}\right) \cos\left(\frac{\alpha - \beta}{2}\right) \]
\[ \sin\alpha - \sin\beta = 2\cos\left(\frac{\alpha + \beta}{2}\right) \sin\left(\frac{\alpha - \beta}{2}\right) \]
Exercises for Chapter 8D - Trigonometric Identities

1. Using the fact that \( \sin \theta = -\sin(-\theta) \), and \( \cos \theta = \cos(-\theta) \), show that \( \tan \theta \) is an odd function of \( \theta \).

2. If \( \theta \) lies in the third quadrant, i.e., \( \pi < \theta < 3\pi/2 \), and \( |\cos \theta| = 1/3 \), what is the value of \( \sin \theta \)?

3. Express \( \sin(15 + 27) \) in terms of sines and cosines of 15 and 27 degrees.

4. Using the formulas for \( \sin(\alpha + \beta) \) and \( \cos(\alpha + \beta) \), show that

\[
\tan(\alpha + \beta) = \frac{\tan \alpha + \tan \beta}{1 - \tan \alpha \tan \beta}
\]

\[
\tan(\alpha - \beta) = \frac{\tan \alpha - \tan \beta}{1 + \tan \alpha \tan \beta}
\]

5. Show that \( \sin^2 \alpha = \frac{1 - \cos 2\alpha}{2} \) and \( \cos^2 \alpha = \frac{1 + \cos 2\alpha}{2} \).

6. If \( \theta \) is an acute angle and \( \sin \theta = 0.6 \), compute \( \cos \theta, \sin(\theta/2), \) and \( \cos(\theta/2) \).

7. Verify the following identities

\[
2 \cos \alpha \cos \beta = \cos(\alpha - \beta) + \cos(\alpha + \beta)
\]

\[
2 \sin \alpha \cos \beta = \sin(\alpha - \beta) + \sin(\alpha + \beta)
\]

8. Given that \( \cos 270^\circ = 0 \), find \( \sin 135^\circ \) and \( \cos 135^\circ \).

9. Verify the following trig identity

\[
\frac{\sin(\alpha + \beta)}{\cos \alpha \cos \beta} = \tan \alpha + \tan \beta
\]

10. The two points \((\cos \alpha, \sin \alpha)\) and \((\cos \beta, \sin \beta)\) are on the unit circle.
   a. What is the angle formed by the radial lines from the center of the unit circle to these points?
   b. What is the length of the chord joining these two points?
   c. Rotate the unit circle so that one of the two points now lies on the point \((1,0)\). So one of the original points has coordinates \((1,0)\). What are the coordinates of the second point, and what is the length of the chord joining these two points?
   d. Show how the fact that the lengths of the chords in parts b. and c. are equal implies the formula

\[
\cos(\alpha + \beta) = \cos \alpha \cos \beta - \sin \alpha \sin \beta
\]

11. Exactly evaluate \( \sin(x + \frac{\pi}{4}) \) where \( \sin x = \frac{4}{5} \) and \( x \) is in Quadrant II.

12. If \( \cot x = -\frac{16}{13} \), where \( x \) is in Quadrant IV, what is the exact value of \( \sin 2x \)?

13. Exactly evaluate \( \cos(x + \frac{\pi}{4}) \) where \( \cos x = \frac{3}{4} \) and \( x \) is in Quadrant IV.
Answers to Exercises for Chapter 8D - Trigonometric Identities

1. \[ \tan(-\theta) = \frac{\sin(-\theta)}{\cos(-\theta)} = -\frac{\sin(\theta)}{\cos(\theta)} = -\tan\theta \]

2. Since \( \theta \) is in the third quadrant, \( \sin \theta \) is negative. Thus,

\[ \sin \theta = -\sqrt{1 - \cos^2 \theta} = -\sqrt{1 - 1/9} = \frac{-2\sqrt{2}}{3} \approx -0.94 \]

3. \[ \sin(15^\circ + 27^\circ) = \sin(15^\circ)\cos(27^\circ) + \sin(27^\circ)\cos(15^\circ) \]

4. \[
\begin{align*}
\tan(\alpha + \beta) &= \frac{\sin(\alpha + \beta)}{\cos(\alpha + \beta)} = \frac{\sin \alpha \cos \beta + \sin \beta \cos \alpha}{\cos \alpha \cos \beta - \sin \alpha \sin \beta} \\
&= \frac{\frac{\sin \alpha}{\cos \alpha} + \frac{\sin \beta}{\cos \beta}}{1 - \frac{\sin \alpha \sin \beta}{\cos \alpha \cos \beta}} = \frac{\tan \alpha + \tan \beta}{1 - \tan \alpha \tan \beta} \\
\tan(\alpha - \beta) &= \tan(\alpha + (-\beta)) = \frac{\tan \alpha + \tan(-\beta)}{1 - \tan \alpha \tan(-\beta)} = \frac{\tan \alpha - \tan \beta}{1 + \tan \alpha \tan \beta}
\end{align*}
\]

5. Start with the sum of angles formula for cosine.

\[ \cos 2\alpha = \cos(\alpha + \alpha) = \cos \alpha \cos \alpha - \sin \alpha \sin \alpha \\
= \cos^2 \alpha - \sin^2 \alpha = 1 - 2\sin^2 \alpha. \]

Solving for \( \sin^2 \alpha \), we have

\[ \sin^2 \alpha = \frac{1 - \cos 2\alpha}{2}. \]

The second formula follows from the identity \( \cos 2\alpha = \cos^2 \alpha - \sin^2 \alpha = 2\cos^2 \alpha - 1. \)

6. \[
\begin{align*}
\cos \theta &= \sqrt{1 - \sin^2 \theta} = \sqrt{1 - 0.36} = \sqrt{0.64} = 0.8 \\
\sin(\theta/2) &= \sqrt{\frac{1 - \cos \theta}{2}} = \sqrt{\frac{1 - 0.8}{2}} = \sqrt{0.1} \approx 0.316 \\
\cos(\theta/2) &= \sqrt{\frac{1 + \cos \theta}{2}} = \sqrt{\frac{1 + 0.8}{2}} = \sqrt{0.9} \approx 0.949
\end{align*}
\]
7. For the first identity, start with the sum of two angles formula for cosine.

\[ \cos(\alpha - \beta) = \cos \alpha \cos \beta + \sin \alpha \sin \beta \]
\[ \cos(\alpha + \beta) = \cos \alpha \cos \beta - \sin \alpha \sin \beta \]

Now add these two equations to get the first identity. To get the second identity do the same thing with the sum of two angles formula for sine.

8. Note that 135 ° is 1/2 of 270 °, and that 135 ° is in the second quadrant, which means that sin 135 ° is positive, and cos 135 ° is negative.

\[ \sin 135° = \sqrt{\frac{1 - \cos 270°}{2}} = \sqrt{\frac{1}{2}} \approx 0.707 \]
\[ \cos 135° = -\sqrt{\frac{1 + \cos 270°}{2}} = -\sqrt{\frac{1}{2}} \approx -0.707 \]

9. \[ \frac{\sin(\alpha + \beta)}{\cos \alpha \cos \beta} = \frac{\sin \alpha \cos \beta + \sin \beta \cos \alpha}{\cos \alpha \cos \beta} \]
\[ = \frac{\sin \alpha \cos \beta}{\cos \alpha \cos \beta} + \frac{\sin \beta \cos \alpha}{\cos \alpha \cos \beta} \]
\[ = \frac{\sin \alpha}{\cos \alpha} + \frac{\sin \beta}{\cos \beta} \]
\[ = \tan \alpha + \tan \beta \]

10. 
   a. Either \( \alpha - \beta \) or \( \beta - \alpha \).
   
   b. 
   \[ \text{length} = \sqrt{(\cos \alpha - \cos \beta)^2 + (\sin \alpha - \sin \beta)^2} \]
   \[ = \sqrt{\cos^2 \alpha - 2 \cos \alpha \cos \beta + \cos^2 \beta + \sin^2 \alpha - 2 \sin \alpha \sin \beta + \sin^2 \beta} \]
   \[ = \sqrt{2(1 - \cos \alpha \cos \beta - \sin \alpha \sin \beta)} \]
   
   c. Assume that the angle formed by the radial lines joining the original two points is \( \beta - \alpha \). Then after rotating the unit circle, the coordinates of the second point are \((\cos(\beta - \alpha), \sin(\beta - \alpha))\), and the length of the rotated chord is 
   \[ \text{length} = \sqrt{(1 - \cos(\beta - \alpha))^2 + \sin^2(\beta - \alpha)} \]
   \[ = \sqrt{1 - 2 \cos(\beta - \alpha) + \cos^2(\beta - \alpha) + \sin^2(\beta - \alpha)} \]
   \[ = \sqrt{2(1 - \cos(\beta - \alpha))} \]
   
   d. Since the lengths of both chords are equal we have 
   \[ \sqrt{2(1 - \cos \alpha \cos \beta - \sin \alpha \sin \beta)} = \sqrt{2(1 - \cos(\beta - \alpha))} \]
   \[ 1 - \cos \alpha \cos \beta - \sin \alpha \sin \beta = 1 - \cos(\beta - \alpha) \]
   \[ \cos(\beta - \alpha) = \cos \alpha \cos \beta + \sin \alpha \sin \beta \]
11. Since \( \sin x = 8/9 \) and \( x \) is in Quadrant II, we can determine that \( \cos x = -\sqrt{17}/9 \). Using the sum of two angles formula for sine we have

\[
\sin(x + \pi/4) = \sin x \cos(\pi/4) + \cos x \sin(\pi/4)
\]

\[
= (8/9)(\sqrt{2}/2) + (-\sqrt{17}/9)(\sqrt{2}/2)
\]

\[
= \frac{8\sqrt{2}}{18} - \frac{\sqrt{34}}{18}
\]

\[
= \frac{8\sqrt{2} - \sqrt{34}}{18}
\]

12. Since \( \cot x = -\frac{16}{15} \) and \( x \) is in Quadrant IV, we can determine that \( \cos x = \frac{16}{\sqrt{481}} \) and \( \sin x = -\frac{15}{\sqrt{481}} \). Using the double angle formula for sine we have

\[
\sin(2x) = 2 \sin x \cos x
\]

\[
= 2 \left( -\frac{15}{\sqrt{481}} \right) \left( \frac{16}{\sqrt{481}} \right)
\]

\[
= \frac{480}{481}
\]

13. Since \( \cos x = 3/4 \) and \( x \) is in Quadrant IV, we can determine that \( \sin x = -\sqrt{7}/4 \). Using the sum of two angles formula for cosine we have

\[
\cos(x + \pi/3) = \cos x \cos(\pi/3) - \sin x \sin(\pi/3)
\]

\[
= (3/4)(1/2) - (-\sqrt{7}/4)(\sqrt{3}/2)
\]

\[
= \frac{3}{8} + \frac{\sqrt{21}}{8}
\]

\[
= \frac{3 + \sqrt{21}}{8}
\]
Chapter 8E - Inverse Sine Function

In addition to the original 6 trig functions there are several more that are extremely useful in mathematics, science and engineering. One of these functions is defined in this section and a few of its properties are also discussed. The others will be defined in the next section.

In order for a function $f$ to have an inverse function it must be one-to-one. None of the trig functions as previously defined satisfy this condition. So our first item is to address this issue, which we do in some detail with the sine function, whose graph is shown below.

It’s clear from the horizontal line test that $\sin x$ is not one-to-one. However, if we restrict the domain of this function from all real numbers to just the interval $[-\pi/2, \pi/2]$, this new function is one-to-one. See the plot below

The range of this function is the closed interval $[-1, 1]$. We define the arcsin function and denote it as follows:

for any $x \in [-1, 1]$, the range of $\sin x$,

$$\arcsin x = \sin^{-1} x = y \text{ if and only if } \sin y = x.$$
The tables below may help to clarify this definition

<table>
<thead>
<tr>
<th>$x$</th>
<th>$\sin x$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$\frac{\pi}{4}$</td>
<td>$\frac{1}{\sqrt{2}}$</td>
</tr>
<tr>
<td>$\frac{\pi}{2}$</td>
<td>1</td>
</tr>
<tr>
<td>$-\frac{\pi}{4}$</td>
<td>$-\frac{1}{\sqrt{2}}$</td>
</tr>
<tr>
<td>$-\frac{\pi}{2}$</td>
<td>$-1$</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>$x$</th>
<th>$\sin^{-1} x$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$\frac{1}{\sqrt{2}}$</td>
<td>$\frac{\pi}{4}$</td>
</tr>
<tr>
<td>1</td>
<td>$\frac{\pi}{2}$</td>
</tr>
<tr>
<td>$-\frac{1}{\sqrt{2}}$</td>
<td>$-\frac{\pi}{4}$</td>
</tr>
<tr>
<td>$-1$</td>
<td>$-\frac{\pi}{2}$</td>
</tr>
</tbody>
</table>

Since $\sin(\pi/4) = 1/\sqrt{2}$ we must have $\sin^{-1}(1/\sqrt{2}) = \pi/4$.

Note: $\sin^{-1}$ does not mean the reciprocal of the sine function. It is standard notation for the inverse function. Geometrically if the point $(x, y)$ is on the graph of $\sin x$, i.e., $\sin x = y$, then the point $(y, x)$ is on the graph of the arcsin function, whose graph we show below.

$$\text{arcsin} x = \sin^{-1} x, -1 \leq x \leq 1$$

The following plot shows both $\sin x$, $\sin^{-1} x$, and the line $y = x$. Note that the graph of the arcsin function is obtained by reflecting the graph of $\sin x$ through the line $y = x$. 

$\sin x$ (blue) and $\sin^{-1} x$ (red)
Before restating the definition of the arcsin function we point out that the domain of the sine function does not have to be the interval \([-\pi/2, \pi/2]\) in order to force the function to be one-to-one. We could have taken many different intervals. For example \(\sin x\) is one-to-one on each of the following intervals:

\[ \left(-\frac{\pi}{2}, \frac{\pi}{2}\right), \left(\frac{\pi}{2}, \frac{3\pi}{2}\right) \]

In fact the sine function is one-to-one on any interval of the form \(\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)\) for any integer \(k\). Setting \(k = 0, 1,\) and \(-1\) gives the three intervals we’ve mentioned. The reason for picking the interval \([-\pi/2, \pi/2]\) is that this is more useful than other possible choices.

We summarize below the above discussion.

\[
\sin x : \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \rightarrow [-1,1], \\
\sin^{-1} x : [-1,1] \rightarrow \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]
\]

where

\[
\sin x = y \text{ if and only if } \sin^{-1} y = x.
\]

Another way to write this last line is the following:

\[
\sin(\sin^{-1} x) = x \text{ for } x \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \text{ and } \\
\sin^{-1}(\sin x) = x \text{ for } x \in [-1,1].
\]

**Example 1:** The sine of \(60^\circ\) equals \(\frac{\sqrt{3}}{2}\). Thus, \(\sin^{-1}(\frac{\sqrt{3}}{2}) = \frac{\pi}{3}\). Note that the value of arcsin is a number with out units. So \(\sin^{-1}(\frac{\sqrt{3}}{2}) = 60^\circ\) would not be correct as this has units, and \(\sin^{-1}(\frac{\sqrt{3}}{2}) = 60\) is also not correct, as 60 is not between \(-\pi/2\) and \(\pi/2\).

**Example 2:** The sine of \(225^\circ\) equals \(-\frac{1}{\sqrt{2}}\). What does the arcsin of \(-1/\sqrt{2}\) equal?

Solution: \(225^\circ\) is equivalent to \(5\pi/4\) radians. However, \(5\pi/4\) is greater than \(\pi/2\), so the answer cannot be \(\sin^{-1}(1/\sqrt{2}) = 5\pi/4\). We need a number, \(\theta\), between \(-\pi/2\) and \(\pi/2\) such that \(\sin \theta = -1/\sqrt{2}\).

Let’s start noting that \(5\pi/4 = \pi + \pi/4\), and use the sum of two angles formula for the sine function.

\[
-\frac{1}{\sqrt{2}} = \sin \frac{5\pi}{4} = \sin \left(\pi + \frac{\pi}{4}\right) \\
= \sin \pi \cos \frac{\pi}{4} + \sin \frac{\pi}{4} \cos \pi \\
= -\sin \frac{\pi}{4} \\
= \sin \left(-\frac{\pi}{4}\right).
\]

Thus, we have \(\sin^{-1}\left(-\frac{1}{\sqrt{2}}\right) = -\frac{\pi}{4}\).
Exercises for Chapter 8E - Inverse Sine Function

1. \( \sin^{-1}(0.4) = ? \) Give your answer in degrees as well as radians. (A calculator may be useful.)

2. If \( \sin \theta = 0.4 \) and \( \pi/2 < \theta < \pi \) what is \( \theta \)? (You should use a calculator to determine \( \sin^{-1}(0.4) \))

3. If \( \sin \theta = 0.25 \), what does \( \cos \theta \) equal?

4. \( \tan(\sin^{-1}(1/3)) = ? \)

5. \( \sin^{-1}(-1) = ? \)

6. Exactly evaluate \( \sin^{-1}\left(\frac{\sqrt{2}}{2}\right) \).

7. \( \sin^{-1}(\sin \frac{5\pi}{3}) = ? \)

8. \( \sin\left(\sin^{-1}\left(\frac{\sqrt{3}}{3}\right)\right) = ? \)
Answers to Exercises for Chapter 8E - Inverse Sine Function

1. Using a calculator we have:

$$\sin^{-1}(0.4) = 0.41152.$$  

Since my calculator is set to radian mode the answer given is in radians. The answer in degrees is

$$\theta = 0.41152 \times \frac{180}{\pi}$$
$$= 23.578 \text{ degrees}.$$  

2. Looking at a plot of the sine function, see below,

![Graph of sin(x)](image)

we see that \(\sin x\) is symmetric about the line \(x = \pi/2\). So if \(\sin \theta = 0.4\) and \(\pi/2 < \theta < \pi\), and if \(\psi\) is such that \(\sin \psi = \sin \theta\) and \(0 < \psi < \pi/2\) then we must have

$$\frac{\pi}{2} - \psi = \theta - \frac{\pi}{2}.$$  

Thus, \(\theta = \pi - \psi = \pi - 0.41152 \approx 2.7301\)

3. Draw a right triangle with hypotenuse of length one, and an angle labeled \(\theta\). Then the side opposite \(\theta\) will have length 0.25. This implies that the side adjacent to \(\theta\) will have length

$$\sqrt{1 - (0.25)^2} \approx 0.968.$$  

Thus,

$$\cos \theta \approx 0.968.$$
4. As in the preceding problem, draw a right triangle with hypotenuse of length three, and an angle labeled $\theta$. See the figure below.

Thus,

$$\tan \theta = \frac{1}{\sqrt{8}}.$$ 

5. Since $\sin(-\pi/2) = -1$, we must have $\sin^{-1}(-1) = -\frac{\pi}{2}$.

6. Since $\sin(\pi/4) = \sqrt{2}/2$ and $\pi/4 \in [-\pi/2, \pi/2]$, we conclude that $\sin^{-1}\left(\frac{\sqrt{2}}{2}\right) = \frac{\pi}{4}$.

7. Since $\frac{5\pi}{3} = 2\pi - \frac{\pi}{3}$, we know that $\sin \frac{5\pi}{3} = \sin \left(-\frac{\pi}{3}\right)$. Thus,

$$\sin^{-1}\left(\sin \frac{5\pi}{3}\right) = \sin^{-1}\left(\sin \left(-\frac{\pi}{3}\right)\right) = -\frac{\pi}{3}$$

8. $\sin\left(\sin^{-1}\left(\frac{\sqrt{3}}{3}\right)\right) = \frac{\sqrt{3}}{3}$
In this section we define the inverse cosine and inverse tangent functions. The last three inverse trig functions: arcsec, arccot, and arccsc are not discussed.

**Inverse Cosine**

The cosine function is also not one-to-one until we restrict its domain. The graph of this function shows that there are many possible restrictions.

We could use any of the following intervals: \([-\pi, 0], [0, \pi], \) or \([\pi, 2\pi]\). The usual choice is to use the interval \([0, \pi]\) for the domain of the cosine function. When this choice is made the arccos function, written as \(\cos^{-1}\) is defined for \(x \in [-1, 1]\) by

\[
\cos^{-1} x = y,
\]

if and only if \(y \in [0, \pi]\) and \(\cos y = x\).

Note that the range of the arccos function is the restricted domain of \(\cos\) and the domain of arccos is the range of the \(\cos\) function.

The table below lists some values of the arccos function.

<table>
<thead>
<tr>
<th>(x)</th>
<th>(\cos^{-1} x)</th>
</tr>
</thead>
<tbody>
<tr>
<td>-2</td>
<td>not defined</td>
</tr>
<tr>
<td>-1</td>
<td>(\pi)</td>
</tr>
<tr>
<td>(-\frac{1}{2})</td>
<td>(\frac{2\pi}{3})</td>
</tr>
<tr>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>(\frac{1}{2})</td>
<td>(\frac{\pi}{3})</td>
</tr>
<tr>
<td>1</td>
<td>0</td>
</tr>
</tbody>
</table>

The graph of the inverse cosine function is shown below. You should sketch the graph of the cosine function with \(x\) restricted to lie in the interval \([0, \pi]\), draw the line \(y = x\) and reflect the graph of cosine about this line. You should get the plot below.
Summarizing the above discussion we have:

For $x$, such that $-1 \leq x \leq 1$,
\[
\cos^{-1} x = y \quad \text{if and only if} \quad \cos y = x,
\]
and $0 \leq y \leq \pi$. That is, for $-1 \leq x \leq 1$ and $0 \leq y \leq \pi$, we have
\[
\cos^{-1} (\cos y) = y
\]
\[
\cos (\cos^{-1} x) = x.
\]

**Inverse Tangent Function**

The graph of the tangent function below shows, that if the domain of $\tan x$ is restricted to $-\pi/2 < x < \pi/2$, then the tangent function is one-to-one, with range the set of all real numbers.

The inverse tangent function is defined as follows. For $x$ any real number $\tan^{-1} x = y$ if and only if $\tan y = x$ and $-\pi/2 < y < \pi/2$. A plot of $\arctan x$ is shown below.
The table below lists some values for the arctan function.

<table>
<thead>
<tr>
<th>x</th>
<th>( \tan^{-1}x )</th>
</tr>
</thead>
<tbody>
<tr>
<td>(-\sqrt{3})</td>
<td>(-\frac{\pi}{3})</td>
</tr>
<tr>
<td>(-1)</td>
<td>(-\frac{\pi}{4})</td>
</tr>
<tr>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>1</td>
<td>(\frac{\pi}{4})</td>
</tr>
<tr>
<td>(\sqrt{3})</td>
<td>(\frac{\pi}{3})</td>
</tr>
</tbody>
</table>

To better help your understanding of this function sketch the graph of \(\tan x\) for \(x\) between \(-\frac{\pi}{2}\) and \(\frac{\pi}{2}\), draw the line \(y = x\), and then sketch the reflection of the graph of \(\tan x\) through the line \(y = x\). You should get the graph of the arctan function.

**Example 1:** If \(0 \leq x \leq \frac{\pi}{2}\), compute \(\cos^{-1}(\sin x)\).

**Solution:** Let \(\theta = \cos^{-1}(\sin x)\). Then \(0 \leq \theta \leq \pi\), and \(\cos \theta = \sin x\). We know that \(\sin x = \cos(\frac{\pi}{2} - x)\). Since both \(\theta\) and \(\frac{\pi}{2} - x\) are in the interval \([0, \pi]\) and \(\cos\) is one-to-one on this interval we may conclude that \(\theta = \frac{\pi}{2} - x\). That is,

\[
\cos^{-1}(\sin x) = \frac{\pi}{2} - x.
\]

**Example 2:** Compute \(\cos(\sin^{-1}x)\).

**Solution:** Since \(\cos \theta\) is an even function of \(\theta\), we may as well assume that \(\sin^{-1}x\) is positive, and set \(\theta = \sin^{-1}x\). Then \(\sin \theta = x\). Draw a right triangle with angle \(\theta\), and label the sides as shown below.

It is clear from this picture that \(\cos(\sin^{-1}x) = \cos \theta = \sqrt{1 - x^2}\).
Exercises for Chapter 8F - Inverse Trig Functions

1. \( \cos^{-1}(1/2) = ? \)
2. \( \tan^{-1}(1/2) = ? \)
3. \( \tan^{-1}(-1) = ? \)
4. \( \sin(\tan^{-1}(1/3)) = ? \)
5. \( \cos(\tan^{-1}x) = ? \)
6. \( \sin(\cos^{-1}x + \tan^{-1}y) = ? \)
7. \( \tan(\sin^{-1}x) = ? \)
8. \( \cos(\sin^{-1}x + \cos^{-1}x) = ? \)
9. Show that \( \sin^{-1}x + \sin^{-1}y = \sin^{-1}(\sqrt{1-y^2} + y\sqrt{1-x^2}) \).
10. Exactly evaluate \( \sin(\cos^{-1}(3x)) \) if \( 0 \leq x \leq \frac{1}{3} \).
Answers to Exercises for Chapter 8F - Inverse Trig Functions

1. Looking at the 30/60 degree right triangle we see that \( \cos 60^\circ = 1/2 \), and since 60\(^\circ\) corresponds to \( \frac{\pi}{3} \) radians, which is between 0 and \( \pi \), we have

\[
\cos^{-1}(1/2) = \frac{\pi}{3}.
\]

2. None of the standard right triangles have an angle whose tangent is 1/2, so we are forced to resort to a calculator, which gives

\[
\tan^{-1} \left( \frac{1}{2} \right) \approx 0.464 \text{ radians}.
\]

3. Since \( \tan \pi/4 = 1 \) and \( \tan \) is an odd function we know that \( \tan(-\pi/4) = -1 \). Thus,

\[
\tan^{-1}(-1) = -\frac{\pi}{4}.
\]

4. Sketch the right triangle with legs of length 1 and 3. Call the angle opposite the leg of length 1, \( \theta \). See the sketch below.

![Sketch of right triangle](image)

The Pythagorean theorem implies that the hypotenuse of this triangle has length \( \sqrt{10} \). Thus, \( \sin \theta = 1/\sqrt{10} \), and

\[
\sin(\tan^{-1}(1/3)) = \sin \theta = \frac{1}{\sqrt{10}}.
\]

5. As in the previous problem draw the appropriate right triangle. See sketch below.

![Sketch of right triangle](image)

From this sketch we have

\[
\cos(\tan^{-1}x) = \cos \theta = \frac{1}{\sqrt{1 + x^2}}.
\]
6. Be sure to draw the appropriate right triangles as we did in the previous exercises.

\[
\sin(\cos^{-1}x + \tan^{-1}y) = \sin(\cos^{-1}x)\cos(\tan^{-1}y) + \sin(\tan^{-1}y)\cos(\cos^{-1}x)
\]

\[
= \frac{\sqrt{1 - x^2}}{\sqrt{1 + y^2}} + \frac{xy}{\sqrt{1 + y^2}}
\]

7. 

\[
\tan(\sin^{-1}x) = \frac{x}{\sqrt{1 - x^2}}.
\]

8. 

\[
\cos(\sin^{-1}x + \cos^{-1}x) = \cos(\sin^{-1}x)\cos(\cos^{-1}x) - \sin(\sin^{-1}x)\sin(\cos^{-1}x)
\]

\[
= \sqrt{1 - x^2} x - x\sqrt{1 - x^2}
\]

\[
= 0
\]

9. Show that the sine of both sides is equal, and since the sine function is one-to-one, the two sides must be equal.

\[
\sin(\sin^{-1}x + \sin^{-1}y) = \sin(\sin^{-1}x)\cos(\sin^{-1}y) + \sin(\sin^{-1}y)\cos(\sin^{-1}x)
\]

\[
= x\sqrt{1 - y^2} + y\sqrt{1 - x^2}.
\]

Since \(\sin(\sin^{-1}\theta) = \theta\), we see that the sine of the right hand side of the original equation equals the sine of its left hand side.

10. Draw a right triangle with one leg \(3x\) and an hypotenuse of length 1. By the pythagorean theorem, the length of the other leg is \(\sqrt{1 - 9x^2}\). Label the angle opposite this side \(\theta\). Thus, we have

\[
\sin(\cos^{-1}3x) = \sin\theta = \sqrt{1 - 9x^2}
\]
Chapter 8G - Law of Sines and Law of Cosines

A classic series of problems in geometry and trigonometry fall under the heading "to solve the triangle". What this means is that some data about a triangle is known, and from this information all of the lengths of the triangle’s sides and all of angles are to be determined. There are two formulas that are very helpful in this matter, and they are

\[
\text{Law of Sines: } \frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}
\]

\[
\text{Law of Cosines: } c^2 = a^2 + b^2 - 2ab \cos C
\]

The notation is explained in the figure below.

The Law of Sines:

This law states, for any triangle, that the ratio of the sine of an angle to the length of the side opposite the angle does not depend on which vertex of the triangle we use. That is, see the figure above:

\[
\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}
\]

A proof of this formula is relatively easy and proceeds as follows. Draw a perpendicular from the vertex with angle \(C\) to the side opposite this vertex. Call the length of this line segment \(h\).

We then have

\[
\sin A = \frac{h}{b} \quad \text{and} \quad \sin B = \frac{h}{a}.
\]

Solving both equations for \(h\) and setting the resulting two expressions for \(h\) equal to each other we have the equation

\[
b \sin A = a \sin B,
\]

which leads to

\[
\frac{\sin A}{a} = \frac{\sin B}{b}.
\]

To derive the other equality(ies) drop a perpendicular from a different vertex. While the picture changes if angle \(B\) is obtuse (> \(\pi/2\)) the argument is the same as in this case, where angle \(B\) is acute (< \(\pi/2\)).
Example 1: A triangle has two sides of length 3 inches and 5 inches with the angle opposite the side of length 5 inches equal to 25°. Determine the length of the third side.

Solution: The first step is to sketch and label a triangle, which models this data.

The ratio of the sine to the length of the opposite side is \( R = \frac{\sin 25^\circ}{5} = 8.4524 \times 10^{-2} \). Knowledge of this ratio enables us to determine angle \( B \). \( \sin B = 3 \left( \frac{\sin 25^\circ}{5} \right) = 3 \times 8.4524 \times 10^{-2} = 0.25357 \). There are two angles between 0 and \( \pi \) whose sine equals this number, one of which is less than \( \pi/2 \) and the second, which is greater than \( \pi/2 \). Since \( b \) is less than \( c \), angle \( B \) cannot be greater than \( \pi/2 \). Thus, \( B = \arcsin 0.25357 = 0.25637 \) radians, or \( 0.25637 \times \left( \frac{180}{\pi} \right) = 14.689 \) degrees, and angle \( A \) equals \( 180^\circ - 25^\circ - 14.689^\circ = 140.31^\circ \), and its sine equals 0.63863. Finally we are able to determine \( a \).

\[
\frac{\sin A}{a} = \frac{\sin C}{c} \\
\frac{a}{\sin(\sin C/c)} \approx \frac{0.63863}{8.4524 \times 10^{-2}} \approx 7.56 \text{ inches}
\]

It’s clear from the answer that angle \( A \) is greater than 90°, and that the picture drawn is not a good representation of the triangle in question. Note that in the process of determining the length of the unknown side we have solved the triangle.

The Law of Cosines:
This law is a generalization of the Pythagorean Theorem for right triangles. If \( a, b, c \) are the lengths of the sides of any triangle and \( C \) is the angle opposite the side of length \( c \) (equivalently \( C \) is the angle included by the sides with lengths \( a \) and \( b \)), we have

\[
c^2 = a^2 + b^2 - 2ab \cos C.
\]

Note that if \( C \) is 90° degrees, then \( \cos C = 0 \), and we have the Pythagorean Theorem.

The following figure will be used in the proof of the Law of Cosines. Note that the line of length \( h \) from vertex \( C \) is perpendicular to the side of length \( c \).

Verify the following equations.
The following proof of the Law of Cosines is a bit more complicated than it needs to be (see the exercises for a simpler derivation of this law), and it depends upon the fact that angle $B$ is an acute angle. However, our goal here is two fold: to use previous identities and definitions, and to prove the Law of Cosines. See the exercises for a proof of the case when $B$ is obtuse.

$$c^2 = (c_1 + c_2)^2 = c_1^2 + 2c_1c_2 + c_2^2$$
$$= b^2\sin^2\theta_1 + 2ab\sin\theta_1\sin\theta_2 + a^2\sin^2\theta_2$$
$$= b^2(1 - \cos^2\theta_1) + 2ab\sin\theta_1\sin\theta_2 + a^2(1 - \cos^2\theta_2)$$
$$= a^2 + b^2 - a^2\cos^2\theta_2 - b^2\cos^2\theta_1 + 2ab\sin\theta_1\sin\theta_2$$
$$= a^2 + b^2 - a^2\cos^2\theta_2 - b^2\cos^2\theta_1 + 2ab(\cos\theta_1\cos\theta_2 - \cos C)$$
$$= a^2 + b^2 - 2ab\cos C - (a^2\cos^2\theta_2 - 2ab\cos\theta_1\cos\theta_2 + b^2\cos^2\theta_1)$$
$$= a^2 + b^2 - 2ab\cos C - (a\cos\theta_2 - b\cos\theta_1)^2$$
$$= a^2 + b^2 - 2ab\cos C.$$ 

**Example 2:** A triangle has two sides of length 3 inches and 5 inches with the angle included between the sides of known length equal to 25°. Determine the length of the third side. Note, the difference between this example and the previous example is the relation of the known angle to the known sides.

![Diagram of a triangle with sides b = 3, a, and c = 5, and angle A = 25°](image)

**Solution:** Using the law of cosines we have:

$$a^2 = b^2 + c^2 - 2bc\cos A$$
$$\approx 9 + 25 - 30(0.90631)$$
$$\approx 6.8107$$

Thus,

$$a \approx \sqrt{6.8107} \approx 2.6097$$
Example 3: Solve the triangle given in Example 2.

Solution: We know the following about this triangle

\[ a = 2.6097, \ b = 3, \ c = 5 \]
\[ A = 25^\circ, \ B = ?, \ C = ? \]

To determine angles \( B \) and \( C \) we’ll use the Law of Sines. To this end we first compute the ratio \( \frac{\sin A}{a} \approx 0.1619 \)

\[ \sin B = b \frac{\sin A}{a} \approx 3(0.1619) \approx 0.4857 \]
\[ \sin C = c \frac{\sin A}{a} \approx 5(0.1619) \approx 0.8095 \]

Thus, we have

\[ B \approx \arcsin 0.4857 \approx 29.058^\circ \]
\[ C \approx \arcsin 0.8095 \approx 53.697^\circ \]

As a check let’s make sure that the sum of the angles equals 180°.

\[ A + B + C = 25 + 29.058 + 53.697 = 107.76^\circ \]

This is not good. So where did we go wrong? See the exercises.
Exercises for Chapter 8G - Law of Sines and Law of Cosines

1. Prove the Law of Sines for the case when angle \( B \) is not less than \( \pi/2 \). There are two cases here: \( B = \pi/2 \) and \( B > \pi/2 \).

2. Find a simpler derivation of the law of cosines. Hint, place vertex \( C \) at the origin, vertex \( A \) on the \( x \)-axis, with vertex \( B \) in the first quadrant, then use the distance formula to calculate \( c \).

3. We did not show in the Law of Sines proof that \( \sin A/a = \sin C/c \). Using the picture in the proof, drop a perpendicular from \( B \) to the opposite side and show that \( \sin A/a = \sin C/c \).

4. Let \( \triangle ABC \) be such that \( a = 2, b = 5, c = 15 \). Find angle \( B \).

5. Let \( \triangle ABC \) be such that \( a = 12, A = 60^\circ \), and \( B = 100^\circ \). Solve this triangle.

6. Let \( \triangle ABC \) be such that \( a = 2, b = 5, c = 4 \). Find angle \( B \).

7. Let \( \triangle ABC \) be such that \( b = 3, c = 5 \), and \( B = 25^\circ \). Solve this triangle. (Be careful)

8. Let \( \triangle ABC \) be such that \( a = 3, b = 4 \), and \( c = 6 \). Solve this triangle.

9. Find where we went wrong in Example 3, and fix it.

10. Let \( \triangle ABC \) be such that \( a = 5, b = 5\sqrt{2} \), and \( A = 30^\circ \). Solve this triangle.

11. Let \( \triangle ABC \) be such that \( a = 5, b = 5\sqrt{2} \), and \( C = 30^\circ \). Solve this triangle.
Answers to Exercises for Chapter 8G - Law of Sines and Law of Cosines

1. Prove the Law of Sines for the case when angle $B$ is not less than $\pi/2$.

   The first case is when $B = \pi/2$, and we have (See the figure below)
   \[
   \frac{\sin A}{a} = \frac{a/b}{b} = \frac{1}{b} \\
   \frac{\sin C}{c} = \frac{\cos A}{c} = \frac{c/b}{c} = \frac{1}{b} \\
   \frac{\sin B}{b} = \frac{1}{b}
   \]

   The second case occurs if $B > \pi/2$. From the picture below we have:
   \[
   h = b \sin A \\
   h = a \sin(\pi - B) = a \sin B
   \]
   Setting these two expressions for $h$ equal to one another we have
   \[
   \frac{\sin A}{b} = \frac{\sin B}{a}
   \]
   If we also draw a perpendicular from $B$ to the opposite side we have
   \[
   h' = c \sin A = a \sin C
   \]
   These equations give us
   \[
   \frac{\sin A}{a} = \frac{\sin C}{c}
   \]

2. Find a simpler derivation of the law of cosines. Hint, place vertex $C$ at the origin, vertex $A$ on the $x$-axis, with vertex $B$ in the first quadrant, then use the distance formula to calculate $c$. 

   \[
   B (a \cos(C), a \sin(C)) \\
   C (0, 0) b A (b, 0)
   \]
\[ c^2 = (a \cos C - b)^2 + a^2 \sin^2 C \]
\[ = a^2 \cos^2 C - 2ab \cos C + b^2 + a^2 \sin^2 C \]
\[ = a^2 + b^2 - 2ab \cos C. \]

3. We did not show in the Law of Sines proof that \( \frac{\sin A}{a} = \frac{\sin C}{c} \). Using the picture in the proof, drop a perpendicular from \( B \) to the opposite side and show that \( \frac{\sin A}{a} = \frac{\sin C}{c} \).

\[ \text{We have the following: } \sin A = \frac{h}{c} \text{ and } \sin C = \frac{h}{a}. \text{ These lead to the equations } h = c \sin A = a \sin C, \text{ from which we have } \]
\[ \frac{\sin A}{a} = \frac{\sin C}{c}. \]

4. Let \( \triangle ABC \) be such that \( a = 2, b = 5, c = 15 \). Find angle \( B \).

From the Law of Cosines we have
\[ b^2 = a^2 + c^2 - 2ac \cos B. \]
Thus,
\[ \cos B = \frac{a^2 + c^2 - b^2}{2ac} = \frac{204.0}{60}. \]

The cosine of an angle bigger than 1, not possible. Clearly there is no triangle with the given sides. Try to draw one.

5. Let \( \triangle ABC \) be such that \( a = 12, A = 60^\circ \), and \( B = 100^\circ \). Solve this triangle.

Angle \( C = 180 - 60 - 100 = 20 \). Now use the Law of Sines to compute the lengths \( b \) and \( c \).
\[ b = \frac{a \sin B}{\sin A} = \frac{12 \sin 100}{\sin 60} \approx 13.646 \]
\[ c = \frac{a \sin C}{\sin A} \approx 4.7392 \]

6. Let \( \triangle ABC \) be such that \( a = 2, b = 5, c = 4 \). Find angle \( B \).

From the Law of Cosines we have
\[ b^2 = a^2 + c^2 - 2ac \cos B. \]
Thus,
\[ \cos B = \frac{a^2 + c^2 - b^2}{2ac} = \frac{4 + 16 - 25}{16} = -0.3125 \]
\[ B = \arccos(-0.3125) \approx 108.21^\circ \]

7. Let \( \triangle ABC \) be such that \( b = 3, c = 5, \) and \( B = 25^\circ \). Solve this triangle. (Be careful)

The common ratio of \( \frac{\sin \text{angle}}{\text{length of opposite side}} \) equals \( \frac{\sin B}{b} \approx 0.14087. \) Thus, we have
\[ \frac{\sin C}{c} = 0.14087, \text{ or } \]
\[ \sin C \approx 5(0.14087) = 0.70435 \]
\[ C \approx \arcsin(0.70435) \approx 44.777^\circ. \]

This is where we need to be careful. There are two angles between 0 and \( \pi \) whose sine equals 0.70435. One is the \( C \) computed above, and the other is \( \pi - C \). It may be possible that the given data
does not uniquely determine a triangle, which is indeed the case here; there are two different triangles that match the given data. Let’s see what happens.

Case $C \approx 44.777^\circ$.

In this case we have $A = 180 - 25 - 44.777 = 110.22^\circ$, and

$$a = \sqrt{b^2 + c^2 - 2bc \cos A}$$

$$\approx \sqrt{9 + 25 - 30 \cos(110.22)}$$

$$\approx \sqrt{44.369} \approx 6.661$$

The figure below shows this triangle.

![Triangle with angles and sides labeled]

Case $C \approx 135.22^\circ$

$A$ now equals $19.78^\circ$ and $a$ equals

$$a \approx \sqrt{9 + 25 - 30 \cos(19.78)}$$

$$\approx \sqrt{5.77} \approx 2.4021$$

This triangle is shown below.

![Triangle with angles and sides labeled]

8. Let $\triangle ABC$ be such that $a = 3$, $b = 4$, and $c = 6$. Solve this triangle.

This is pretty straight forward. Use the Law of Cosines to find the three angles of this triangle.

$$A = \arccos \left( \frac{b^2 + c^2 - a^2}{2bc} \right) = \arccos \left( \frac{16 + 36 - 9}{48} \right) \approx 26.385^\circ$$

$$B = \arccos \left( \frac{a^2 + c^2 - b^2}{2ac} \right) = \arccos \left( \frac{9 + 36 - 16}{36} \right) \approx 36.336^\circ$$

$$C = \arccos \left( \frac{a^2 + b^2 - c^2}{2ab} \right) = \arccos \left( \frac{9 + 16 - 36}{24} \right) \approx 117.28^\circ$$

As a check we note that the sum of these three angles equals $180.001$.

9. Find where we went wrong in Example 3, and fix it.

It Example 3 the mistake is the assumption that angles $B$ and $C$ are both acute angles. A few minutes thought about the shape of the triangle seems to indicate that $C$ is an obtuse angle. From the work in Example 3, we have that $\sin C \approx 0.8095$, and the acute angle with this value of sine is $53.697^\circ$. Thus, the obtuse angle is $180 - 53.697 = 126.3^\circ$.

Using the previously computed value of $B$, let’s check the sum of these angles

$$A + B + C = 25 + 29.058 + 126.3 = 180.36$$

This is much better. As an additional check let’s use the Law of Cosines to compute $C$. 

© WEBA1G : Pre-Calculus
Chapter 8H - Miscellaneous Exercises

1. Find all angles \( \theta \) between 0 and \( 2\pi \) for which \( \sin \theta = 0.15 \).
2. Find all angles \( \theta \) between 0 and \( 2\pi \) for which \( \cos \theta = 0.15 \).
3. Find all angles \( \theta \) between 0 and \( 2\pi \) for which \( \tan \theta = 1.4 \).
4. If \( 0 \leq x \leq \frac{1}{4} \), what does \( \sin(\cos^{-1}x) \) equal?
5. If \( 0 \leq x \leq \frac{1}{5} \), what does \( \cos(\cos^{-1}x) \) equal?
6. If \( 0 \leq x \leq \frac{1}{6} \), what does \( \cos(\sin^{-1}x) \) equal?
7. If \( 0 \leq x \leq \frac{1}{7} \), what does \( \sin(\sin^{-1}x) \) equal?
8. Solve the equation \( 2 \sin x + \sin 2x = 0 \).
9. Solve the equation \( \cos x \tan x = \tan x \).
10. Solve the equation \( 2 \sin x \cos x - \sqrt{3} \sin x + 2 \cos x - \sqrt{3} = 0 \) on the interval \([0, 2\pi]\).
11. Solve the equation \( 2 \sin x \tan x - \tan x - 2 \sin x + 1 = 0 \) on the interval \([0, 2\pi]\).
12. Solve the equation \( \sin x + \cos x = 0 \) on the interval \([0, 2\pi]\).
13. Solve the equation \( \sin x \cos x = 0 \) on the interval \([0, 2\pi]\).
14. Solve the equation \( \sin x \cos x = \frac{1}{4} \) on the interval \([0, 2\pi]\).
15. Solve the equation \( \sin^2 x - \cos^2 x = \frac{1}{2} \) on the interval \([0, 2\pi]\).
16. Solve the equation \( 2 \sin x \cos x + \sin x \cos x = \frac{-1}{\sqrt{2}} \) on the interval \([0, 2\pi]\).
17. Solve the equation \( \sin^2 x + \cos^2 x = \frac{1}{2} \) on the interval \([0, 2\pi]\).
18. Solve the equation \( \cos 2x \cos x + \sin 2x \sin x = 1 \) on the interval \([0, 2\pi]\).
Chapter 8H - Solutions to Miscellaneous Exercises

1. Find all angles $\theta$ between 0 and $2\pi$ for which $\sin \theta = 0.15$.
   Since $\sin \theta$ is positive, $\theta$ must lie between 0 and $\pi$, and there are two values of $\theta$, which satisfy this condition. They are
   
   $\theta = \arcsin 0.15 \approx 0.15057$ radians, and
   
   $\theta \approx \pi - 0.15057 \approx 2.991$ radians.

2. Find all angles $\theta$ between 0 and $2\pi$ for which $\cos \theta = 0.15$.
   Since $\cos \theta$ is positive, $\theta$ must lie between 0 and $\pi/2$ or between $3\pi/2$ and $2\pi$. There are two such $\theta$ and they are
   
   $\theta = \arccos 0.15 \approx 1.4202$ radians, and
   
   $\theta \approx 2\pi - 1.4202 \approx 4.8630$ radians.

3. Find all angles $\theta$ between 0 and $2\pi$ for which $\tan \theta = 1.4$.
   $\tan \theta$ is positive, which implies that $\theta$ lies between $\pi/2$ and $\pi$ or between $3\pi/2$ and $2\pi$. Thus,
   
   $\theta = \arctan 1.4 \approx 0.95055$ radians, or
   
   $\theta \approx \pi + 0.95055 \approx 4.0921$ radians.

4. If $0 \leq x \leq \frac{1}{4}$, what does $\sin(\cos^{-1}4x)$ equal?
   If we set $\theta = \cos^{-1}4x$, then $\cos \theta = 4x$. Thus, $\sin(\cos^{-1}3x) = \sin \theta = \sqrt{1 - 16x^2}$. (Draw the right triangle with one of the angles equal to $\theta$.)

5. If $0 \leq x \leq \frac{1}{5}$, what does $\cos(\cos^{-1}5x)$ equal?
   $\cos(\cos^{-1}5x) = 5x$.

6. If $0 \leq x \leq \frac{1}{6}$, what does $\cos(\sin^{-1}6x)$ equal?
   Set $\theta = \sin^{-1}6x$. Then $\sin \theta = 6x$, and
   
   $\cos(\sin^{-1}6x) = \cos \theta = \sqrt{1 - 36x^2}$.

7. If $0 \leq x \leq \frac{1}{7}$, what does $\sin(\sin^{-1}7x)$ equal?
   $\sin(\sin^{-1}7x) = 7$.

8. Solve the equation $2\sin x + \sin 2x = 0$.
   
   $0 = 2\sin x + \sin 2x = 2\sin x + 2\sin x \cos x$
   
   $= 2\sin x(1 + \cos x)$.
   
   Thus, we must have $\sin x = 0$ or $1 + \cos x = 0$.
   
   Case $\sin x = 0$: This implies that $x = 0, \pm\pi, \pm2\pi, \ldots$.
   
   Case $1 + \cos x = 0$: If $\cos x = -1$, then $x = \pm\pi, \pm3\pi, \pm5\pi, \ldots$. 

© WeB.ArG : Pre-Calculus
9. Solve the equation $\cos x \tan x = \tan x$.
   If $\tan x \neq 0$, then $\cos x = 1$.
   Case $\tan x = 0$: in this situation $x = 0, \pm \pi, \pm 2\pi, \ldots$.
   Case $\cos x = 1$: here we have $x = 0, \pm 2\pi, \pm 4\pi, \ldots$.

10. Solve the equation $2 \sin x \cos x - \sqrt{3} \sin x + 2 \cos x - \sqrt{3} = 0$ on the interval $[0, 2\pi]$.
    
    $$0 = 2 \sin x \cos x - \sqrt{3} \sin x + 2 \cos x - \sqrt{3} = 0 = 2 \cos x (1 + \sin x) - \sqrt{3} \sin x - \sqrt{3}$$
    
    $$= 2 \cos x (1 + \sin x) - \sqrt{3} (\sin x + 1) = (1 + \sin x)(2 \cos x - \sqrt{3}).$$
    
    Thus, either $1 + \sin x = 0$ or $2 \cos x - \sqrt{3} = 0$.
    
    Case $1 + \sin x = 0$: this implies that $x = \frac{3\pi}{2}$.
    
    Case $2 \cos x - \sqrt{3} = 0$: here $\cos x = \frac{\sqrt{3}}{2}$. Hence $x = \frac{\pi}{6}$, or $\frac{11\pi}{6}$.

11. Solve the equation $2 \sin x \tan x - \tan x - 2 \sin x + 1 = 0$ on the interval $[0, 2\pi]$.
    
    $$0 = 2 \sin x \tan x - \tan x - 2 \sin x + 1 = 2 \sin x (\tan x - 1) - (\tan x - 1)$$
    
    $$= (2 \sin x - 1)(\tan x - 1).$$
    
    Case $2 \sin x - 1 = 0$. From $\sin x = \frac{1}{2}$, we have $x = \frac{\pi}{6}$, or $\frac{5\pi}{6}$.
    
    Case $\tan x - 1 = 0$. $\tan x = 1$ implies that $x = \frac{\pi}{4}$, or $\frac{5\pi}{4}$.

12. Solve the equation $\sin x + \cos x = 0$ on the interval $[0, 2\pi]$.
    
    If either $\cos x$ or $\sin x$ is zero, then the other term must also equal 0, since there is no value of $x$ for which both $\sin x$ and $\cos x$ are zero, we can divide this equation by $\cos x$ (or $\sin x$) and derive $\tan x = -1$. Thus, $x = \frac{3\pi}{4}$, or $x = \frac{7\pi}{4}$.

13. Solve the equation $\sin x \cos x = 0$ on the interval $[0, 2\pi]$.
    
    Either $\sin x = 0$ or $\cos x = 0$. If it’s the former, then $x = 0$ or $\pi$. If it’s the latter then $x = \frac{\pi}{2}$ or $\frac{3\pi}{2}$.

14. Solve the equation $\sin x \cos x = \frac{1}{4}$ on the interval $[0, 2\pi]$.
    
    $$\frac{1}{4} = \sin x \cos x = \frac{1}{2} \sin 2x$$
    
    $$\frac{1}{2} = \sin 2x.$$ 
    
    Thus $2x = \frac{\pi}{6}$, $\frac{5\pi}{6}$, $\frac{13\pi}{6}$, $\frac{17\pi}{6}$, and $x = \frac{\pi}{12}$, $\frac{5\pi}{12}$, $\frac{13\pi}{12}$, and $\frac{17\pi}{12}$.

15. Solve the equation $\sin^2 x - \cos^2 x = \frac{1}{2}$ on the interval $[0, 2\pi]$.
    
    $$\frac{1}{2} = \sin^2 x - \cos^2 x = 1 - 2 \cos^2 x$$
    
    $$\cos^2 x = \frac{1}{4}$$
    
    $$\cos x = \pm \frac{1}{2}.$$
    
    If $\cos x = \frac{1}{2}$, then $x = \frac{\pi}{3}$ or $\frac{5\pi}{3}$, if $\cos x = -\frac{1}{2}$, then $x = \frac{2\pi}{3}$ or $\frac{4\pi}{3}$.
16. Solve the equation \( \sin 2x \cos x + \sin x \cos 2x = \frac{-1}{\sqrt{2}} \) on the interval \([0, 2\pi]\).

The expression \( \sin 2x \cos x + \sin x \cos 2x \) equals \( \sin 3x \). If \( x \) is in \([0, 2\pi]\), then \( 3x \) is in \([0, 6\pi]\).

The solutions to \( \sin \theta = -1/\sqrt{2} \) for \( \theta \) in \([0, 2\pi]\) are \( \theta = \frac{5\pi}{4}, \frac{7\pi}{4}, \frac{13\pi}{4}, \frac{15\pi}{4}, \frac{21\pi}{4}, \frac{23\pi}{4} \).

Thus, \( x = \theta/3 \) must equal

\[
\frac{5\pi}{12}, \frac{7\pi}{12}, \frac{13\pi}{12}, \frac{5\pi}{12}, \frac{21\pi}{12}, \frac{23\pi}{12}.
\]

17. Solve the equation \( \sin^2 x + \cos^2 x = \frac{1}{2} \) on the interval \([0, 2\pi]\).

Since \( \sin^2 x + \cos^2 x \) equals 1 for all \( x \), this equation has no solution.

18. Solve the equation \( \cos 2x \cos x + \sin 2x \sin x = 1 \) on the interval \([0, 2\pi]\).

\[
1 = \cos 2x \cos x + \sin 2x \sin x = \cos(2x - x) = \cos x.
\]

Thus, we have \( x = 0 \).