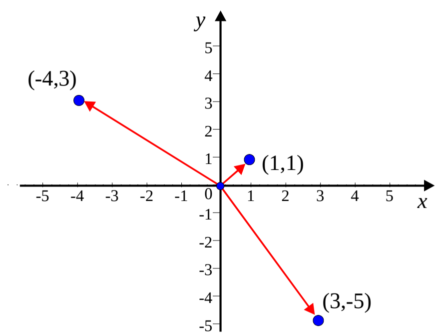
Chapter 9A - A Brief Description

As life becomes more complicated, entities arise that not only have a ***magnitude***, but also a ***direction*** associated with them. Such objects are called ***vectors***. A common example of this occurs when you give someone directions to get to your home: go 5 blocks west, then 16 blocks north, and my house is on the corner. There are two vectors here. The first has a magnitude of 5 units and a direction, west; the second has a magnitude of 16 and a direction, north. These verbal descriptions of a vector are too cumbersome, and a better notation has been invented. It is the same notation which is used to locate points in the plane, R^2 or in three-space, R^3.

Representing Vectors.

Example 1: A vector which is one unit long and points due north can be represented by the ordered pair $\langle 1, 0\rangle$, and a vector which has magnitude 5 and points **northwest** is $\left\langle -\frac{5}{\sqrt{2}}, \frac{5}{\sqrt{2}} \right\rangle$

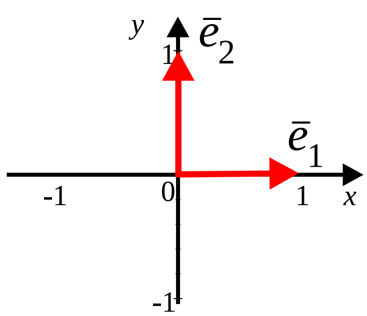
The individual values are called ***components***. Since these vectors have only two components, we think of them as belonging to or lying in a plane. We often picture vectors as arrows. The tail of the arrow is at the origin and the arrow tip is located at the point whose Cartesian coordinates are the same as the ordered pair (triple) which represents the vectors. The vectors $\langle 1,1\rangle$, $\langle -4,3\rangle$, and $\langle 3,-5\rangle$ are plotted below.



Two dimensions

In R^2 (the plane) there are two special vectors. The two that are parallel to the coordinate axes. They are commonly denoted by the symbols $\vec{i}$ and $\vec{j}$ or by the pair of symbols $\vec{e}_1$ and $\vec{e}_2$. That is,

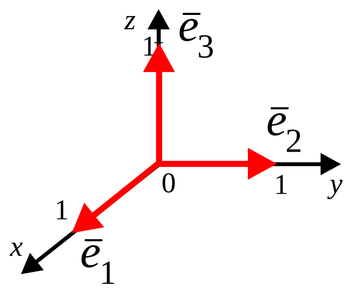
$$\vec{e}_1 = \vec{i} = \langle 1,0\rangle$$
$$\vec{e}_2 = \vec{j} = \langle 0,1\rangle$$



Three dimensions

In R^3 we have three special vectors which point in directions parallel to the coordinate axes.

$$\vec{e}_1 = \vec{i} = \langle 1,0,0\rangle$$
$$\vec{e}_2 = \vec{j} = \langle 0,1,0\rangle$$
$$\vec{e}_3 = \vec{k} = \langle 0,0,1\rangle$$



There is an ambiguity in this notation. What does $\vec{e}_2$ mean? Is it the ordered pair of numbers $\langle 0,1\rangle$ or the ordered triple of numbers $\langle 0,1,0\rangle$. The answer is determined by the context of the particular discussion in which this symbol appears.

Coordinate vectors

These vectors $\{\vec{e}_1, \vec{e}_2\}$ are sometimes referred to as the coordinate vectors. That is, in R^2 the standard coordinate vectors are $\vec{e}_1$ and $\vec{e}_2$.

Question: What are the coordinate vectors of R^3?

Answer: The coordinate vectors in R^3 are:

$$\vec{e}_1,\ \vec{e}_2,\ \text{and } \vec{e}_3$$

Example 2: Suppose you tell someone the way to get to your house is to go 7 blocks **east**, then 1 block south, and finally 4 blocks east. One way to indicate this in vector notation is

$$\langle 7,0\rangle + \langle 0,-1\rangle + \langle 4,0\rangle$$

Or using the coordinate vectors

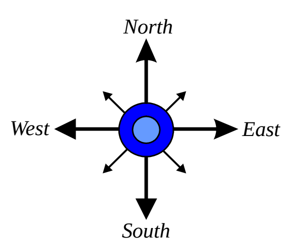
$$7\vec{e}_1 - \vec{e}_2 + 4\vec{e}_1 = 11\vec{e}_1 - \vec{e}_2$$

Example 3: Suppose you were given the following directions:

$$\langle 1,1\rangle + \langle 0,-4\rangle + \langle 3,0\rangle + \langle 0,-2\rangle.$$

The English translation of the above would be go north-east, square root of 2 units, then south, 4 units, then east, three more units and finally south 2 units. Where would you be in the plane?

Answer: You would be 4 units east and 5 units south from the starting point. The coordinates are $\langle 4,-5\rangle$. In vector notation: $\langle 1,1\rangle + \langle 0,-4\rangle + \langle 3,0\rangle + \langle 0,-2\rangle = \langle 4,-5\rangle$



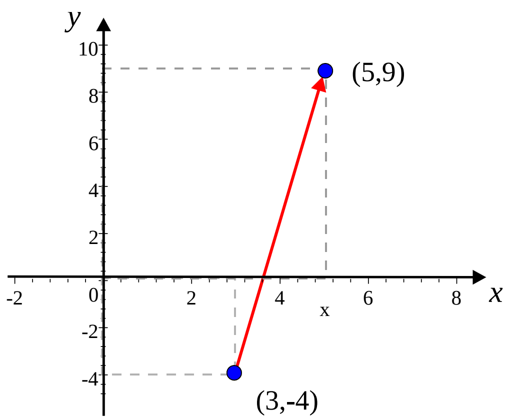
Another use of vectors is to show a change of position. For example if something moves from point $P = (a, b)$ to a point $Q = (c, d)$, this change in postion is commonly denoted by the vector $\overrightarrow{PQ} = Q - P = \langle c - a, d - b \rangle$.

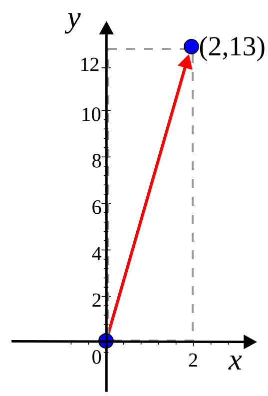
Example 4: Suppose you trace a curve which starts at the point $(3, -4)$ and ends at the point $(5, 9)$. What vector describes the resultant of this motion?

Solution: Take the ending position and subtract the starting position from it. The desired vector is

$$\langle 5 - 3, 9 - (-4) \rangle = \langle 2, 13 \rangle.$$

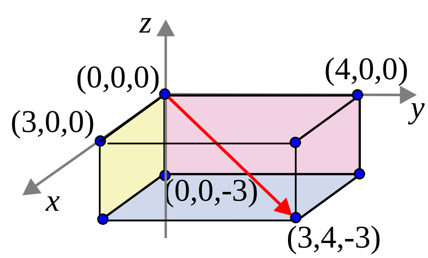
The following pictures give a graphic representation of this vector. The first is the actual vector; the second is the way to represent it. It is usually correct to represent any vector with originating point at the origin.





Example 5: Represent the vector from $P = (1, 3, 2)$ to $Q = (4, 7, -1)$

Solution: First compute the difference $\langle 4 - 1, 7 - 3, -1 - 2 \rangle = \langle 3, 4, -3 \rangle$. The picture is shown below.



Coordinates of the vector (3,4,-3)

Exercises for Chapter 9A - A Brief Description

1. Find a vector which represents moving from the point P to the point Q for each of the following pairs of points. For a - d, sketch the vector.

a. $P = (1,1)$, $Q = (5,-3)$

b. $P = (-3,7)$, $Q = (-8,-12)$

c. $P = (-2,7,4)$, $Q = (14,6,10)$

d. $P = (19,5,23)$, $Q = (13,-4,17)$

e. $P = (3,0,-6,3)$, $Q = (-4,4,-5,0)$

2. A dog runs 4.5 miles in a direction 35^o degrees north of east. Assuming that the origin is the dog's starting point and that east is the positive x axis, what are the coordinates of the dog's location?

3. A flea is on a cat. If the cat goes 5 feet in a direction 40^o degrees (measured counter clockwise from the positive x axis) and the flea then gets off the cat and goes 1 foot in a direction of 90^o degrees with respect to the positive x axis, what are the locations of the flea and the cat?

4. If $\vec{x} = \langle 2,-5 \rangle$, what angle does $\vec{x}$ make with the positive x axis?

5. If an ant starts at the point $(1,2)$ and stops at the point $(-4,5)$, what vector represents the change of position of the ant?

6. If a vector starts at the point $(3,-7)$ and terminates at the point $(14,5)$, what is the tangent of the angle this vector makes with the positive x axis?

Answers to Exercises for Chapter 9A - A Brief Description

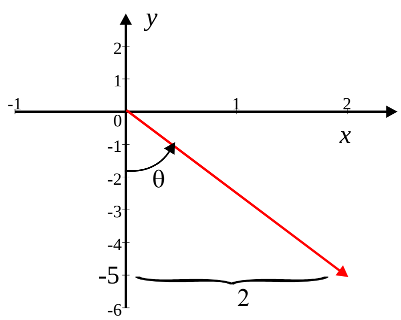
1.

- **a.** $\overrightarrow{PQ} = \langle 5-1, -3-1 \rangle = \langle 4, -4 \rangle$
- **b.** $\overrightarrow{PQ} = \langle -8-(-3), -12-7 \rangle = \langle -5, -19 \rangle$
- **c.** $\langle 14-(-2), 6-7, 10-4 \rangle = \langle 16, -1, 6 \rangle$
- **d.** $\langle 13-19, -4-5, 17-23 \rangle = \langle -6, -9, -6 \rangle$
- **e.** $\langle -4-3, 4-0, -5-(-6), 0-3 \rangle = \langle -7, 4, 1, -3 \rangle$

2. $35^o = \frac{\pi}{180}35 = .61087$. The x coordinate of the dog's position is $4.5\cos .61087 = 3.6862$ and the y coordinate is $4.5\sin .61087 = 2.5811$

3. The cat's location is given by the vector $\left\langle 5\cos\left(\frac{40\pi}{180}\right), 5\sin\left(\frac{40\pi}{180}\right)\right\rangle = \langle 3.8302, 3.2139 \rangle$. The location of the flea is found by adding the vector $\langle 0, 1 \rangle$ to the postion vector of the cat. So the flea's position is represented by the vector $\langle 3.8302, 4.2139 \rangle$.

4. The vector $\langle 2, -5 \rangle$ is sketched below. Then read below sketch.



Let θ be the angle the vector makes with the negative y-axis. Then $\tan\theta = \frac{2}{5}$. Hence, $\theta = \arctan\frac{2}{5} = .38051$ radians. or $0.38051\frac{180}{\pi} = 21.802°$ degrees. Thus, the angle the vector makes with the positive x-axis is $291.802°$ degrees.

5. $\langle -4-1, 5-2 \rangle = \langle -5, 3 \rangle$

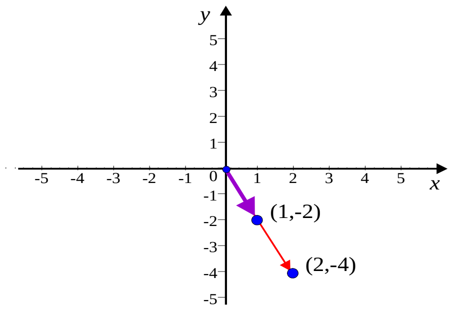
6. This vector equals $\langle 11, 12 \rangle$. If θ is the angle this vector makes with the positive x axis, then $\tan\theta = \frac{12}{11}$.

Chapter 9B - Scalar Multiplication

The operation of scalar multiplication takes a scalar, that is a real number, and multiplies a vector by it. Here is one instance: $2\langle 1,-2\rangle = \langle 2,-4\rangle$. Notice that we multiplied each component of the vector $\langle 1,-2\rangle$ by 2 to get the scalar product of the vector $\langle 1,-2\rangle$ and the number 2. This is the general formula

$$\alpha\langle x_1,x_2\rangle = \langle \alpha x_1,\alpha x_2\rangle,$$

where α represents an arbitrary real number and the symbols x_1 and x_2 also represent arbitrary real numbers. The pattern is the same if we are in R^3 or R^n. The picture below gives a geometrical interpretation of this operation.

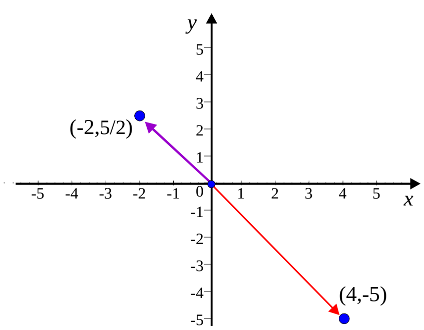


Scalar multiplication of vectors

Notice that the direction of the product is the same as the original vector $\langle 1,-3\rangle$, and that it has twice the length. If we multiply the vector by a negative number, then the resulting product has a direction 180 degrees opposite the original direction. In the future whenever a vector is a scalar multiple of another vector we will say the two vectors are parallel.

Example 1: Find the scalar multiple of $\langle 4,-5\rangle$ by the scalar $\alpha = -\frac{1}{2}$.

Solution: We have $(-\frac{1}{2})\langle 4,-5\rangle = \left\langle -2, \frac{5}{2}\right\rangle$ (See illustration below.)



Example 2: Find the scalar multiple of $\langle 12,-1\rangle$ by the scalar $\alpha = 2$.

Answer: $2\langle 12,-1\rangle = \langle 24,-2\rangle$

Example 3: Find the scalar multiple of $\langle -4,-6\rangle$ by the scalar $\alpha = -7$.

Answer: $-7\langle -4,-6\rangle = \langle 28,42\rangle$

A similar definition holds for vectors in R^n, that is

$$\alpha\langle x_1, x_2, \cdots, x_n\rangle = \langle \alpha x_1, \alpha x_2, \cdots, \alpha x_n\rangle.$$

Example 4: Find the scalar multiple of $\langle 2, 12, -3, -4\rangle$ by the scalar $\alpha = 3$.

Solution: Following the definition, we have $3\langle 2, 12, -3, -4\rangle = \langle 6, 36, -9, -12\rangle$, which is obtained by multiplying 3 by the individual components of the vector.

Exercises for Chapter 9B - Scalar Multiplication

1. If a woman was planing on walking 2 miles in a northern direction, but a companion persuades her to walk twice as far, how far will she walk and in which direction?

2. If the product of a vector $\vec{x}$ with the number 3 is equal to $\langle 4, 15\rangle$, what is $\vec{x}$?

3. Compute the following scalar products:

a. $5\langle 6, 13\rangle$

b. $-3\langle 13, 4, -7\rangle$

c. $2\langle 3, 12, 4, -5, 8\rangle$

4. Does the equation $\alpha\langle 3, 4\rangle = \langle -15, -20\rangle$ have a solution? If yes, find it.

5. Does the equation $\alpha\langle 3, 4\rangle = \langle -15, 20\rangle$ have a solution? If yes, find it.

6. For each of the following, mentally compute the answer, then check the answer.

a. $2\langle 3, -4, 5, 1\rangle$

b. $-1\langle 1, 2, 3, -5\rangle$

c. $7\langle 23, -2\rangle$

7. For each of the following draw the vector, and then draw the result of the scalar product.

a. $2\langle -1, -1\rangle$

b. $-3\langle -1, 2\rangle$

8. For each of the vectors below draw the first vector with base at the origin and the second with its base at the point $(2, 0)$. Note that the second vector is a scalar multiple of the first, and that the two vectors are parallel.

a. $\langle -1, -2\rangle$ and $-2\langle -1, -2\rangle$

b. $\langle 1, 1\rangle$ and $3\langle 1, 1\rangle$

9. Sketch the set of vectors which have the form $t\langle 1, 1\rangle$ for all values of t which lie between -2 and 1.

10. Is there a scalar α such that $\alpha\langle 1, 1\rangle = \langle 1, 2\rangle$?

Answers to Exercises for Chapter 9B - Scalar Multiplication

1. 4 miles in a northern direction.

2. We have $3\vec{x} = \langle 4, 15 \rangle$. Thus, $\vec{x} = \frac{1}{3}\langle 4, 15 \rangle = \langle \frac{4}{3}, 5 \rangle$.

3. Compute the following scalar products:

a. $5\langle 6, 13 \rangle = \langle 30, 65 \rangle$

b. $-3\langle 13, 4, -7 \rangle = \langle -39, -12, 21 \rangle$

c. $2\langle 3, 12, 4, -5, 8 \rangle = \langle 6, 24, 8, -10, 16 \rangle$

4. $\alpha\langle 3, 4 \rangle = \langle -15, -20 \rangle$ is true for $\alpha = -5$.

5. $\alpha\langle 3, 4 \rangle = \langle -15, 20 \rangle$ does not have a solution. The components of any multiple of $\langle 3, 4 \rangle$ will always have the same sign, or both components will be zero.

6. For each of the following, mentally compute the answer, then check the answer.

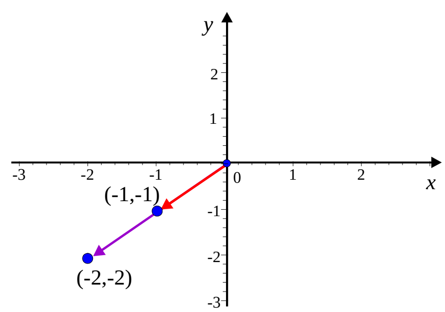
a. $2\langle 3, -4, 5, 1 \rangle = \langle 6, -8, 10, 2 \rangle$

b. $-1\langle 1, 2, 3, -5 \rangle = \langle -1, -2, -3, 5 \rangle$

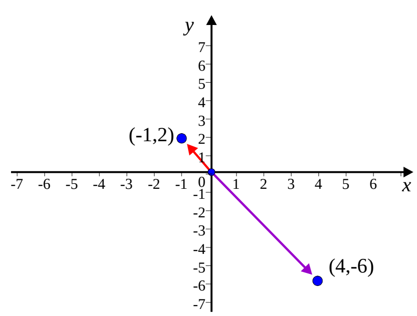
c. $7\langle 23, -2 \rangle = \langle 161, -14 \rangle$

7.

a.

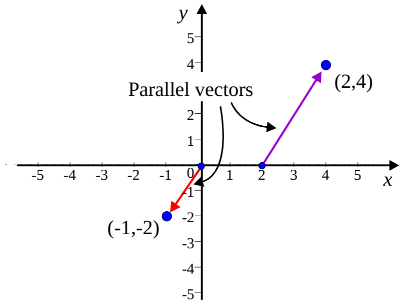


b. $-3\langle -1, 2 \rangle = \langle 3, -6 \rangle$

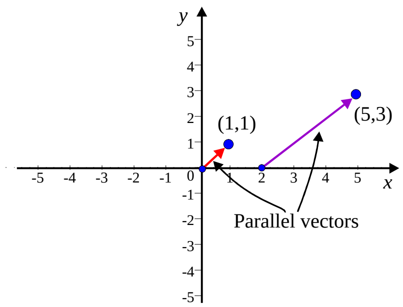


8.

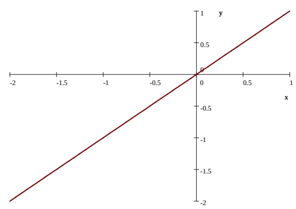
a.



b.



9.



$t\langle 1,1\rangle$ for $-2 \leq t \leq 1$

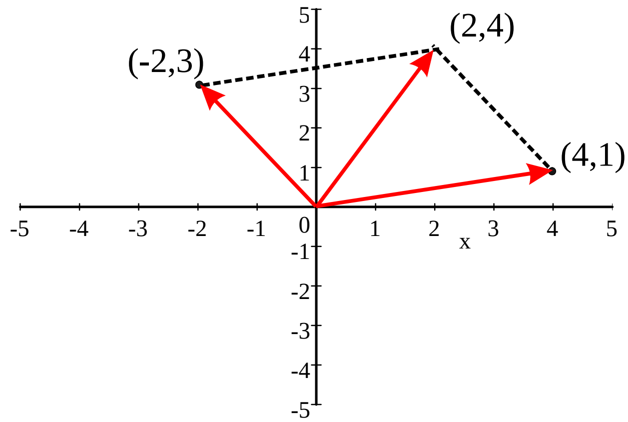
10. No, any vector of the form $\alpha\langle 1,1\rangle$ must have its components equal. The components of $\langle 1,2\rangle$ are not equal.

Chapter 9C - Vector Addition

The second algebraic operation is vector addition. This operation takes two vectors and adds them together to get a third vector. For example $\langle -2,3\rangle + \langle 4,1\rangle = \langle 2,4\rangle$. To get the components of the sum of these two vectors we add the corresponding components of the summands. Thus, $-2+4 = 2$ and $3+1 = 4$. This is the general rule as we state below

$$\langle x_1, x_2\rangle + \langle y_1, y_2\rangle = \langle x_1 + y_1, x_2 + y_2\rangle.$$

Here the symbols x_1, x_2, y_1, and y_2 represent arbitrary real numbers. This rule of addition is commonly referred to as the parallelogram law of addition. The reason for this is shown in the plot below where we sketch the sum of two vectors.



The sum of $\langle -2,3\rangle$ and $\langle 4,1\rangle$ is $\langle 2,4\rangle$.

Notice that the sum of these two vectors is the diagonal of the parallelogram formed by the two summands.

Example 1: Find the sum of $\langle 12,-1\rangle$ and $\langle 5,-2\rangle$.

Solution: We have, adding the individual components, $\langle 12+5, -1+(-2)\rangle = \langle 17,-3\rangle$

Example 2: Find the sum of $\langle 6,-2\rangle$ and $\langle -5,-2\rangle$.

Solution: $\langle 6+(-5), -2+(-2)\rangle = \langle 1,-4\rangle$

Addition of vectors in any number of dimensions

Vector addition is defined similarly for two vectors in R^n.

$$\langle x_1, x_2, \cdots, x_n\rangle + \langle y_1, y_2, \cdots, y_n\rangle = \langle x_1 + y_1, x_2 + y_2, \cdots, x_n + y_n\rangle.$$

The reason for these particular definitions of scalar multiplication and vector addition is that vectors are used to model physical objects, and experience has demonstrated that these definitions accurately model how these objects combine with each other. Think back to your first physics course where forces, velocities, and accelerations were described by vectors. A simpler model surfaces when we remember how vectors are used to give directions on moving from one location to another.

Be aware that addition of two vectors is only defined if the vectors have the same number of components. Thus, a sum of the form $\langle 2,7\rangle + \langle 1,1,1\rangle$ is not defined. The reason for not defining such an operation is that no one has seen a need for it.

Properties of the Algebraic Operations

We first list the algebraic properties which are possessed by the operations of vector addition and scalar multiplication. In the following $\vec{X}$, $\vec{Y}$, and $\vec{Z}$ represent arbitrary vectors in the same vector space. That is, all three vectors are in the same R^n; terms of the form α and β will represent arbitrary scalars.

Let $\vec{X}$, $\vec{Y}$, and $\vec{Z}$ represent arbitrary vectors and let α and β be scalars. Then,

1. $\vec{X}+\vec{Y} = \vec{Y}+\vec{X}$

2. $\left(\vec{X}+\vec{Y}\right)+\vec{Z} = \vec{X}+\left(\vec{Y}+\vec{Z}\right)$

3. $\alpha\left(\vec{X}+\vec{Y}\right) = \alpha\vec{X}+\alpha\vec{Y}$

4. $(\alpha+\beta)\vec{X} = \alpha\vec{X}+\beta\vec{X}$

5. $1\vec{X} = \vec{X}$

Proof of all five facts.

We will prove the truth of these equalities for vectors in R^2. The proofs for the general case in R^n are essentially the same. We let $\vec{X} = \langle x_1,x_2\rangle$, $\vec{Y} = \langle y_1,y_2\rangle$ and $\vec{Z} = \langle z_1,z_2\rangle$.

1. $\vec{X}+\vec{Y} = \langle x_1,x_2\rangle+\langle y_1,y_2\rangle = \langle x_1+y_1,x_2+y_2\rangle$

$= \langle y_1+x_1,y_2+x_2\rangle = \langle y_1,y_2\rangle+\langle x_1,x_2\rangle = \vec{Y}+\vec{X}.$

2. $\left(\vec{X}+\vec{Y}\right)+\vec{Z} = (\langle x_1,x_2\rangle+\langle y_1,y_2\rangle)+\langle z_1,z_2\rangle$

$= \langle x_1+y_1,x_2+y_2\rangle+\langle z_1,z_2\rangle$

$= \langle (x_1+y_1)+z_1,(x_2+y_2)+z_2\rangle$

$= \langle x_1+(y_1+z_1),x_2+(y_2+z_2)\rangle$

$= \langle x_1,x_2\rangle+\langle y_1+z_1,y_2+z_2\rangle$

$= \vec{X}+\left(\vec{Y}+\vec{Z}\right)$

3. $\alpha\left(\vec{X}+\vec{Y}\right) = \alpha(\langle x_1,x_2\rangle+\langle y_1,y_2\rangle) = \alpha\langle x_1+y_1,x_2+y_2\rangle$

$= \langle \alpha x_1+\alpha y_1,\alpha x_2+\alpha y_2\rangle$

$= \langle \alpha x_1,\alpha x_2\rangle+\langle \alpha y_1,\alpha y_2\rangle$

$= \alpha\langle x_1,x_2\rangle+\alpha\langle y_1,y_2\rangle$

$= \alpha\vec{X}+\alpha\vec{Y}$

4. $(\alpha+\beta)\vec{X} = (\alpha+\beta)\langle x_1,x_2\rangle = \langle(\alpha+\beta)x_1,(\alpha+\beta)x_2\rangle$

$= \langle \alpha x_1+\beta x_1,\alpha x_2+\beta x_2\rangle$

$= \langle \alpha x_1,\alpha x_2\rangle+\langle \beta x_1,\beta x_2\rangle$

$= \alpha\langle x_1,x_2\rangle+\alpha\langle y_1,y_2\rangle$

$= \alpha\vec{X}+\beta\vec{X}$

5. $1\vec{X} = 1\langle x_1,x_2\rangle = \langle 1x_1,1x_2\rangle = \langle x_1,x_2\rangle = \vec{X}$

Example 3: $\langle 2,-3,7\rangle+\langle 1,0,-5\rangle = \langle 3,-3,2\rangle = 3\vec{e}_1-3\vec{e}_2+2\vec{e}_3$

Example 4: $-3\langle 1,2\rangle+5\langle -1,4\rangle = \langle -8,14\rangle = -8\vec{e}_1+14\vec{e}_2$

Exercises for Chapter 9C - Vector Addition

1. Rewrite each of the following vectors in terms of the $\vec{i}$, $\vec{j}$, and $\vec{k}$ vectors.

a. $2\langle 3,-7\rangle - 5\langle 13,5\rangle$
b. $\langle 1,3\rangle + 5\langle -2,1\rangle - 7\langle 6,5\rangle$
c. $\langle 1,2,0\rangle + \langle 1,0,-4\rangle$
d. $-3\langle 3,-1,5\rangle + 2\langle 5,5,3\rangle$

2. Find all vectors $\vec{x}$ which satisfy the vector equation $2\vec{x} + 5\langle 7,8\rangle = \langle -3,7\rangle$.

3. Find all vectors $\vec{x}$ which satisfy the vector equation $2\vec{x} + 5\langle 1,-2,4\rangle = \langle 5,12,17\rangle$.

4. Sketch the set of vectors $s\langle 1,2\rangle + t\langle 0,1\rangle$ for all values of s and t which satisfy the constraints $0 \le s \le 1$ and $-1 \le t \le 0$.

5. Are there scalars α and β such that $\alpha\langle 1,2\rangle + \beta\langle -2,3\rangle = \langle 7,9\rangle$?

6. For each of the following, mentally compute the answer, then check your answer.

a. $\langle 3,-4,5,1\rangle + \langle 1,12,-4,5\rangle$
b. $\langle 1,2\rangle - \langle 7,15\rangle$
c. $\langle 23,-2\rangle + \langle -5,11\rangle$

7. For each of the following draw the individual vectors, and then draw the result of the vector addition.

a. $\langle 1,-1\rangle + \langle 2,5\rangle$
b. $\langle -1,2\rangle - \langle -2,-3\rangle$

8. For each of the following pairs $\vec{x}$ and $\vec{y}$ compute $\vec{x} + \vec{y}$, $\vec{x} - \vec{y}$, and $2\vec{x} + 3\vec{y}$.

a. $\vec{x} = \langle 1,2\rangle$, $\vec{y} = \langle -1,2\rangle$
b. $\vec{x} = \langle -3,2,5\rangle$, $\vec{y} = \langle 5,1,-4\rangle$
c. $\vec{x} = \langle 0,1,3,-7\rangle$, $\vec{y} = \langle 12,-3,2,9\rangle$

9. If a vector $\vec{x}$ is written as the sum of scalar multiples of vectors $\vec{a}$ and $\vec{b}$, i.e., $\vec{x} = \alpha\vec{a} + \beta\vec{b}$, we say that $\vec{x}$ is a linear combination of $\vec{a}$ and $\vec{b}$. For each of the following pairs of vectors $\vec{a}$ and $\vec{b}$, write the vector $\vec{i} = \langle 1,0\rangle$ as a linear combination of $\vec{a}$ and $\vec{b}$.

a. $\langle 1,1\rangle$ and $\langle 0,1\rangle$
b. $\langle 1,1\rangle$ and $\langle -1,1\rangle$
c. $\langle 2,5\rangle$ and $\langle -5,2\rangle$

Answers to Exercises for Chapter 9C - Vector Addition

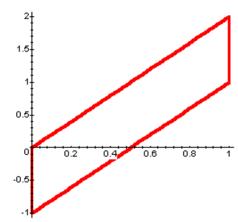
1.

a. $2\langle 3,-7\rangle - 5\langle 13,5\rangle = \langle -59,-39\rangle = -59i - 39j$
b. $\langle 1,3\rangle + 5\langle -2,1\rangle - 7\langle 6,5\rangle = \langle -51,-27\rangle = -51i - 27j$
c. $\langle 1,2,0\rangle + \langle 1,0,-4\rangle = \langle 2,2,-4\rangle = 2i + 2j - 4k$
d. $-3\langle 3,-1,5\rangle + 2\langle 5,5,3\rangle = \langle 1,13,-9\rangle = i + 13j - 9k$

2. $2\vec{x} + 5\langle 7,8\rangle = \langle -3,7\rangle$
$2\vec{x} = \langle -3,7\rangle - 5\langle 7,8\rangle = \langle -38,-33\rangle$
$\vec{x} = \frac{1}{2}\langle -38,-33\rangle = \langle -19,-\frac{33}{2}\rangle$

3. $2\vec{x} + 5\langle 1,-2,4\rangle = \langle 5,12,17\rangle$
$2\vec{x} = \langle 5,12,17\rangle - 5\langle 1,-2,4\rangle = \langle 0,22,-3\rangle$
$\vec{x} = \frac{1}{2}\langle 0,22,-3\rangle = \langle 0,11,-\frac{3}{2}\rangle$

4.



5. $\alpha\langle 1,2\rangle + \beta\langle -2,3\rangle = \langle 7,9\rangle$. Implies that $\alpha - 2\beta = 7$ and $2\alpha + 3\beta = 9$. The solution to this system is $\alpha = 39/7$, $\beta = -5/7$.

6.

a. $\langle 3,-4,5,1\rangle + \langle 1,12,-4,5\rangle = \langle 4,8,1,6\rangle$
b. $\langle 1,2\rangle - \langle 7,15\rangle = \langle -6,-13\rangle$
c. $\langle 23,-2\rangle + \langle -5,11\rangle = \langle 18,9\rangle$

7.

a. $\langle 1,-1\rangle + \langle 2,5\rangle = \langle 3,4\rangle$
b. $\langle -1,2\rangle - \langle -2,-3\rangle = \langle 1,5\rangle$

8. For each of the following pairs $\vec{x}$ and $\vec{y}$ compute $\vec{x}+\vec{y}$, $\vec{x}-\vec{y}$, and $2\vec{x}+3\vec{y}$.

a. $\vec{x}+\vec{y} = \langle 1,2\rangle + \langle -1,2\rangle = \langle 0,4\rangle$
$\vec{x}-\vec{y} = \langle 1,2\rangle - \langle -1,2\rangle = \langle 2,0\rangle$
$2\vec{x}+3\vec{y} = 2\langle 1,2\rangle + 3\langle -1,2\rangle = \langle -1,10\rangle$
b. $\vec{x}+\vec{y} = \langle -3,2,5\rangle + \langle 5,1,-4\rangle = \langle 2,3,1\rangle$
$\vec{x}-\vec{y} = \langle -3,2,5\rangle - \langle 5,1,-4\rangle = \langle -8,1,9\rangle$

$2\vec{x} + 3\vec{y} = 2\langle -3, 2, 5\rangle + 3\langle 5, 1, -4\rangle = \langle 9, 7, -2\rangle$

c. $\vec{x} + \vec{y} = \langle 0, 1, 3, -7\rangle + \langle 12, -3, 2, 9\rangle = \langle 12, -2, 5, 2\rangle$

$\vec{x} - \vec{y} = \langle 0, 1, 3, -7\rangle - \langle 12, -3, 2, 9\rangle = \langle -12, 4, 1, -16\rangle$

$2\vec{x} + 3\vec{y} = 2\langle 0, 1, 3, -7\rangle + 3\langle 12, -3, 2, 9\rangle = \langle 36, -7, 12, 13\rangle$

9.

a. $\langle 1, 0\rangle = \langle 1, 1\rangle - \langle 0, 1\rangle$

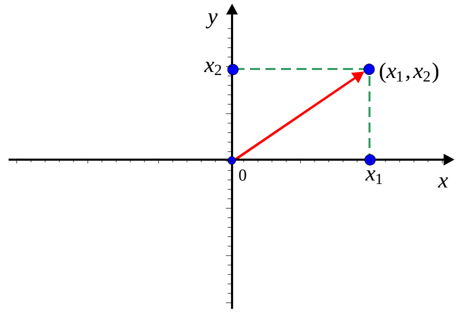
b. $\langle 1, 0\rangle = \frac{1}{2}\langle 1, 1\rangle - \frac{1}{2}\langle -1, 1\rangle$

c. $\langle 1, 0\rangle = \frac{2}{29}\langle 2, 5\rangle - \frac{5}{29}\langle -5, 2\rangle$

Chapter 9D - Length

The **length** of any vector $\langle x_1, x_2 \rangle$ in R^2 is defined to be the distance from the origin to the point which has the coordinates (x_1, x_2). That is,

$$||\langle x_1, x_2 \rangle|| = \sqrt{x_1^2 + x_2^2}$$



Thus, $||\langle 5, -1 \rangle|| = \sqrt{5^2 + 1^2} = \sqrt{26}$.

A similar formula for the length of a vector is also defined for vectors in R^n.

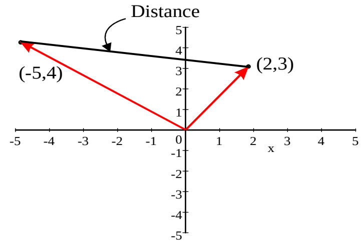
$$||\langle x_1, x_2, \cdots, x_n \rangle|| = \sqrt{x_1^2 + x_2^2 + \cdots x_n^2} \ .$$

$||\langle 2, -2, 4 \rangle|| = \sqrt{2^2 + (-2)^2 + 4^2} = 2\sqrt{6}$, and $||\langle 1, 1, 1, 1, 3 \rangle|| = \sqrt{1^2 + 1^2 + 1^2 + 3^2} = \sqrt{13}$.

The distance between two vectors $\vec{X}$ and $\vec{Y}$ is defined to be the length of the vector $\vec{X} - \vec{Y}$. Since the length of a vector is the same as the length of the negative of the vector, it doesn't matter if we compute $\left|\left|\vec{X} - \vec{Y}\right|\right|$ or $\left|\left|\vec{Y} - \vec{X}\right|\right|$. Thus, the distance between the vector $\langle 2, 3 \rangle$ and the vector $\langle -5, 4 \rangle$ is $||\langle 2, 3 \rangle - \langle -5, 4 \rangle|| = ||\langle 7, -1 \rangle|| = \sqrt{50}$.

Note: In case you were wondering why we use $|| \quad ||$ to denote the length of a vector, it is to remind ourselves that we are dealing with a vector and not a real number.

If we draw the triangle which has sides $\langle 2, 3 \rangle$, $\langle -5, 4 \rangle$, the third side is $\langle 7, -1 \rangle$, and we see that the distance between these two vectors is just the length of the third side $\langle 7, -1 \rangle$.



The distance between

Properties

Let $\vec{X}$ and $\vec{Y}$ represent arbitrary vectors and let α and β be scalars. Then,

1. $\left|\left|\vec{X}\right|\right| = 0$ if and only if $\vec{X}$ is the zero vector. (Note: The zero vector is the vector in which all of its components are zero.)

2. $\left|\left|\alpha\vec{X}\right|\right| = |\alpha|\left|\left|\vec{X}\right|\right|$ That is, the length of a rescaled vector is the same (in magnitude) rescaling of the length.

3. $\left|\left|\vec{X}+\vec{Y}\right|\right| \le \left|\left|\vec{X}\right|\right| + \left|\left|\vec{Y}\right|\right|$ This inequality is called the triangle inequality. The reason for this is that if one labels the lengths of the sides of a triangle as $\left|\left|\vec{X}\right|\right|$ and $\left|\left|\vec{Y}\right|\right|$, then the third side has length $\left|\left|\vec{X}+\vec{Y}\right|\right|$. The inequality is just another way of saying the shortest distance between two points is the straight line between them.

Proof: These statements wil be verified for vectors in R^2. The proofs for R^3 and R^n are somewhat similar.

1. If $\left|\left|\vec{X}\right|\right| = 0$, where $\vec{X} = \langle x_1, x_2\rangle$, then $x_1^2 + x_2^2 = 0$. Since the sum of two nonnegative numbers (x_1^2 and x_2^2) can equal zero if and only if each of them is zero, we must have $x_1^2 = 0$ and $x_2^2 = 0$. Hence $\vec{X} = \langle 0,0\rangle = \vec{0}$. Conversely if $\vec{X} = \langle 0,0\rangle$, then $\left|\left|\vec{X}\right|\right| = \sqrt{0+0} = 0$.

2.
$$\begin{aligned}\left|\left|\alpha\vec{X}\right|\right| &= ||\alpha\langle x_1,x_2\rangle|| = ||\langle \alpha x_1, \alpha x_2\rangle||\\ &= \sqrt{(\alpha x_1)^2 + (\alpha x_2)^2}\\ &= \sqrt{\alpha^2\left((x_1)^2 + (x_2)^2\right)}\\ &= \sqrt{\alpha^2}\sqrt{(x_1)^2+(x_2)^2} = |\alpha|\left|\left|\vec{X}\right|\right|\end{aligned}$$

3. Before verifying this inequality we need a few preliminary inequalities. The first one, $0 \le (x_2y_1 - x_1y_2)^2$ is obvious and is used to verify the inequality $x_1y_1 + x_2y_2 \le \sqrt{(x_1)^2+(x_2)^2}\sqrt{(y_1)^2+(y_2)^2}$.

This last inequality looks formidable, but it follows from the first inequality. (Square both sides of the second inequality and simplify.). We are now ready to verify the triangle inequality.

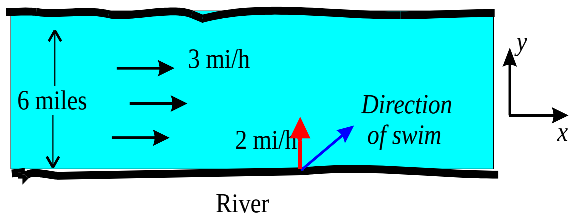
$$\begin{aligned}\left|\left|\vec{X}+\vec{Y}\right|\right|^2 &= (x_1+y_1)^2 + (x_2+y_2)^2\\ &= (x_1^2 + 2x_1y_1 + y_1^2) + (x_2^2 + 2x_2y_2 + y_2^2)\\ &= x_1^2 + x_2^2 + 2(x_1y_1 + x_2y_2) + y_1^2 + y_2^2\\ &\le x_1^2 + x_2^2 + 2\sqrt{(x_1)^2+(x_2)^2}\sqrt{(y_1)^2+(y_2)^2} + y_1^2 + y_2^2\\ &= \left(\sqrt{x_1^2+x_2^2} + \sqrt{y_1^2+y_2^2}\right)^2 = \left|\left|\vec{X}+\vec{Y}\right|\right|^2\end{aligned}$$

Thus, we have shown (take the square root of both sides) that the triangle inequality $\left|\left|\vec{X}+\vec{Y}\right|\right| \le \left|\left|\vec{X}\right|\right| + \left|\left|\vec{Y}\right|\right|$ is valid.

Since the sum of two vectors is the vector equal to the sum of the corresponding components, we can use this fact to resolve vectors into horizontal and vertical components as in the following example.

Example 1: Consider a plane which is flying at 25,000 feet due west with a speed of 500 miles per hour. Thus, the velocity of the plane can be represented by the vector $\vec{V} = \langle -500, 0, 0 \rangle$. Suppose our plane suddenly experiences a down draft whose velocity is 125 miles per hour. Then, the resultant velocity of the plane is the sum of the two velocity vectors. That is, the resultant velocity equals $\langle -500, 0, 0 \rangle + \langle 0, 0, -125 \rangle = \langle -500, 0, -125 \rangle$. The resultant speed of the plane is the length of this vector which is $\sqrt{(-500)^2 + 0^2 + (-125)^2} \sim 515.39$ miles per hour.

Example 2: A swimmer sets out to cross a river 6 miles wide with a downstream current at 3mi/h. He swims in a direction perpendicular to the bank at 2mi/h, At what point downstream will the swimmer reach the other bank.



Solution: The net velocity of the swimmer is the vector $3\vec{e}_1 + 2\vec{e}_2$. To cross the river the swimmer must travel 6 miles in the vertical direction. At a speed of 2 mi/h in the vertical direction, this will require 3 hours. Thus, the position vector after 3 hours will be $3(3\vec{e}_1 + 2\vec{e}_2) = 9\vec{e}_1 + 6\vec{e}_2$, From this we see that the swimmer will be 9 miles downstream.

Question: What is the total distance travelled by the swimmer?

Answer: $\|9\vec{e}_1 + 6\vec{e}_2\| = \|\langle 9, 6 \rangle\| = \sqrt{9^2 + 6^2} = \sqrt{81 + 36} = \sqrt{117} = 3\sqrt{13}$ miles

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Example 3: Find the unit vector in the direction of $\langle 4, -9 \rangle$. Note: A unit vector is a vector that has a length of 1.

Solution: We start by finding the length of the vector $\langle 4, -9 \rangle$:

$$\|\langle 4, -9 \rangle\| = \sqrt{4^2 + (-9)^2} = \sqrt{16 + 81} = \sqrt{97}$$

Thus, to find the vector of length one in the direction of $\langle 4, -9 \rangle$ we must multiply the vector by $\frac{1}{\sqrt{97}}$.

Thus, our desired vector is:

$$\frac{1}{\sqrt{97}} \langle 4, -9 \rangle = \left\langle \frac{4}{\sqrt{97}}, \frac{-9}{\sqrt{97}} \right\rangle$$

Exercises for Chapter 9D - Length

1. Compute the lengths of the following vectors.

a. $\langle 2,3\rangle$
b. $-4\langle -3,7\rangle$
c. $7\langle 3,3\rangle + 5\langle -1,8\rangle$
d. $\langle 2,3,-5\rangle$

2. If a dog runs from the point with coordinates $(-2,6)$ to the point $(5,-1)$, how far has the dog run? If it takes the dog four hours to run between the two points, how fast is the dog running. Assume that the units in the coordinate system are miles.

3. An airplane flies from the point $(1,1,3)$ to the point $(5,-5,0.1)$. Once again the unit distance is one mile. If it takes the plane 4 minutes to make this descent, how fast is the plane descending in miles per hour?

4. Find a unit vector which is parallel to the vector which points from the point $(5,13)$ to $(17,11)$.

5. A flea is walking on a coordinate plane. If the flea starts at the point $(1,0)$, and then hops to each of the points $(2,1)$, $(3,0)$, $(1,-2)$, and finally $(-1,-3)$ in succession, how far has the flea hopped, and what vector points from the flea's initial position to its final position?

6. You are flying an airplane at an altitude of 5000 feet and heading in a north-east direction. The control tower tells you to descend to an altitude of 1000 feet while simultaneously heading straight south. Find a unit vector which points in the direction you should head the plane.

7. For each pair of vectors $\vec{x}$ and $\vec{y}$ compute their lengths and the lengths of $\vec{x}+\vec{y}$ and $\vec{x}-\vec{y}$. Then sketch the four vectors.

a. $\vec{x} = \langle 1,1\rangle$ and $\vec{y} = \langle 1,-1\rangle$
b. $\vec{x} = \langle 1,1\rangle$ and $\vec{y} = \langle 1,0\rangle$
c. $\vec{x} = \langle 2,-1\rangle$ and $\vec{y} = \langle 1,2\rangle$
d. $\vec{x} = \langle 1,0\rangle$ and $\vec{y} = \langle 0,-1\rangle$

8. For each of the vectors given, find a unit vector which points in the same direction as the vector.

a. $\langle 1,2\rangle$
b. $\langle 3,5\rangle$
c. $\langle 2,3,-5\rangle$
d. $\langle 2,-4,7,12\rangle$

Answers to Exercises for Chapter 9D - Length

1.

a. $\|\langle 2,3\rangle\| = \sqrt{13} = 3.6056$

b. $\|-4\langle -3,7\rangle\| = 4\sqrt{58} = 30.463$

c. $\|7\langle 3,3\rangle + 5\langle -1,8\rangle\| = \|\langle 16,61\rangle\| = \sqrt{3977} = 63.063$

d. $\|\langle 2,3,-5\rangle\| = \sqrt{38} = 6.1644$

2. A vector which represents the dog's change of position is $\langle 5-(-2),-1-6\rangle = \langle 7,-7\rangle$. The dog has traveled a distance equal to the length of this vector. That is, $\|\langle 7,-7\rangle\| = 7\sqrt{2}$ miles. The dog's average velocity is $\dfrac{7\sqrt{2}}{4} = 2.4749$ miles per hour.

3. The plane descends a distance of $\|\langle 5,-5,0.1\rangle - \langle 1,1,3\rangle\| = \|\langle 4,-6,-2.9\rangle\| = 7.7724$ miles. Thus, the rate of descent is $\dfrac{7.7724}{4} = 1.9431$ miles per minute. This equals $60(1.9431) = 116.59$ miles per hour.

4. The vector which points from $(5,13)$ to $(17,11)$ is $\langle 17-5,11-13\rangle = \langle 12,-2\rangle$. A unit vector parallel to this direction is $\dfrac{\langle 12,-2\rangle}{\|\langle 12,-2\rangle\|} = \dfrac{\langle 12,-2\rangle}{2\sqrt{37}}$.

5. The flea has hopped a total distance of
$\|\langle 2-1,1-0\rangle\| + \|\langle 3-2,0-1\rangle\| + \|\langle 1-3,-2-0\rangle\| + \|\langle -1-1,-3-(-2)\rangle\|$
$= \sqrt{2} + \sqrt{2} + 2\sqrt{2} + \sqrt{5} = 7.8929$. The vector which points from the starting place to the final position of the flea is $\langle -1-1,-3-0\rangle = \langle -2,-3\rangle$.

6. A vector which point in the "descent" direction is $\langle 0,0,-1\rangle$ and one which points due south is $\langle 0,-1,0\rangle$. Thus, a vector which points in the planes direction is $\langle 0,-1,-1\rangle$, and a unit vector in this direction is $\dfrac{\langle 0,-1,-1\rangle}{\sqrt{2}}$.

7.

a. $\|\vec{x}\| = \|\langle 1,1\rangle\| = \sqrt{2}$ $\quad \|\vec{y}\| = \|\langle 1,-1\rangle\| = \sqrt{2}$
$\|\vec{x}+\vec{y}\| = \|\langle 2,0\rangle\| = 2$ $\quad \|\vec{x}-\vec{y}\| = \|\langle 0,2\rangle\| = 2$

b. $\|\vec{x}\| = \|\langle 1,1\rangle\| = \sqrt{2}$ $\quad \|\vec{y}\| = \|\langle 1,-1\rangle\| = \sqrt{2}$
$\|\vec{x}+\vec{y}\| = \|\langle 2,1\rangle\| = \sqrt{5}$ $\quad \|\vec{x}-\vec{y}\| = \|\langle 0,1\rangle\| = 1$

c. $\|\vec{x}\| = \|\langle 2,-1\rangle\| = \sqrt{5}$ $\quad \|\vec{y}\| = \|\langle 1,2\rangle\| = \sqrt{5}$
$\|\vec{x}+\vec{y}\| = \|\langle 3,1\rangle\| = \sqrt{10}$ $\quad \|\vec{x}-\vec{y}\| = \|\langle 1,-3\rangle\| = \sqrt{10}$

d. $\|\vec{x}\| = \|\langle 1,0\rangle\| = 1$ $\quad \|\vec{y}\| = \|\langle 0,-1\rangle\| = 1$

$$\|\vec{x}+\vec{y}\| = \|\langle 1,-1\rangle\| = \sqrt{2} \qquad \|\vec{x}-\vec{y}\| = \|\langle 1,1\rangle\| = \sqrt{2}$$

8. For each of the vectors given, find a unit vector which points in the same direction as the vector.

a. $\dfrac{\langle 1,2\rangle}{\|\langle 1,2\rangle\|} = \dfrac{\langle 1,2\rangle}{\sqrt{5}}$

b. $\dfrac{\langle 3,5\rangle}{\|\langle 3,5\rangle\|} = \dfrac{\langle 3,5\rangle}{\sqrt{34}}$

c. $\dfrac{\langle 2,3,-5\rangle}{\|\langle 2,3,-5\rangle\|} = \dfrac{\langle 2,3,-5\rangle}{\sqrt{38}}$

d. $\dfrac{\langle 2,-4,7,12\rangle}{\|\langle 2,-4,7,12\rangle\|} = \dfrac{\langle 2,-4,7,12\rangle}{\sqrt{213}}$

Chapter 9E - Dot Product

Unlike vector addition, which takes two vectors and gives back a third vector, the dot product of two vectors is a scalar not a vector. For this reason the dot product is sometimes referred to as the scalar product. The formula used to define and compute the dot product of two vectors is

$$\langle x_1, x_2\rangle \cdot \langle y_1, y_2\rangle = x_1y_1 + x_2y_2.$$

Analogous formulas are defined for the dot product of two vectors in R^n. The formula is

$$\langle x_1, x_2, \cdots, x_n\rangle \cdot \langle y_1, y_{2,} \cdots, y_n\rangle = x_1y_1 + x_2y_2 + \cdots + x_ny_n.$$

We point out that the length of a vector is related to the dot product of the vector with itself.

$$\vec{X} \cdot \vec{X} = \left|\left|\vec{X}\right|\right|^2$$

Another way to say the same thing is: $\left|\left|\vec{X}\right|\right| = \left(\vec{X} \cdot \vec{X}\right)^{1/2}$.

The reason for the usefulness of the dot product is that there is a relationship between the lengths of the vectors, the angle between them and their dot product. If we let $\vec{x}$ and $\vec{y}$ denote vectors in R^2 or R^3 then this relationship is:

$$\vec{x} \cdot \vec{y} = ||\vec{x}||\,||\vec{y}||\cos\theta$$

or

$$\cos\theta = \frac{\vec{x} \cdot \vec{y}}{||\vec{x}||\,||\vec{y}||}$$

where θ is the smaller of the two angles determined by the two vectors $\vec{x}$ and $\vec{y}$.

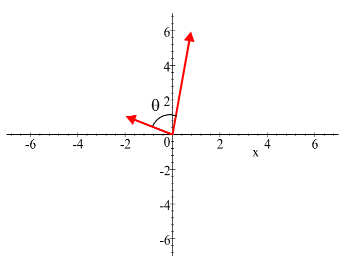
Memorize this formula!!

Example 1: Use this formula to calculate the cosine of the angle between the two vectors $\langle 1, 6\rangle$ and $\langle -2, 1\rangle$.

Solution: From the second form of the formula, we have

$$\cos\theta = \frac{\langle 1, 6\rangle \cdot \langle -2, 1\rangle}{||\langle 1, 6\rangle||\ ||\langle -2, 1\rangle||} = \frac{4}{\sqrt{37}\sqrt{5}} \approx 0.29409\,.$$

Thus, $\theta = \arccos(0.29409) \approx 1.2723$ radians which is approximately 72.9 degrees.



An interesting consequence of this formula is the fact that two (non-zero) vectors are perpendicular if and only if their dot product is zero.

Here's why: Clearly the dot product of two non-zero vectors is zero if and only if the cosine of the angle between them is zero. That is, if and only if the angle between the two vectors is 90 degrees. In symbols, if $\vec{x} \cdot \vec{y} = ||\vec{x}||\ ||\vec{y}||\cos\theta = 0,$ and if $||\vec{x}||$ and $||\vec{y}||$ are both non-zero, then $\cos\theta$ must be 0. This happen only if θ is $\pm\frac{\pi}{2}$ radians or $\pm 90^o$ degrees.

Example 2: Find the dot product of the following vectors:

- $\langle 2,5\rangle \cdot \langle -1,3\rangle$.
- $\langle 7,-3\rangle \cdot \langle -2,3\rangle$.
- $\langle 2,-3,0\rangle \cdot \langle 13,12,17\rangle$
- $\langle 12,-8\rangle \cdot \langle 9,-3\rangle$
- $\langle 23,45,67\rangle \cdot \langle 0,1,0\rangle$

Solution:

- $\langle 2,5\rangle \cdot \langle -1,3\rangle = 2(-1)+5(3) = 13$
- $\langle 7,-3\rangle \cdot \langle -2,3\rangle = -23$
- $\langle 2,-3,0\rangle \cdot \left\langle 13,\ 12,17\right\rangle = 2*13+(-3)*12+0 = -10$
- $\langle 12,-8\rangle \cdot \langle 9,-3\rangle = 12*9+(-8)*(-3) = 132$
- 45

Properties of the Dot Product

It is often useful to note the following relationship between the dot product of a vector with itself and its length

$$\vec{X} \cdot \vec{X} = x_1^2 + x_2^2 + \cdots + x_n^2 = \left|\left|\vec{X}\right|\right|^2.$$

The dot product also satisfies the following relations:

1. $\vec{X} \cdot \vec{Y} = \vec{Y} \cdot \vec{X}$.
2. $\vec{X} \cdot \left(\vec{Y}+\vec{Z}\right) = \vec{X} \cdot \vec{Y} + \vec{X} \cdot \vec{Z}$.
3. $\left(\alpha\vec{X}\right) \cdot \vec{Y} = \vec{X} \cdot \left(\alpha\vec{Y}\right) = \alpha\left(\vec{X} \cdot \vec{Y}\right)$

Example 3: Compute the dot product of the vectors $\vec{X} = \langle 2,11\rangle$ and $\vec{Y} = \langle -3,1\rangle$, and show that property 1 is satisfied.

Solution:

$\vec{X} \cdot \vec{Y} = \langle 2,11\rangle \cdot \langle -3,1\rangle = 2(-3)+11(1) = 5 = (-3)2+1(11) = \langle -3,1\rangle \cdot \langle 2,11\rangle = \vec{Y} \cdot \vec{X}$

Example 4: Verify property 2 using the following vectors. $\vec{X} = \langle 2,11\rangle$, $\vec{Y} = \langle -3,1\rangle$, and $\vec{Z} = \langle 1,6\rangle$.

Solution: $\vec{X} \cdot \left(\vec{Y}+\vec{Z}\right) = \langle 2,11\rangle \cdot (\langle -3,1\rangle + \langle 1,6\rangle) = \langle 2,11\rangle \cdot \langle -2,7\rangle = 73$

$\vec{X} \cdot \vec{Y} + \vec{X} \cdot \vec{Z} = \langle 2,11\rangle \cdot \langle -3,1\rangle + \langle 2,11\rangle \cdot \langle 1,6\rangle = 5+68 = 73$

Example 5: Verify property 3 using the vectors $\vec{X} = \langle 2,11\rangle$, $\vec{Y} = \langle -3,1\rangle$, and $\alpha = 3$.

Solution: $\left(\alpha\vec{X}\right) \cdot \vec{Y} = (3\langle 2,11\rangle) \cdot \langle -3,1\rangle = \langle 6,33\rangle \cdot \langle -3,1\rangle = 15$

$\vec{X} \cdot \left(\alpha\vec{Y}\right) = \langle 2,11\rangle \cdot (3\langle -3,1\rangle) = \langle 2,11\rangle \cdot \langle -9,3\rangle = 15$

$\alpha\left(\vec{X} \cdot \vec{Y}\right) = 3(\langle 2,11\rangle \cdot \langle -3,1\rangle) = 3(5) = 15$

Exercises for Chapter 9E - Dot Product

1. Compute the dot product for the following pairs of vectors.

a. $\langle 1,2\rangle$, $\langle 3,-7\rangle$

b. $\langle 2,14,5\rangle$,$\langle -2,0,7\rangle$

c. $(\langle 1,-2\rangle + \langle -3,-8\rangle)$, $\langle 3,2\rangle$

d. $\langle 3,4,-3,1\rangle$, $\langle 5,-3,0,6\rangle$

2. Find all **unit** vectors that are perpendicular to the vector $\langle 2, -3\rangle$.

3. A man wants to walk in a direction which is perpendicular to the vector $\langle 7,13\rangle$ and for which the x coordinate is decreasing. Find a unit vector which points in this direction.

4. Find the angle between the two vectors $\langle 1, 3\rangle$ and$\langle 4,-2\rangle$.

5. Find the angle between the two vectors $\langle 2,3\rangle$ and $\langle -5,7\rangle$.

6. Find the angle between the following pairs of vectors

a. $\langle 1,1\rangle$,$\langle 9,-4\rangle$

b. $\langle 2,3,-8\rangle$, $\langle 2,2,13\rangle$

c. $\langle 1,2,-3,7\rangle$,$\langle -1,17,5,2\rangle$

7. Find all vectors which make an angle of 45 degrees with the vector $\langle 1,1\rangle$ and have length 1.

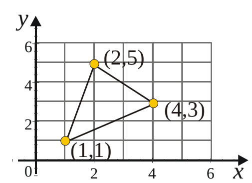
8. Determine if the following pairs of vectors are perpendicular.

a. $\langle 1,2\rangle$, $\langle 3,3\rangle$

b. $\langle 2,1,3\rangle$, $\langle 0,3,-1\rangle$

c. $\langle 4,3,0,-10,2\rangle$, $\langle 1,-1,5,-6,2\rangle$

9. Suppose the vertices of a triangle are $A(1,1)$, $B(2,5)$, $C(4,3)$. What are the three angles of this triangle.?



Answers to Exercises for Chapter 9E - Dot Product

1.

- **a.** $\langle 1,2\rangle \cdot \langle 3,-7\rangle = -11$
- **b.** $\langle 2,14,5\rangle \cdot \langle -2,0,7\rangle = 31$
- **c.** $(\langle 1,-2\rangle + \langle -3,-8\rangle) \cdot \langle 3,2\rangle = \langle -2,-10\rangle \cdot \langle 3,2\rangle = -26$
- **d.** $\langle 3,4,-3,1\rangle \cdot \langle 5,-3,0,6\rangle = 9$

2. If $\langle a,b\rangle$ is perpendicular to $\langle 2,-3\rangle$, we must have

$$\langle a, b\rangle \cdot \langle 2,-3\rangle = 2a - 3b = 0$$

This means $a = \frac{3}{2}b$. Since this equation has more unknowns, we can select one and solve for the other. We take $b = 2$. Then $a = \frac{3}{2}2 = 3$. Thus the vector $\langle 3,2\rangle$ is perpendicular to $\langle 2,-3\rangle$. Now a unit vector in that direction is

$$\frac{\langle 3,2\rangle}{\|\langle 3,2\rangle\|} = \frac{1}{\sqrt{13}}\langle 3,2\rangle$$

The other unit vector perpendicular to $\langle 2,-3\rangle$ is the negative $-\frac{1}{\sqrt{13}}\langle 3,2\rangle$.

3. The perpendicular to the vector $\langle 7,13\rangle$ is given by interchanging the coordinates and changing the sign of one component. Thus

$$\langle -13,7\rangle \perp \langle 7,13\rangle$$

The unit vector is $\frac{\langle -13,7\rangle}{\|\langle -13,7\rangle\|} = \frac{\langle -13,7\rangle}{\sqrt{218}}$ Moving in this direction, the x-coordinate decreases.

4. If θ is the angle between the two vectors, then we have

$$\begin{aligned}\cos\theta &= \frac{\langle 1,3\rangle \cdot \langle 4,-2\rangle}{\|\langle 1,3\rangle\|\,\|\langle 4,-2\rangle\|} \\ &= \frac{-2}{2\sqrt{10}\,\sqrt{5}} = -\frac{1}{10}\sqrt{2} \\ &\approx -.14142\end{aligned}$$

Thus,

$$\begin{aligned}\theta &= \arccos(-.14142) \\ &= 1.7127 \text{ radians} \\ &\approx \frac{180}{\pi}1.7127 \text{ degrees} \\ &= 98.13 \text{ degrees}\end{aligned}$$

5. If θ is the angle between the two vectors, then we have $\cos\theta = \frac{\langle 2,3\rangle \cdot \langle -5,7\rangle}{\|\langle 2,3\rangle\|\,\|\langle -5,7\rangle\|} =$ $\frac{11}{\sqrt{13}\,\sqrt{74}} \approx .35465$.

Thus,

$$\theta = \arccos(.35465) = 1.2083 \text{ radians, or} = \frac{180}{\pi}(1.2083) \approx 69.23 \text{ degrees.}$$

6.

a. $\cos\theta = \dfrac{\langle 1,1\rangle \cdot \langle 9,-4\rangle}{\|\langle 1,1\rangle\|\,\|\langle 9,-4\rangle\|} = \dfrac{5}{\sqrt{2}\sqrt{97}}$

$\theta = \arccos\left(\dfrac{5}{\sqrt{2}\sqrt{97}}\right) = 1.2036 \text{ radians} = 68.961 \text{ degrees}$

b. $\cos\theta = \dfrac{\langle 2,3,-8\rangle \cdot \langle 2,2,13\rangle}{\|\langle 2,3,-8\rangle\|\,\|\langle 2,2,13\rangle\|} = \dfrac{-94}{\sqrt{77}\sqrt{177}}$

$\theta = \arccos\left(\dfrac{-94}{\sqrt{77}\sqrt{177}}\right) = 2.5068 \text{ radians} = 143.63 \text{ degrees}$

c. $\cos\theta = \dfrac{\langle 1,2,-3,7\rangle \cdot \langle -1,17,5,2\rangle}{\|\langle 1,2,-3,7\rangle\|\,\|\langle -1,17,5,2\rangle\|} = \dfrac{32}{3\sqrt{7}\sqrt{319}}$

$\theta = \arccos\left(\dfrac{32}{3\sqrt{7}\sqrt{319}}\right) = 0.22573 \text{ radians} = 12.933 \text{ degrees}$

7. The easiest way to solve this problem is to draw the vector $\langle 1,1\rangle$ and observe that it makes an angle of 45 degrees with both the positive x and positive y axes. Thus, the desired unit vectors are $i = \langle 1,0\rangle$ and $j = \langle 0,1\rangle$.

8.

a. $\langle 1,2\rangle \cdot \langle 3,3\rangle = 9 \neq 0$. The vectors are not perpendicular.

b. $\langle 2,1,3\rangle \cdot \langle 0,3,-1\rangle = 0$. The vectors are perpendicular

c. $\langle 4,3,0,-10,2\rangle \cdot \langle 1,-1,5,-6,2\rangle = 65 \neq 0$. Thus, the vectors are not perpendicular.

9. Let a, b, and c denote the angles at the vertices A, B, and C respectively. Then we can think of a as the angle between the vectors $\overrightarrow{AB} = \langle 1,4\rangle$ and $\overrightarrow{AC} = \langle 3,2\rangle$. Thus,

$a = \arccos\left(\dfrac{\langle 1,4\rangle \cdot \langle 3,2\rangle}{\|\langle 1,4\rangle\|\,\|\langle 3,2\rangle\|}\right) = \arccos(.73994) = 42.274$ degrees. Similarly

$b = \arccos\left(\dfrac{\langle -1,-4\rangle \cdot \langle 2,-2\rangle}{\|\langle -1,-4\rangle\|\,\|\langle 2,-2\rangle\|}\right) = 59.038$ degrees, and

$c = \arccos\left(\dfrac{\langle -3,-2\rangle \cdot \langle -2,2\rangle}{\|\langle -3,-2\rangle\|\,\|\langle -2,2\rangle\|}\right) = 78.69$ degrees.