Math 150 Lecture Notes for Chapter 4 Functions

Math 150 Lecture Notes for Section 4A Introduction to Functions

Definition
A function is a
- relation in which no two different ordered pairs have the same first element
- rule that assigns exactly one element in set B (range) to each element in set A (domain)
- relation where there is only one output, *y*-value, for each input, *x*-value.

Is $\{(5,-6),(3,-3),(-1,-1),(0,-6),(5,4)\}$ a function?

Is $x^2 + y^2 = 9$ a function?

Is $\{(5,6),(3,-3),(-1,-1),(0,-6),(4,4)\}$ a function?

Your calculator has some functions programmed in it.

$y1 = \sqrt{5-x}$

$y1(1) =$ $y1(-4) =$ $y1(2) =$ $y1(8) =$

Function Notation

$f(x)$, read "*f* of *x*," says that *f* is a function with input *x*. That is $y = f(x)$.

If $f(x) = 6x^2 - 3$, what does $f(4)$ equal? What is the input? What is the output? What is the associated ordered pair?

Evaluating Functions

If $f(x) = -3x^2 + 6x - 7$, find the following.

a. $f(-2)$

b. $f(-x)$

c. $f(a+h)$

The **difference quotient** is the slope of the line that goes through the points $(x, f(x))$ and $(x+h, f(x+h))$ on the graph of a function. The slope of the line $m = \dfrac{f(x+h) - f(x)}{(x+h) - x} = \dfrac{f(x+h) - f(x)}{h}$ is the difference quotient, such that $h \neq 0$. This is also known as the **average rate of change** of the *y*-values with respect to their *x*-values.

If $f(x) = 5x^2 - 6x + 8$, evaluate the difference quotient.

If $f(x)=\dfrac{2}{x+5}$, evaluate the difference quotient.

Domain of a Function

The domain of most algebraic functions is the set of all real numbers except:

1. any real numbers that causes the denominator to be zero
2. any real numbers that cause the radicand (number under the root) of an even index to be negative

What is the domain of the following functions?

a. $f(x)=\dfrac{x-5}{x^2-25}$

b. $f(x)=\dfrac{x^2-9}{x+3}$

c. $f(x)=\sqrt[3]{x^5-2x}$

d. $f(x)=\dfrac{6-x}{5x+8}$

e. $f(x)=\sqrt{x^2+2x-35}$

f. $f(x)=\dfrac{\sqrt[4]{7-x}}{\sqrt{x+8}}$

Applying Functions

Word Problems

- Understand the problem; identify what you are trying to find and give it a variable name; are there restrictions on the variable
- Draw a picture, table, or graph; or look for a pattern
- Write an equation
- Solve the equation (remember your units in the answer)
- Reflect; check your answer; is the answer reasonable; can you generalize your answer; did you learn a concept

Billy Bob, a rancher, wants to enclose a rectangular field with fencing that costs $15 per foot. How much would it cost to enclose a field that is three times as long as it is wide? What is the domain?

A hot air balloonist charges 120 Euros for a ride, plus 52 Euros for each hour the balloon is in the air. If the longest time the balloonist will give a balloon ride for is 5 hours, what is the function for the price of a hot air balloon ride? What is the domain?

A boat can travel 10 miles downstream in the same time it can travel 5 miles upstream. If the speed of the current is 3 miles per hour, what is the speed of the boat?

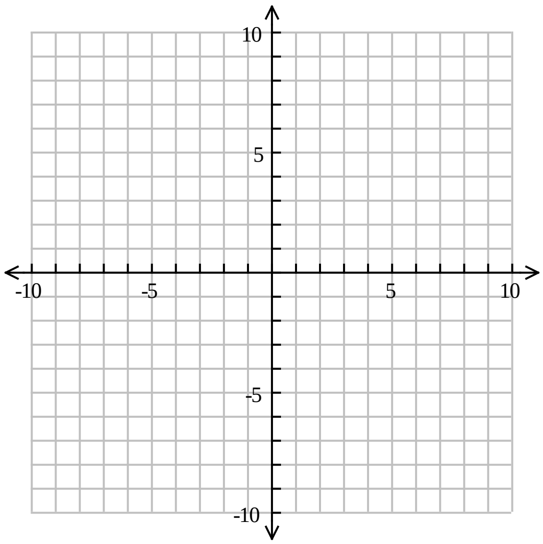
Jennifer wants to make 4 liters of an 18% alcohol solution by mixing a 21% alcohol solution with a 14% alcohol solution. How many liters of 14% alcohol should be used? What is the domain?

It takes Charles 2 hours longer to mow and trim a large field than it does Houston. If they work together, it takes $\frac{63}{16}$ hours to complete the job. How long does it take Charles to mow and trim the field by himself? What is the domain?

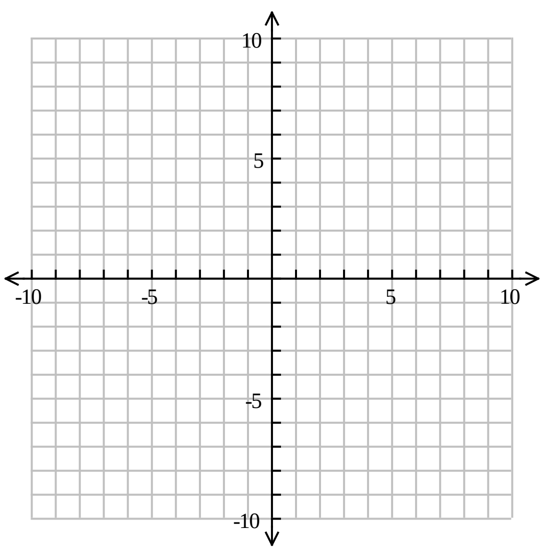
Math 150 Lecture Notes for Section 4B Graphs of Functions

Graphs of Functions

Graph $f(x) = |5 - x| + 2$. Find the *x*- and *y*-intercepts.



Graph $f(x) = \sqrt{x - 5}$. Find the *x*- and *y*-intercepts.

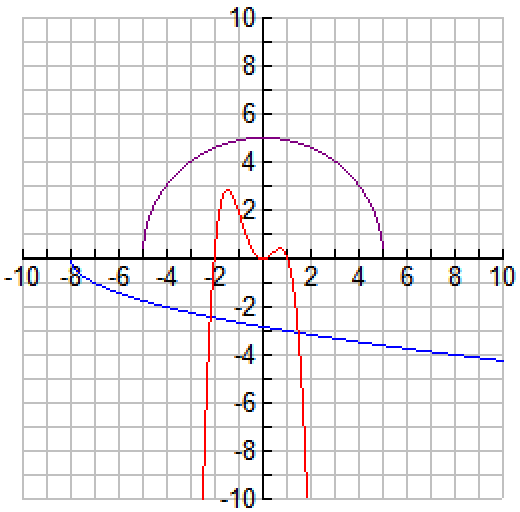


Vertical Line Test – A set of point in the plane represents a function iff no vertical line intersects the graph in more than one point.

Which of the following graphs are functions?

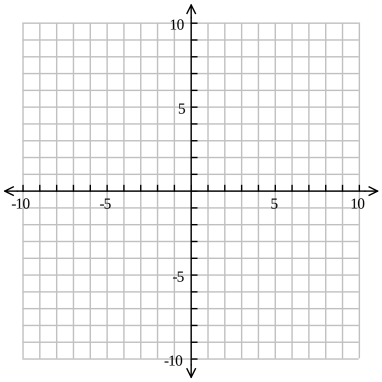
Domain and Range from Graphs

Use the graphs to find the domain and range of each function.



Catalog of Basic Functions

1. Constant Function: $f(x) = c$



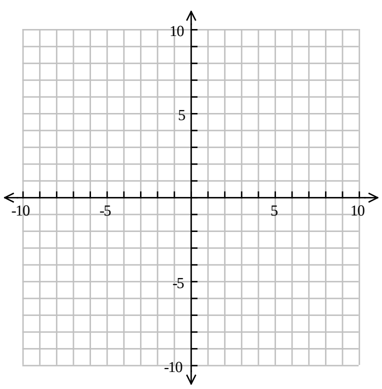
x-intercept:

y-intercept:

domain:

range:

2. Identity Function: $f(x) = x$



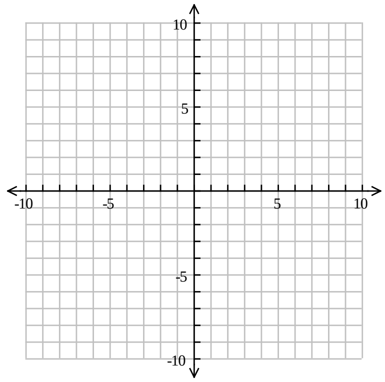
x-intercept:

y-intercept:

domain:

range:

3. Squaring Function: $f(x) = x^2$



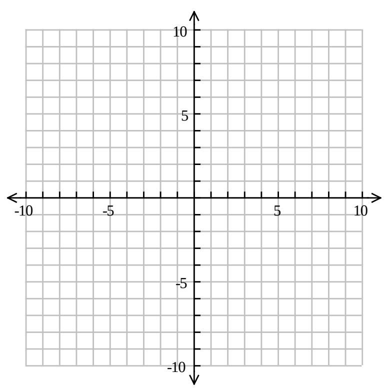
x-intercept:

y-intercept:

domain:

range:

4. Cubing Function: $f(x) = x^3$



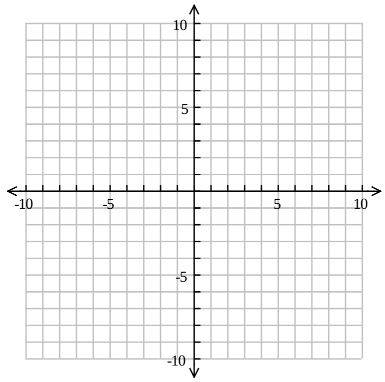
x-intercept:

y-intercept:

domain:

range:

5. Square Root Function: $f(x) = \sqrt{x} = x^{\frac{1}{2}}$



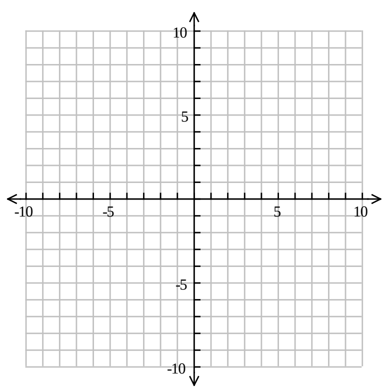
x-intercept:

y-intercept:

domain:

range:

6. Absolute Value Function: $f(x) = |x|$



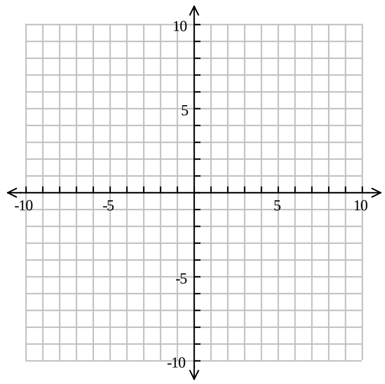
x-intercept:

y-intercept:

domain:

range:

7. Reciprocal Function: $f(x) = \frac{1}{x} = x^{-1}$



x-intercept:

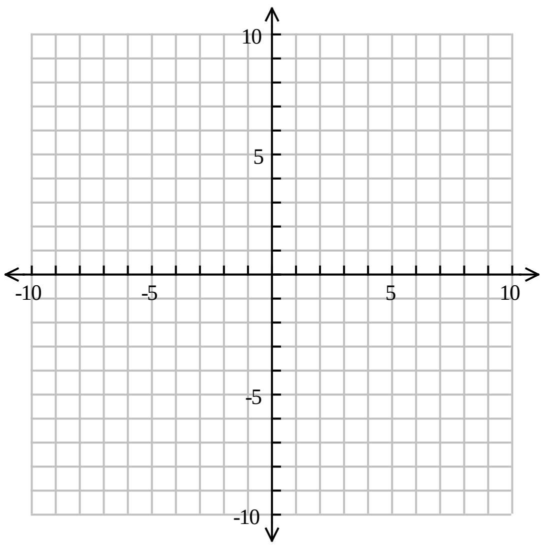
y-intercept:

domain:

range:

Piecewise Functions

Graph $f(x) = \begin{cases} 1 - \dfrac{x^2}{10}, & x < -8 \\ x, & -8 \le x < -3 \\ \sqrt{x+4}, & x \ge -2 \end{cases}$



$f(-10) =$

$f(-8) =$

$f(0) =$

$f(-2.5) =$

What is the domain of *f*?

What is the range of *f*?

What are the intercepts?

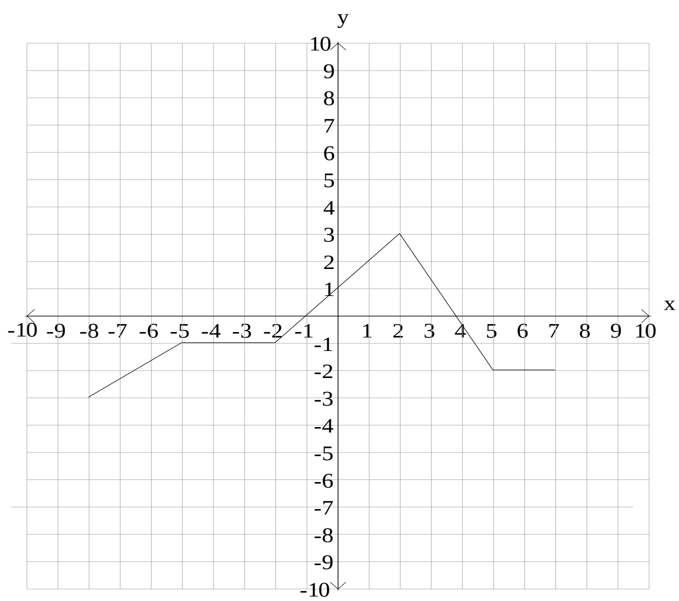
Increasing, Decreasing, and Constant

Examples of an interval of a graph that is increasing:

Examples of an interval of a graph that is decreasing:

Example of an interval of a graph that is constant:

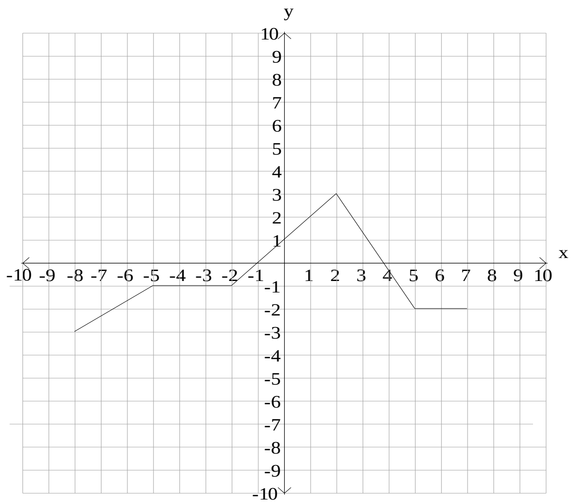
Use the graph to find the domain, range, intercepts, and intervals where *f* is increasing, decreasing and constant.



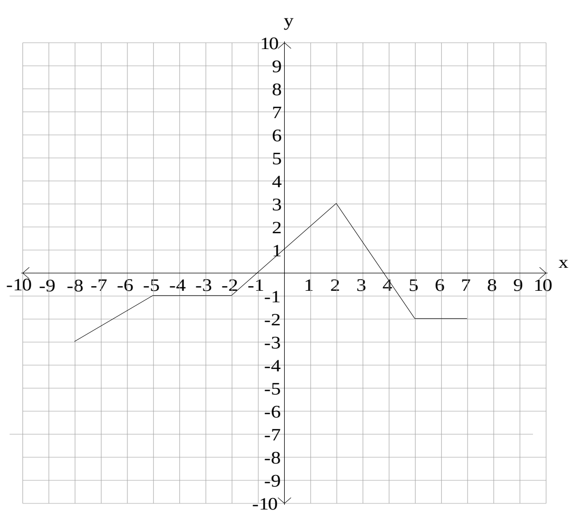
Math 150 Lecture Notes for Section 4C Transformations of Functions

Horizontal and Vertical Shifts

Use the graph of $y = f(x)$ to graph $y = f(x) + 5$ on the same coordinate plane.

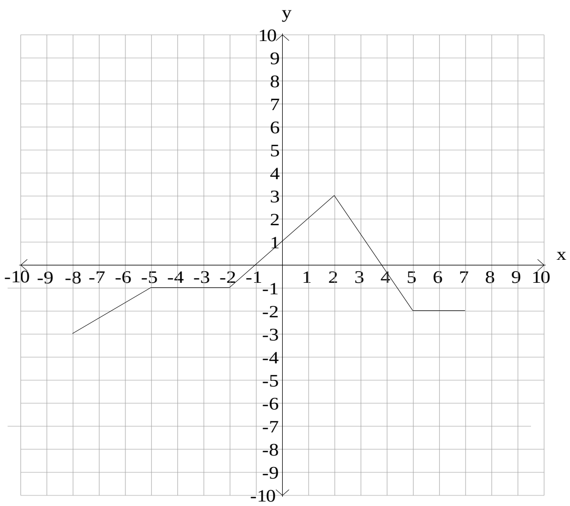


Use the graph of $y = f(x)$ to graph $y = f(x - 3)$ on the same coordinate plane.

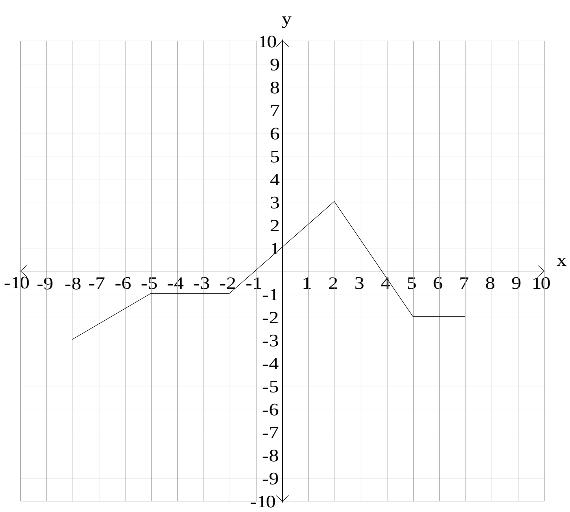


Reflections about the x- and y-axis

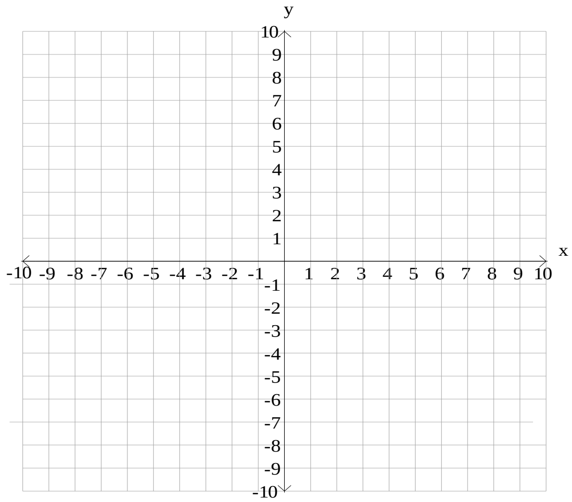
Use the graph of $y = f(x)$ to graph $y = -f(x)$ on the same coordinate plane.



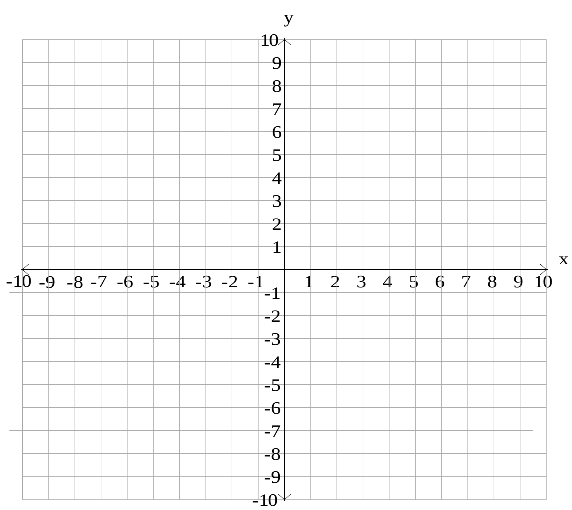
Use the graph of $y = f(x)$ to graph $y = f(-x)$ on the same coordinate plane.



Use the graph of $f(x)=\sqrt{x}$ to graph $y=\sqrt{-x}$ on the same coordinate plane.

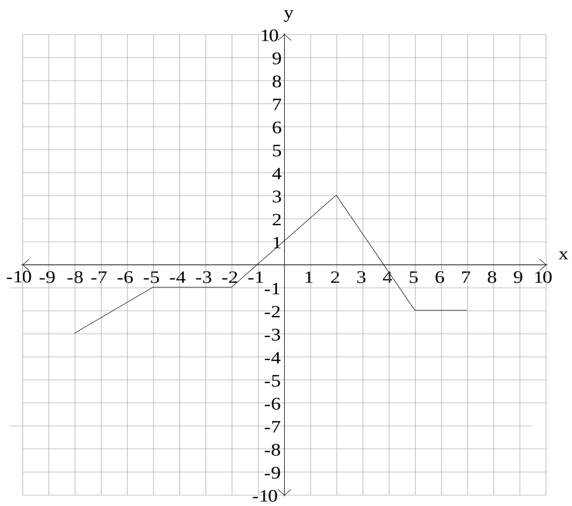


Use the graph of $f(x)=\sqrt{x}$ to graph $y=-\sqrt{x}$ on the same coordinate plane.

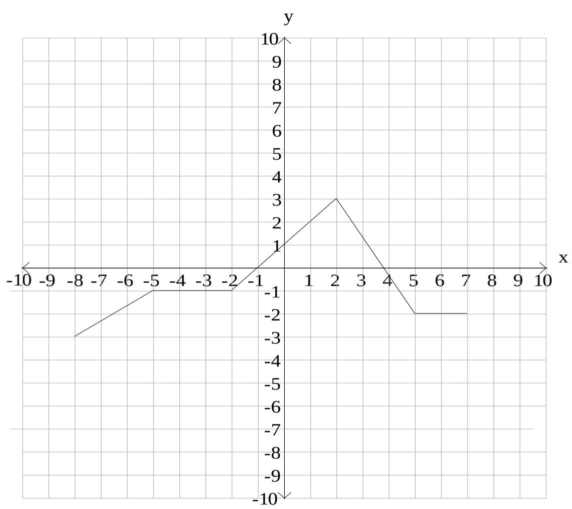


Vertical Stretch and Shrink

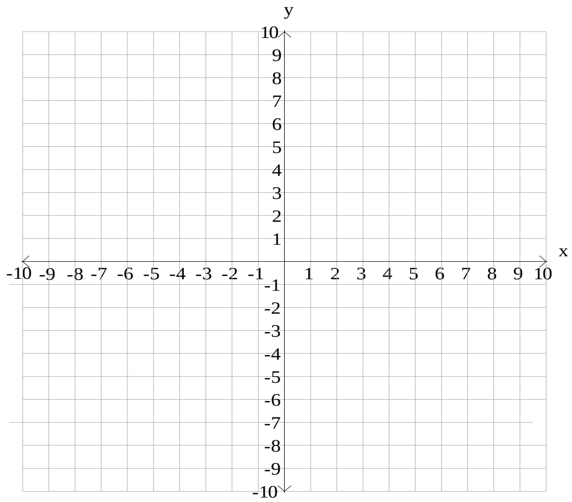
Use the graph of $y = f(x)$ to graph $y = 3f(x)$ on the same coordinate plane.



Use the graph of $y = f(x)$ to graph $y = \frac{1}{4}f(x)$ on the same coordinate plane.



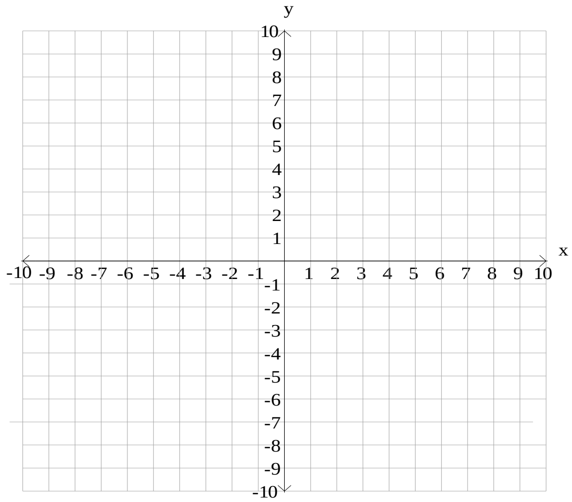
Use the graph of $f(x) = x^2$ to graph $y = \frac{1}{2}x^2$ on the same coordinate plane.



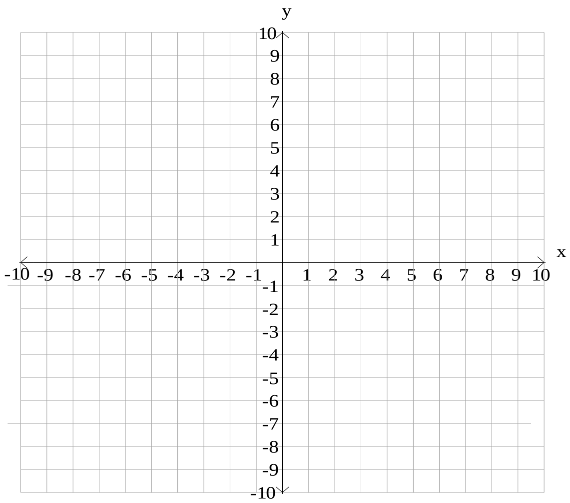
If $|a| > 1$, there is a **vertical stretch** by a factor of $|a|$; if $0 < |a| < 1$, there is a **vertical shrink** by a factor of $|a|$.

Combining Transformations

Use the graph of $f(x) = x^2$ to graph $y = (x+2)^2 - 3$ on the same coordinate plane.



Use the graph of $f(x) = |x|$ to graph $y = -|x-4| + 2$ on the same coordinate plane.



Even and Odd Functions

A function is **even** iff $f(-x) = f(x)$ for all *x* in the domain of *f*. The graph of an even function is symmetric with respect to the *y*-axis.

A function is **odd** iff $f(-x) = -f(x)$ for all *x* in the domain of *f*. The graph of an odd function is symmetric with respect to the origin.

Algebraically show $f(x) = -3x^5 - 2x^3 + 5x$ is even, odd, or neither.

Algebraically show $f(x) = \dfrac{1}{-3x^4 + 5x^2 - 7}$ is even, odd, or neither.

Algebraically show $f(x) = -3x^4 + 2x - 6$ is even, odd, or neither.

Math 150 Lecture Notes for Section 4D Maximum/Minimum Function Values

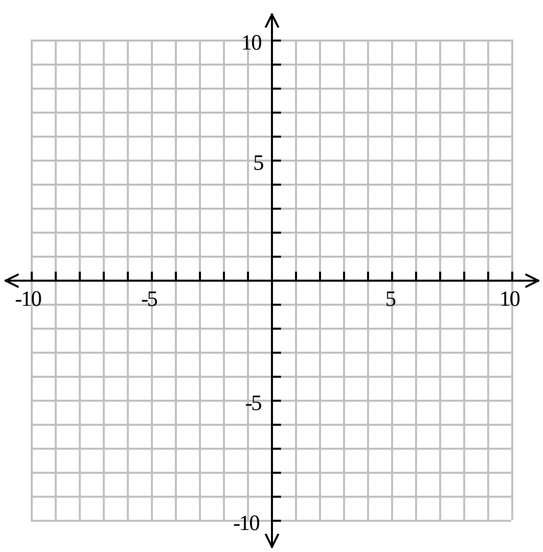
Quadratic Functions

A **quadratic function** can be written in the **general** form of $f(x) = ax^2 + bx + c$ where *a*, *b*, and *c* are real numbers and $a \neq 0$. The graph of a quadratic function is a parabola.

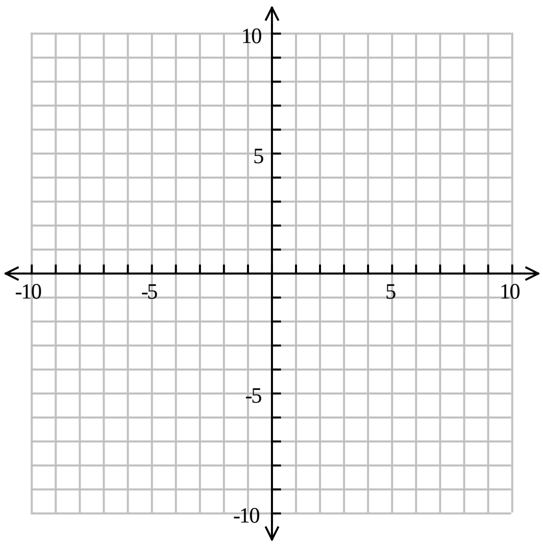
If the parabola opens up, the vertex (*h*, *k*) is the lowest point on the parabola, and *k* is the **minimum** value of *f* and it occurs when *x* = *h*. If the parabola opens down, the vertex (*h*, *k*) is the highest point on the parabola, and *k* is the **maximum** value of *f* and it occurs when *x* = *h*. Maximum and minimum values are called **extreme** values of the function.

The **standard** form of a quadratic is $f(x) = a(x-h)^2 + k$, where $a \neq 0$. If $a > 0$, the parabola opens up, and if $a < 0$, the parabola opens down. The horizontal shift is *h* units, and the vertical shift is *k* units. If $|a| > 1$, there is a vertical stretch by a factor of $|a|$; if $0 < |a| < 1$, there is a vertical shrink by a factor of $|a|$.

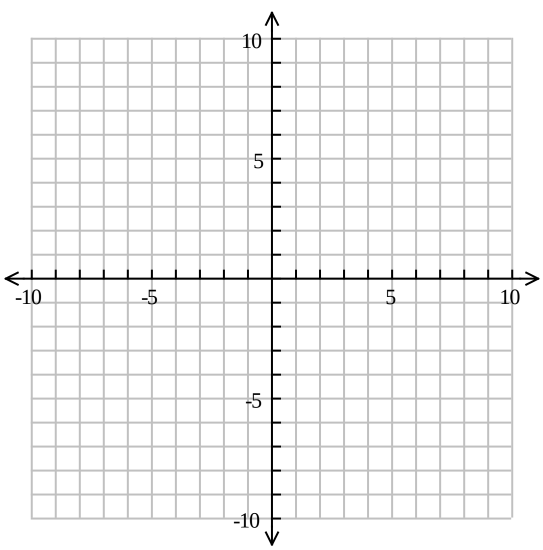
Graph $f(x) = 4x^2 - 48x + 149$ by writing f into standard form and then by transforming $y = x^2$.



Graph $f(x) = -2x^2 - 32x - 131$ by writing f into standard form and then by transforming $y = x^2$.



Sketch an even function that satisfies ***all*** of the following: is increasing on the intervals (2, 3) and (3, 7); is decreasing on the intervals $(0,2)$ and $(7,\infty)$; has *x*-intercepts at 5 and 9; has a *y*-intercept of -4; has a maximum value of 6 and is undefined at $x = 3$.



Axis of Symmetry

If $f(x)$ is a function with vertex $V(h,k)$, the axis of symmetry is $x = h$.

Find the vertex and axis of symmetry of $f(x) = -5x^2 + 10x + 4$.

Alternate Method: Find the vertex and axis of symmetry of $f(x) = ax^2 + bx + c$.

Using the alternate method, find the vertex and axis of symmetry of $f(x) = -5x^2 + 10x + 4$.

Note that $f(x) = ax^2 + bx + c$ written in standard form is

$$f(x) = a\left(x - \frac{-b}{2a}\right)^2 + \frac{4ac - b^2}{4a} \text{ with vertex } \left(\frac{-b}{2a}, \frac{4ac - b^2}{4a}\right).$$

Zeros of Quadratic Functions

If x_0 is a **zero** or **root** of a quadratic function $f(x) = ax^2 + bx + c$, then $f(x_0) = 0$.

Find the zero(s) of $f(x) = x^2 + 2x - 63$.

Find the zero(s) of $f(x) = -3x^2 - 7x + 2$.

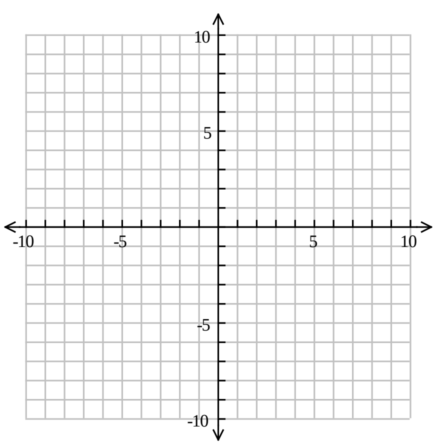
Find the zero(s) of $f(x) = 5x^2 - 20x + 20$.

Find the zero(s) of $f(x) = -x^2 - 4x - 5$.

The solution to any quadratic equation of the form $ax^2 + bx + c = 0$ is $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$. Note that $b^2 - 4ac$ is the **discriminant** of the function $f(x) = ax^2 + bx + c$.

- If $b^2 - 4ac > 0$, then there are two (real) zeros
- If $b^2 - 4ac = 0$, then there is one (real) zero
- If $b^2 - 4ac < 0$, then there are no (real) zeros

Calculate the discriminant of $f(x) = -2x^2 - 20x - 50$ and plot the function.



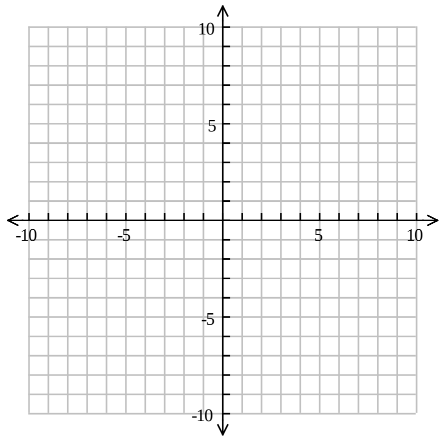
Applications

An object thrown or fired straight upwards at an initial speed of v_0 feet per second will reach a height of *h* feet after *t* seconds such that $h(t) = -16t^2 + v_0 t$.

A round stone is thrown straight upwards at an initial speed of 60 feet per second.

a. When does the stone reach a height of 50 feet?
b. When does the stone reach a height of 60 feet?
c. What is the maximum height reached by the stone?
d. When is the maximum height reached?
e. When does the stone hit the ground?

Local Maximum/Minimum Values



The graph changes from decreasing to increasing at **local minimum** points. The graph changes from increasing to decreasing at **local maximum** points.

While it is easy to determine a maximum or minimum value of a quadratic function, it usually takes methods used in calculus to find **local extrema** for other functions.

Use a graphing calculator to find the local extrema, rounding each coordinate to the nearest hundredth, for the function $y = -x^3 - x^2 + 2x + 3$.

More Applications

An open box is constructed from a 108 cm square sheet of cardboard by cutting squares of equal size from each corner and folding up the sides. What should be the dimensions of the cutout squares for the box to have maximum volume?

Math 150 Lecture Notes for Section 4E Combinations of Functions

Sum, Difference, Product, and Quotient

If *f* and *g* are functions with domains *A* and *B* respectively, then the following operations on the functions are defined

- Sum $(f+g)(x) = f(x) + g(x)$ with domain $A \cap B$
- Difference $(f-g)(x) = f(x) - g(x)$ with domain $A \cap B$
- Product $(fg)(x) = f(x) \cdot g(x)$ with domain $A \cap B$
- Quotient $\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}, g(x) \neq 0$ with domain $A \cap B - \{x \mid g(x) = 0\}$

Let $f(x) = x^2 - 9$ and $g(x) = \frac{x-3}{x+3}$.

a. What is the domain of *f*?

b. What is the domain of *g*?

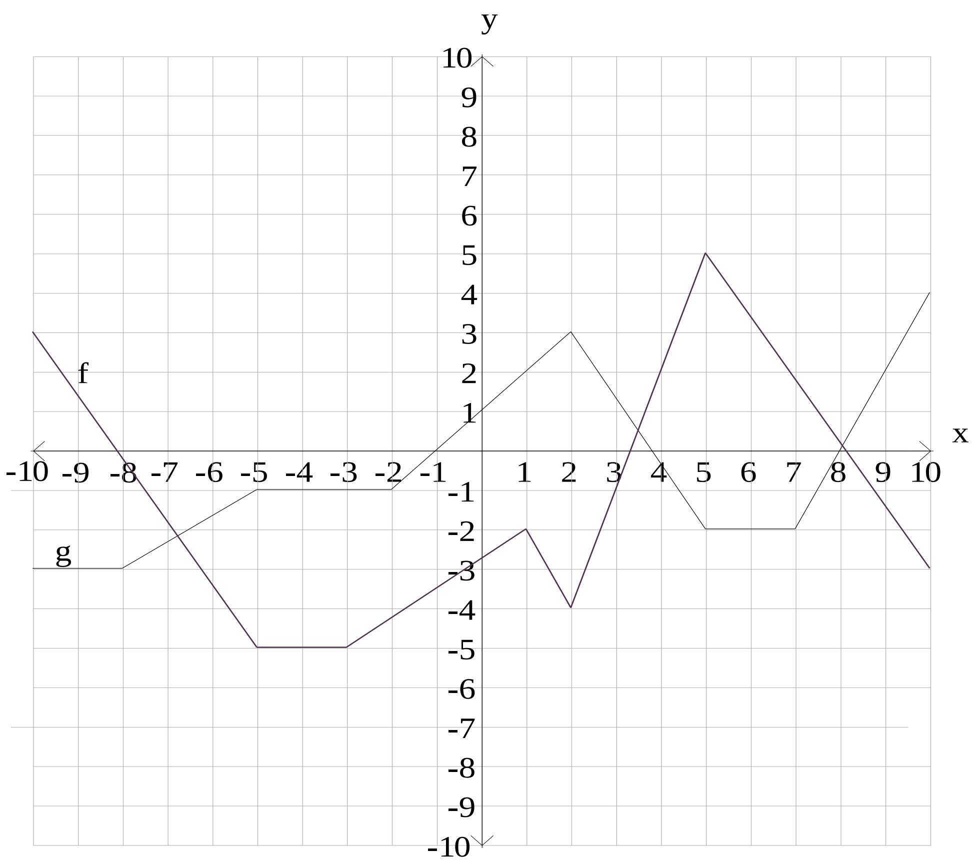
c. Find $(f+g)(x)$ and its domain.

d. Find $(f-g)(x)$ and its domain.

e. Find $(fg)(x)$ and its domain.

f. Find $\left(\frac{g}{f}\right)(x)$ and its domain.

Find the point-wise sum of the graphs shown below.



Function Composition

For functions *f* and *g*, the **composition of functions** $f \circ g$ is defined as $(f \circ g)(x) = f\left(g(x)\right)$. The domain of $f \circ g$ is $\{x \in Dom[g] \mid g(x) \in Dom[f]\}$

Let $f(x) = 5x + 2$ and $g(x) = \frac{1}{2x - 4}$.

a. $(f \circ g)(1) =$

b. $(f \circ g)(x) =$

c. Find the domain of $f \circ g$.

d. $(g \circ f)(3) =$

e. $(g \circ f)(x) =$

f. Find the domain of $g \circ f$.

g. $(g \circ f \circ f)(x) =$

Let $f(x) = \dfrac{1}{x^2}$ and $g(x) = \sqrt{x-5}$.

a. What is the domain of *f*?

b. What is the domain of *g*?

c. $(f \circ g)(x) =$

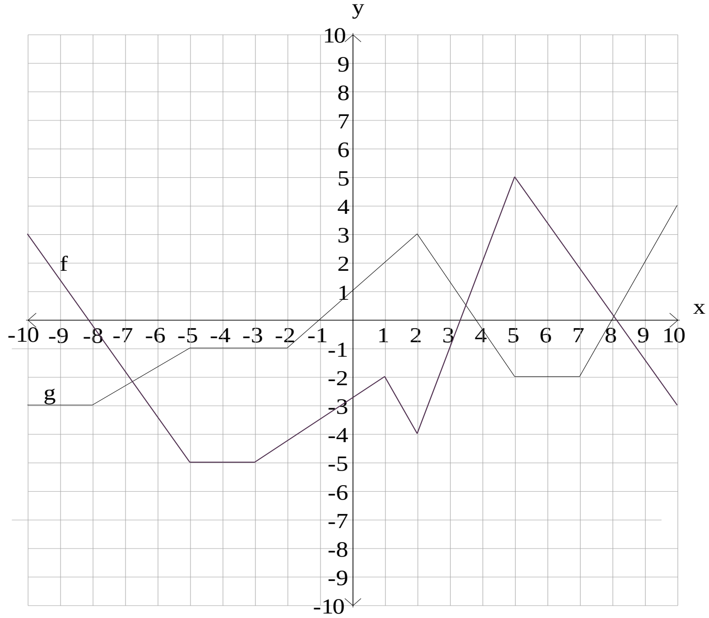
d. What is the domain of $f \circ g$?

e. $(g \circ f)(x) =$

f. Find the domain of $g \circ f$.

Let $h(x) = (x-9)^3$. Find functions *f* and *g* such that $h(x) = (f \circ g)(x)$.

Use the graph below to evaluate the following.



a. $(f \circ g)(-1) =$

b. $(g \circ f)(-1) =$

c. $(g \circ g)(-5) =$

d. $(f \circ g)(0) =$

Math 150 Lecture Notes for Section 4F Inverse Functions

Inverse Relations

A function *f* pairs an output *y* with each input *x*. The inverse of *f* reverses the process. If $(a,b) \in f$, then $(b,a) \in \text{inverse of } f$.

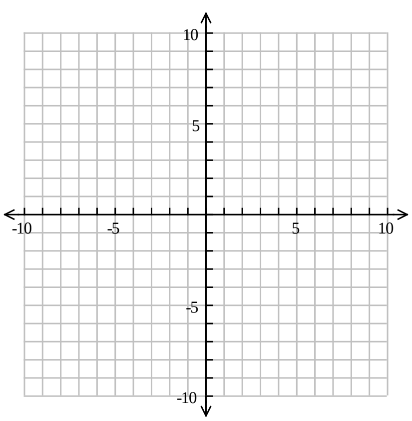
Domain f	**Range** f	**Domain of Inverse**	**Range of Inverse**
1	6		
4	10		
2	8		
5	9		
3	7		

Is the inverse a function?

Domain f	**Range** f	**Domain of Inverse**	**Range of Inverse**
2	1		
10	3		
8	5		
4	1		
6	3		

Is the inverse a function?

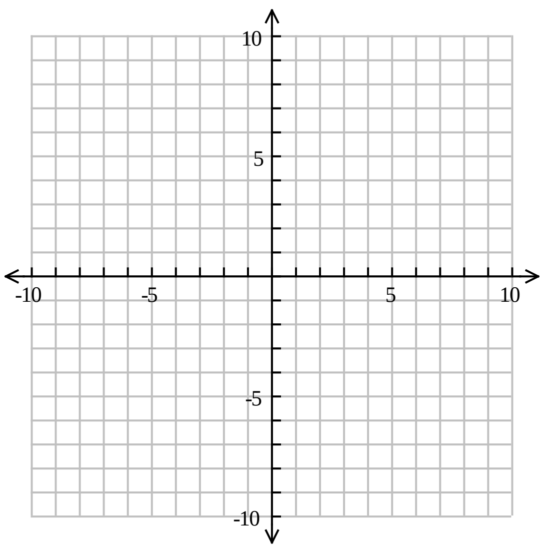
Graph $\{(-5,7),(-3,-5),(0,6),(1,-1),(2,2),(4,9)\}$ and its inverse on the same coordinate plane.



Note:

- Domain of f is the range of its inverse
- Range of f is the domain of its inverse
- Corresponding points have a symmetry about the line $y = x$
- x -intercept of f is the y -intercept of its inverse
- y -intercept of f is the x -intercept of its inverse
- Graphs of a function and its inverse are symmetric about the line $y = x$

Graph $y = \frac{x^3}{5} - 4$. Use symmetry about the line $y = x$ to graph its inverse. Is its inverse a function?



One-to-One Functions

All functions have inverse *relations*, but not all inverse relations are functions. For a function f to have an inverse function each x in the domain must be paired with a different y in the range. This type of function is called a **one-to-one function**.

Vertical Line Test – A set of point in the plane represents a function iff no vertical line intersects the graph in more than one point.

Horizontal Line Test – A function is one-to-one iff no horizontal line intersects the graph in more than one point.

Which of the following are one-to-one functions?

A function is **one-to-one** iff $f(x_1) = f(x_2)$ implies $x_1 = x_2$.

Algebraically prove or disprove $y = 2\sqrt[5]{x} - 4$ is a one-to-one function.

Algebraically prove or disprove $y = 3x^2 + 5$ is a one-to-one function.

Definition of Inverse Function

If f is a one-to-one function with domain A and range B, then, for every $b \in B$, its inverse function, f^{-1}, is defined by $f^{-1}(b) = a$ iff $f(a) = b$. If $(a,b) \in f$, the $(b,a) \in f^{-1}$. The domain of f^{-1} is the range of f and the range of f^{-1} is the domain of f.

If $f(9) = -7$, then $f^{-1}(-7) =$

If $f^{-1}(2) = 3$, then $f(3) =$

If the domain of f is $(-3,2] \cup [5,\infty)$, then the __________ of f^{-1} is _______________.

If the domain of f^{-1} is $(-\infty,\infty)$, then the __________ of f is _______________.

If f and g are inverse functions, then $(f \circ g)(x) = (g \circ f)(x) = x$

Show $f(x) = 5x^3 + 4$ and $g(x) = \sqrt[3]{\frac{x-4}{5}}$ are inverse functions.

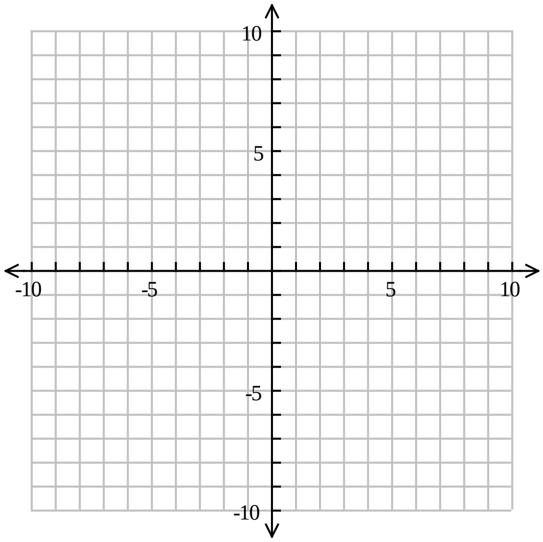
Finding the Inverse of a Function

Finding the inverse of a function f:

1. Find the domain and range of f.
2. Interchange the domain and range of f to find the domain and range of f^{-1}.
3. Write $y = f(x)$.
4. Interchange x and y.
5. Solve for y. The resulting y is f^{-1}.

If $f(x) = 8x + 6$, find its inverse function.

Graph $f(x) = 8x + 6$, its inverse function, and $y = x$ on the same graph (or *square* window).



If $f(x) = -2x^3 - 7$, find its inverse function.

The function $f(x) = \frac{1}{3}x^2 + 5$ is not a one-to-one function. Restrict the domain of f so that its inverse will be a function. Find its inverse.