Right notation: $\frac{1}{2}$, $\frac{1}{x+2}$

Poor notation: ½ because is $1/x+2$ the same as $\frac{1}{x}+2$ or $\frac{1}{x+2}$?

0 is zero, a number; and $\varnothing$ is the empty set, a set

$\therefore$ is the symbol for "therefore."

"Iff" or "iff" stands for "if, and only if."

Examples of ***exact*** answers:

$\frac{16}{41}$, $-29\sqrt[3]{5}$, $\frac{\pi}{2}$, $\frac{-\sqrt{2}-\sqrt{6}}{4}$, $3000(1.013125)^{20}$

Approximate answers of the above list:

0.3902439024, –49.59, 1.57, –0.966, 3893.87

Math 150 Lecture Notes for Chapter 1 Basic Algebraic Concepts

Math 150 Lecture Notes for Section 1A Real Numbers

Properties of Real Numbers

For any two real numbers a and b, the sum $a + b$ and the product $a\ b$ or $a \cdot b$ are uniquely defined real numbers that satisfy the following properties.

I. Commutative Property
  a. Addition $a + b = b + a$
  b. Multiplication $a\ b = b\ a$

II. Associative Property
  a. Addition $(a + b) + c = a + (b + c)$
  b. Multiplication $(a\ b)\ c = a\ (b\ c)$

III. Identity
  a. Additive identity is the unique number 0 such that a + 0 = 0 + a = a
  b. Multiplicative identity is the unique number 1 such that 1a = (a)(1) = a

IV. Inverses

a. Additive inverse of a is the unique number (^{-}a) such that upon addition yields the additive identity: $a + (^{-}a) = (^{-}a) + a = 0$

b. Multiplicative inverse of $a \neq 0$ is the unique number $\frac{1}{a}$ or a^{-1}, called the reciprocal of a, such that upon multiplication yields the multiplicative identity: $(a)\left(\frac{1}{a}\right) = \left(\frac{1}{a}\right)(a) = 1$

V. Subtraction $a - b = a + (-b)$

VI. Division $a \div b = \frac{a}{b} = a\left(\frac{1}{b}\right) = ab^{-1}$

VII. Distributive Property of Multiplication over Addition (used in factoring)

a. $a\,(b + c) = a\,b + a\,c$

b. $(b + c)\,a = b\,a + c\,a$

Types of Real Numbers

Natural Numbers, Positive Integers, or Counting Numbers are $N = \{1, 2, 3, 4, 5, \ldots\}$

Whole Numbers or Nonnegative Integers are $W = \{0, 1, 2, 3, 4, 5, \ldots\}$

Integers $Z = \{\ldots, -3, -2, -1, 0, 1, 2, 3, 4, 5, \ldots\}$

Rational Numbers $Q = \left\{\frac{m}{n} \middle| m, n \in Z; n \neq 0\right\}$ which also include *N*, *W*, and *Z*. Rational numbers have repeating or terminating decimal expansion.

Examples of Rational Numbers:

Irrational Numbers, $R - Q$ or $R \setminus Q$, are non-repeating, non-terminating decimal numbers, and thus cannot be represented by a ratio of an integer and a non-zero integer (and are thus disjoint from the rational numbers.

Examples of Irrational Numbers:
Real numbers, *R*, are the set of all rational and irrational numbers.

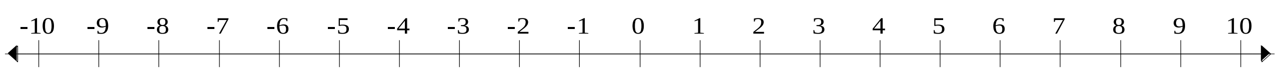
Examples:

Venn Diagram Summary:

Number Lines and Absolute Value

Each point on the real number lines corresponds to exactly one real number.

Find π, $\frac{-7}{3}$, -8, and $\frac{11}{2}$ on the number line.



The **absolute value** of a number *x*, denoted by |*x*|, refers to the distance from that number to the origin or zero.

If a is a real number, then the absolute value of *a* is $|a| = \begin{cases} a & \text{if} \quad a \geq 0 \\ -a & \text{if} \quad a < 0 \end{cases}$

| 0 | =

| 9 | =

| – 4 | =

| 3 – 5 | =

$| 1 - e | =$

$| x - y | =$

The **distance** between two points *a* and *b* on the real number line is $| b - a |$.

What is the distance between the real numbers __________ and __________?

Math 150 Lecture Notes for Section 1B Exponents and Radicals

Order of Operations

Parenthesis Exponents | Multiplication and Division in order from left to right | Addition and Subtraction in order from left to right

Please Excuse | My Dear | Aunt Sally

$-5^2 - 60 \div 5 \cdot 4 - 8 + 3 - 2(5-7)^2 + 3 \cdot 2^3 =$

Properties of Exponents

- If *n* is a positive integer, then $x^n = x \cdot x \cdot x \cdot \ldots \cdot x$ such that there are *n* factors. This can be extended to all real numbers *n*.
- $a^0 = 1, \quad a \neq 0$
- $a^{-n} = \dfrac{1}{a^n}, \quad a \neq 0$
- $a^m a^n = a^{m+n}$
- $\dfrac{a^m}{a^n} = a^{m-n}, \quad a \neq 0$
- $\left(a^m\right)^n = a^{mn}$
- $\left(ab\right)^n = a^n b^n$
- $\left(\dfrac{a}{b}\right)^n = \dfrac{a^n}{b^n}, \quad b \neq 0$
- $\left(\dfrac{a}{b}\right)^{-n} = \dfrac{b^n}{a^n} = \left(\dfrac{b}{a}\right)^n, \quad a, b \neq 0$
- $\dfrac{a^{-n}}{b^{-m}} = \dfrac{b^m}{a^n}, \quad a, b \neq 0$

$6^3 =$

$\left(-9\right)^2 =$

$-9^2 =$

$6^{-1} =$

$3^0 =$

$\left(2^{-4}\right)\left(3x^2\right)^4 =$

$\left(\frac{2x}{5}\right)^3\left(\frac{2x}{5}\right)^6 =$

$\frac{x^9}{x^5} =$

$\frac{-2x^3}{7x^9} =$

$\left(x^{-5}\right)^2 =$

$\left(\frac{-5x}{2}\right)^{-3} =$

$\left(\frac{-3x^{-4}}{2xy^{-6}}\right)^2 =$

$\left(144x^{-5}\right)\left(80x\right)^{11} =$

$\frac{\left(\frac{1}{3}\right)^{-1005} - 27^{334}}{9^{502} + 9^{501}} =$

Radicals and Properties of Radicals

Radicals or roots are related to exponents such that $b = \sqrt[n]{a} = a^{\frac{1}{n}}$ iff $b^n = a$. Here the *n*th root of the number *a* is the number *b*. The number *b* is called an *n*th root of *a*. The number *n* is referred to as the index of the radical (if no index appears, *n* is understood to be 2). The principal *n*th root of a number is the *n*th root of *a* which has the same sign as *a*. Both 3 and –3 satisfy $x^2 = 9$, but 3 is the (principal) square root of 9.

The symbol $\sqrt{\ }$ means "the nonnegative square root of," that is, $\sqrt{a} = b$ means $b^2 = a$ where $b \geq 0$.

Note: $\sqrt{-100}$ is not a real number (it is a complex number) since what real number squared is –100? Also, $\sqrt{0} = 0$.

What is the domain of $\sqrt{x+6}$?

Properties of Roots, Radicals, and Exponents

- $a^{\frac{1}{n}} = \sqrt[n]{a}$
- $a^{\frac{m}{n}} = \sqrt[n]{a^m} = \left(\sqrt[n]{a}\right)^m$
- $\sqrt{a^n} = \left(\sqrt{a}\right)^n$
- $\sqrt[n]{ab} = \sqrt[n]{a}\sqrt[n]{b}$
- $\sqrt[n]{\frac{a}{b}} = \frac{\sqrt[n]{a}}{\sqrt[n]{b}}$
- $\sqrt[m]{\sqrt[n]{a}} = \sqrt[mn]{a}$
- $\sqrt[n]{a^n} = \begin{cases} a & if \quad n \quad is \quad odd \\ |a| & if \quad n \quad is \quad even \end{cases}$

$\sqrt{25} =$

$32^{\frac{2}{5}} =$

$\sqrt[5]{-32} =$

$\sqrt[6]{-64} =$

$(-216)^{\frac{1}{3}} =$

$-9^{\frac{1}{2}} =$

$9^{\frac{1}{2}} =$

$\sqrt[6]{(-3)^6} =$

$\sqrt[4]{16x^4} =$

Simplifying Radicals

A radical expression is simplified when the following conditions hold:
1. All possible factors (“perfect roots”) have been removed from the radical.
2. The index of the radical is as small as possible.
3. No radicals appear in the denominator.

$\sqrt[3]{648x^4y^6} =$

$\sqrt[8]{64x^4y^{12}} =$

$\sqrt[3]{\dfrac{27^n \cdot 3^{4n}}{9^{-n}}} =$

$\left(\dfrac{-3375}{64}\right)^{\frac{-4}{3}} =$

Rationalizing Denominators

If after simplifying an expressions with exponents and/or radicals, there is still a radical in the denominator, we "rationalized the denominator" to remove the radical in the denominator.

Single-Term Radical in Denominator – If after simplifying the expression, the denominator has a single-term radical, multiply both numerator and denominator by something that will produce a perfect root in the denominator (this is just multiplying by a 'fancy' 1).

$$\frac{1}{\sqrt[3]{4}} =$$

$$\frac{8}{\sqrt[4]{54}} =$$

$$\sqrt{\frac{(9x)^3}{50x^8}} =$$

$$\frac{2}{\sqrt[3]{(x-2)^2}} =$$

$$\sqrt[3]{\frac{-54x^2y^{-4}z^{-2}}{50x^{-3}y^{-5}}} =$$

Radical in Denominator is a Sum or Difference of Terms – If after simplifying the expression, the denominator has a sum or difference of a radical, multiply both the numerator and denominator by the conjugate of the denominator. Conjugate pairs are $a+\sqrt{b}$ and $a-\sqrt{b}$, and $\sqrt{a}+\sqrt{b}$ and $\sqrt{a}-\sqrt{b}$. For example, the conjugate of $5-\sqrt{7}$ is $5+\sqrt{7}$.

Rationalize the Denominator

$$\frac{3}{4-5\sqrt{7}}=$$

$$\frac{\sqrt{2}-2}{\sqrt{2}+4}=$$

$$\frac{2}{\sqrt{5}-\sqrt{3}}=$$

Combining Radical Expressions

In a radical expressions, the terms must be alike to be combined by addition or subtraction. Like radical-terms are ones that have the same index and same radicand. The radicand is the expression inside the radical.

$$\sqrt{5}+2\sqrt{7}-3\sqrt{5}-\sqrt{7}=$$

$$\sqrt[3]{16x^4}-3x\sqrt{18x}-x\sqrt[3]{250x}=$$

$$\sqrt{\frac{2}{3}}+\sqrt{\frac{5}{27}}=$$

Math 150 Lecture Notes for Section 1C Polynomials

Definition of Polynomial

Polynomials are expressions which contain terms of the form ax^n, where a is a real constant and n is a nonnegative integer. A polynomial is of the form $a_n x^n + a_{n-1}x^{n-1} + \ldots + a_1 x + a_0$ where all a_i are real numbers such that a_n, the leading coefficient, is nonzero, and n is a positive integer. Also a_0 is known as the constant term, and $a_n x^n$ is called the leading term. The degree of the polynomial is given by n.

Zero Degree or Constant	8
First Degree or Linear	$3x + 5$
Second Degree or Quadratic or Binomial	$x^2 - 5x + 2$, $7x^2$
Cubic, Quartic, Quintic, . . ., nth degree	

Example: The surface area S in cm^2 of a 5 cm high can is given by the polynomial equation $S = 2\pi r^2 + 10\pi r$, where r is radius in cm. Here the leading coefficient is 2π, the leading term is $2\pi r^2$, the degree of the polynomial is 2, and the constant is 0.

Example: The volume V of a box with particular constraints is $V = x^3 + 15x^2 + 71x + 105$.

Leading coefficient:
Leading term:
Degree of polynomial:
Constant term:

Polynomials do not contain negative exponents or radicals. Which of the following are polynomials?

$x^{-2} - 5x + 2$ $\qquad \sqrt{3x^5 - 6x + 2}$ $\qquad \frac{2}{3}x^4 - \sqrt{5}x^2 + \pi$ $\qquad \frac{2}{x^3} - \frac{1}{3x^2} + e$

Sums and Differences of Polynomials

When we add, subtract or simplify polynomials we combine like terms, that is, terms with the same variable and exponent.

Note: $a - b = a + (-b)$, be careful to distribute the negative appropriately!

$5 - (x - 4) = 5 - x + 4$ and note that $5 - (x - 4) \neq 5 - x - 4$

$\left(5x^3 - 3x + 8\right) - \left(6x^3 + 5x^2 - x - 7\right) =$

$\left(x^4 - 3x^2 + 6\right) - \left(4x^3 - 6x^2 + 2x - 5\right) + \left(-x^4 + 7x^3\right) =$

Products of Polynomials

Use the distributive property and initially treat the second factor as a unit:

$(2x-5)(3x+4) = 2x(3x+4) - 5(3x+4) =$

Example: $\left(5x^2 - x + 3\right)(x-4)$

Method 1 using the distributive property and initially treat the second factor as a unit:

$\left(5x^2 - x + 3\right)(x-4) =$

Method 2 using the distributive property using the rectangle or box style:

$\left(5x^2 - x + 3\right)(x-4) =$

$(a-b)(a+b) =$

Special Products

- Difference of Squares: $a^2 - b^2 = (a+b)(a-b)$
- Square of a Binomial: $(a+b)^2 = a^2 + 2ab + b^2$ and $(a-b)^2 = a^2 - 2ab + b^2$
- Sum of Cubes: $a^3 + b^3 = (a+b)(a^2 - ab + b^2)$
- Difference of Cubes: $a^3 - b^3 = (a-b)(a^2 + ab + b^2)$
- Cube of a Binomial: $(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$ and $(a-b)^3 = a^3 - 3a^2b + 3ab^2 - b^3$

$(x+2)(x^2 - 2x + 4) =$

$[(x-3)(x-2)]^2 =$

Quotients of Polynomials

Perform long division on $2511 \div 27$. (divide, multiply, subtract, bring down, and repeat as long as possible).

Perform polynomial long division on $\left(x^3+9x^2-x-105\right)\div\left(x+5\right)$. (divide, multiply, subtract (draw the line; change the signs; add), bring down, repeat as long as possible).

Note that $\left(x^3+9x^2-x-105\right)\div\left(x+5\right)=x^2+4x-21$ also means that

$\left(x^3+9x^2-x-105\right)=\left(x+5\right)\left(x^2+4x-21\right)$

Perform long division on $1040 \div 43$.

Perform polynomial long division on $\left(x^5 - 3x^3 + x^2 + 9\right) \div \left(x^2 + 2x\right)$.

Factoring

I. Common Factors - are factors of every term in an expression

$18x^5 - 75x^2 =$

$5x^2\sqrt{9-x} + x\sqrt{9-x} =$

II. Factor by Grouping – usually used when have more than three terms in an expression

$3x^5 - 5x^4 + 3x - 5 =$

III. Factor by Using Special Products

$x^2 - 10x + 25 =$

$x^3 + 6x^2 + 12x + 8 =$

$x^2 + 9 =$

IV. Factoring Trinomials ($x^2 + bx + c$)

$(x+m)(x+n) = x^2 + bx + c$

$x^2 + (m+n)x + mn = x^2 + bx + c$

So note that $m+n=b$ and $mn=c$.

In the first example we must find two numbers whose sum is 11 and whose product is 18.

$x^2 + 11x + 18 =$

$x^2 + x - 20 =$

V. Factoring Trinomials ($ax^2 + bx + c$)

$(mx+s)(nx+t) = ax^2 + bx + c$

$mnx^2 + (mt+ns)x + st = ax^2 + bx + c$

So note that $mn = a$ and $st = c$. This is done mainly by trial and error. Try to split the middle term bx into two terms such that we can factor by grouping.

$6x^2 + 23x + 15 =$

$10x^2 - x - 21 =$

Factor Completely:

$x^2 + 2x - 15 =$

$16x^4 - 25x^2 =$

$27x^6 + 64 =$

$x^5 - 4x^3 - x^2 + 4 =$

Math 150 Lecture Notes for Section 1D Rational Expressions

Definition of a Rational Expression

A **rational expression** is the quotient of two polynomials, $\frac{p}{q}$, where $q \neq 0$.

Examples: $\frac{6x - x^2}{2 + x}$, $\frac{x^3 - 4x^2 + 6}{5}$, $\frac{x^2 - 2x + 9}{x^3 + x^2 - 6}$, $\frac{x - 4}{x + 3}$

What is the domain of $\frac{x^2 + 5}{4 - 3x}$? Recall that the denominator cannot equal zero.

What is the domain of $\frac{4}{3x^2 + 18}$?

What is the domain of $\frac{x + 3}{x^2 - 10x + 21}$?

Simplifying Rational Expressions

Simplifying a rational expression means to reduce it to lowest terms, which is done by cancelling factors common to the numerator and denominator.

$$\frac{27x^7}{15x^9} =$$

$$\frac{x^2 + 5x + 6}{5x + 10} =$$

$$\frac{x^2 - 9}{x^2 - 11x + 24} =$$

$$\frac{x^2 - 9x}{2x^5 + 5x} =$$

Operations with Rational Expressions

I. Multiplication

$\frac{a}{b}\cdot\frac{c}{d}=\frac{ac}{bd}$, and $\frac{ab}{bd}=\frac{a}{d}$ since $\frac{b}{b}=1$

$$\frac{3x^2}{5x}\cdot\frac{2x}{51}=$$

$$\frac{2x^2-2x-24}{15x^2-10x}\cdot\frac{3x^2-2x}{x+3}=$$

$$\frac{2x^2+x-15}{x^3-1}\cdot\frac{1-x}{2x-5}=$$

II. Division

$\frac{a}{b}\div\frac{c}{d}=\frac{ad}{bc}$

$$\frac{x^2-1}{x^2+2}\div\left(x^2+7x+6\right)=$$

$$\frac{x^3-27}{x+6}\div\frac{x^2+3x+9}{x^2+4x-12}=$$

$$\frac{x^2+3x-10}{x^2+6x+5}\div\frac{2x^2-x-6}{x^2+2x+1}\cdot\frac{2x+3}{2x+2}=$$

III. Addition and Subtraction

Addition and subtraction is done by first finding a common denominator, preferably the least common denominator, LCD.

$$\frac{2}{5-x}+\frac{3}{x}=$$

$$\frac{x+4}{x+3}-\frac{5x+1}{x^2-x-12}=$$

$$\frac{x^2-8x-9}{x^2+8x+7}-\frac{x-8}{x^2+14x+49}=$$

Compound Fractions

A **compound** or **complex** fraction is an expression with contains fractions within fractions. To simplify a compound fraction, first simplify both the numerator and denominator individually, then divide the numerator by the denominator by multiplying by the reciprocal of the denominator.

$$\frac{5-\frac{x}{x+5}}{\frac{x-2}{x+3}}=$$

$$\frac{\frac{1}{x}-\frac{x+1}{x+2}}{\frac{x+5}{x^2-4}+\frac{x}{x^2-5x+6}}=$$

Expressions like the following are common to calculus classes.

$$\frac{3(x+h+5)^{-1}-3(x+5)^{-1}}{h}=$$

Math 150 Lecture Notes for Section 1E Complex Numbers

Definition of a Complex Number

A complex number is a number that can be written as $a+bi$, where *a* and *b* are real numbers and $i=\sqrt{-1}$. The *a* part is known as the real part and *b* is the imaginary part. If $b=0$, then we have a real number. If $b\neq 0$, then we have a non-real complex number. Two complex numbers are equal iff their real parts are equal and their imaginary parts are equal. The standard form of a complex number is $a+bi$.

Examples:

Write $\sqrt{45}-\sqrt{-8}$ in standard form.

Find real numbers *a* and *b* such that $2-5a-6i=8+4b\sqrt{-5}$.

$(2-i)(-6-7i)=$

The **absolute value of a complex number,** $z = a + bi$, is $|z| = |a + bi| = \sqrt{a^2 + b^2}$.

$|5 - 10i| =$

$|\sqrt{-9} + 6| =$

$|-4| =$

$|-8i| =$

Properties of the Absolute Value of a Complex Number

1. The absolute value of $z = a + bi$ represents the distance from the origin (0, 0) to the point (*a*, *b*).
2. If z_1 and z_2 are complex numbers, then $|z_1 z_2| = |z_1||z_2|$ and $\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}$.
3. Triangle inequality: If z_1 and z_2 are complex numbers, then $|z_1 + z_2| \le |z_1| + |z_2|$.

Show the absolute value of the sum of $3 + 7i$ and $4 - 9i$ is less than or equal to the sum of their absolute values.

The **conjugate of the complex number** $z = a + bi$ is $\overline{z} = a - bi$. Note that z and $\overline{z}$ are conjugates of each other.

Compute the conjugate of $12 - 7i$.

Compute the conjugate of $-9 + i$.

Properties of Complex Conjugates

1. $z \cdot \overline{z} = |z|^2$ and
   $(a+bi)(a-bi) = a^2 - abi + abi - b^2i^2 = a^2 - b^2(-1) = a^2 + b^2 = |a+bi|^2$
2. If x is a real number, then $\overline{x} = x$.
3. $\overline{\overline{z}} = z$, the conjugate of the conjugate of z is z.
4. $\overline{(z_1 + z_2)} = \overline{z_1} + \overline{z_2}$
5. $\overline{(z_1 \cdot z_2)} = \overline{z_1} \cdot \overline{z_2}$
6. $\overline{\left(\frac{z_1}{z_2}\right)} = \frac{\overline{z_1}}{\overline{z_2}}$

If $z_1 = 4 + 2i$ and $z_2 = 6 - 3i$, compute

a. $\overline{z_1 + z_2}$

b. $\overline{z_1} + \overline{z_2}$