Math 150 Lecture Notes for Chapter 5 Special Types of Functions

Math 150 Lecture Notes for Section 5A Polynomial Functions

Definition

A **polynomial function** is any function $p(x)$ that can be written in the form of $p(x) = a_n x^n + a_{n-1}x^{n-1} + ... + a_2 x^2 + a_1 x + a_0$ where all exponents are non-negative integers, all a_i are real numbers, and $a_n \neq 0$. The **degree** of the polynomial is n, the **leading coefficient** is a_n, the **leading term** is $a_n x^n$, and the **constant term** is a_0. A non-zero constant function is a polynomial of degree zero. The zero polynomial, $p(x) = 0$ is sometimes said to have degree of $-\infty$.

Polynomial	Degree	Leading Coefficient	Constant Term
$p(x) = 6 - 5x^2$			
$p(x) = -6$			
$p(x) = \sqrt{3}x^4 + x^{48}$			

Behavior of Polynomials

Polynomial	as $x \to -\infty,\ f(x) \to$	as $x \to \infty,\ f(x) \to$
$f(x) = -3x^3 + 6$		
$f(x) = \frac{1}{2}x^3 - 5x$		
$f(x) = -x^4 - 3x^3$		

Proof: Factor out the highest power of x in $p(x)$ and notice how each factor behaves for very small or very large values of x.

$$p(x) = x^n\left(a_n + \frac{a_{n-1}}{x} + \frac{a_{n-2}}{x^2} + ... + \frac{a_1}{x^{n-1}} + \frac{a_0}{x^n}\right)$$

So as $x \to \pm\infty, p(x) \approx a_n x^n$

Describe the end behavior of the polynomial $p(x) = a_n x^n + a_{n-1}x^{n-1} + ... + a_2 x^2 + a_1 x + a_0$.

as $x \to -\infty, p(x) \to$ as $x \to \infty, p(x) \to$

n odd, $a_n > 0$

n odd, $a_n < 0$

n even, $a_n > 0$

n even, $a_n < 0$

Describe the end behavior of the polynomial $p(x) = x^4 - 3x^3$.

Describe the end behavior of the polynomial $p(x) = -15x^{35} + 27x^{30} - 6$.

Describe the end behavior of the polynomial $p(x) = 24x^{56} + x^{55} + 30x^2 - 5$.

Math 150 Lecture Notes for Section 5B Rational Functions

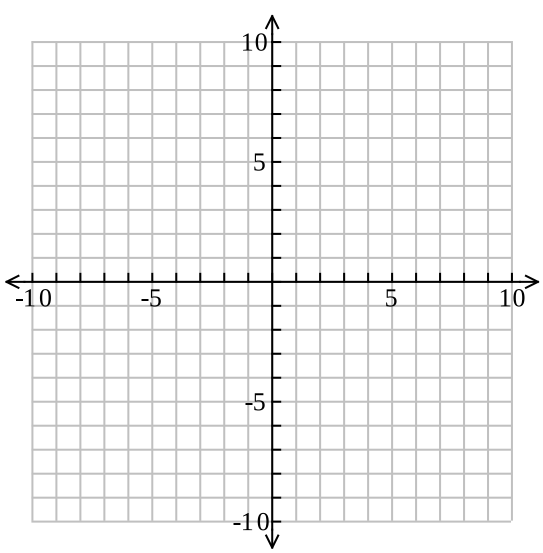
Definition of a Rational Function

A **rational function** is the quotient of two polynomials. That is, if $r(x)$ is a rational function then there are two polynomials $p(x)$ and $q(x) \neq 0$ such that $r(x) = \dfrac{p(x)}{q(x)}$.

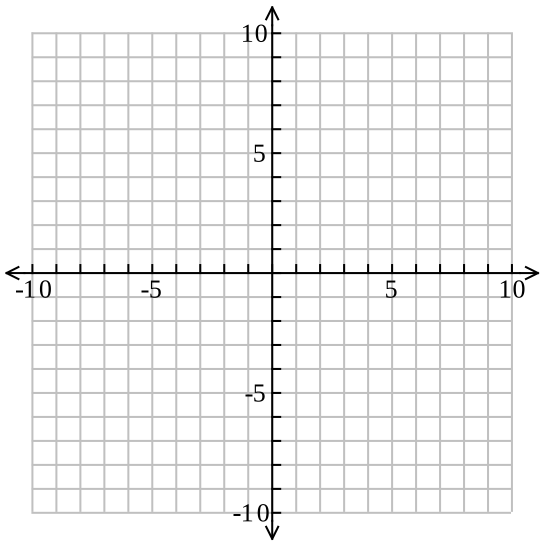
Examples:

The **domain** of any rational function $r(x) = \dfrac{p(x)}{q(x)}$ is the set of all real numbers except the set of x where $q(x) = 0$. The ***x*-intercepts** of a rational function are those real numbers x for which $r(x) = 0$, or that is, the values of x in the domain of $r(x)$ such that $p(x) = 0$. The ***y*-intercept** is y_0 such that $r(0) = y_0$.

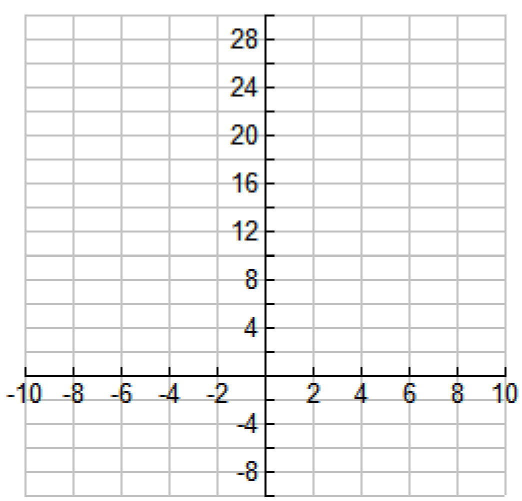
Find the domain, intercepts, and graph of the rational function $r(x) = \dfrac{x^2 - 3x - 10}{x^2 - 2x + 1}$.



Find the domain, intercepts, and graph of the rational function $r(x) = \dfrac{x^2 + x - 20}{x^3 - 8}$.



Find the domain, intercepts, and graph of the rational function
$r(x) = \dfrac{(x+1)\left(x^2 + 12x + 36\right)}{x^2 + 5x - 6}$.



Horizontal Asymptotes

True or False: The graph of a function cannot cross an asymptote.

The horizontal line $y = a$ is a horizontal asymptote of the function $r(x)$ if $\lim_{x \to \infty} r(x) = a$ or if $\lim_{x \to -\infty} r(x) = a$.

Determine the domain, intercepts and horizontal asymptotes of $r(x) = \dfrac{x}{(x-1)^2}$.
Determine the limits as *x* approaches negative and positive infinity by making a table.

Trick: If $r(x) = \dfrac{p(x)}{q(x)}$ is a rational function, divide the numerator and denominator by the highest power of *x*. To find the horizontal asymptote, let the absolute value of *x* get large. Note that any expression of the form $\dfrac{c}{x^n}$, where *c* is any constant and *n* is a positive number, must go to zero as the absolute value of *x* gets large.

Use this trick to determine the horizontal asymptote of $r(x) = \dfrac{x}{(x-1)^2}$.

Determine the domain, intercepts and horizontal asymptotes of $r(x)=\dfrac{8x^2+32x+80}{4x^2+32}$.

Determine the domain, intercepts and horizontal asymptotes of $r(x)=\dfrac{x^3-10x^2+25x}{x^2+x-2}$.

Theorem: Let rational function $r(x)=\dfrac{p(x)}{q(x)}$, where polynomial $p(x)$ is of degree *m* and polynomial $q(x)$ is of degree *n*, such that $p(x)=a_m x^m+a_{m-1}x^{m-1}+...+a_2x^2+a_1x+a_0$ and $q(x)=b_n x^n+b_{n-1}x^{n-1}+...+b_2x^2+b_1x+b_0$.

- If $m<n$, $\lim_{x\to\pm\infty} r(x)=0$.
- If $m=n$, $\lim_{x\to\pm\infty} r(x)=\dfrac{a_m}{b_n}$.
- If $m>n$, then $\lim_{x\to\pm\infty} r(x)$ does not exist.

Evaluate $\lim_{x\to\pm\infty}\dfrac{-5x^4-6x^3+8}{3x^2-6x^4+x}$ by using the 'trick' and again by using the theorem.

Vertical Asymptotes

The function $f(x)$ has a **vertical asymptote** at $x = a$ if the numbers $|f(x)|$ get arbitrarily large as *x* approaches *a* from the left or the right. That is, $\lim_{x \to a^-} f(x) = \pm\infty$ or $\lim_{x \to a^+} f(x) = \pm\infty$.

Let $r(x) = \dfrac{x+5}{x-4}$.

x	3	3.9	3.99999	4	4.00001	4.1	5
$r(x)$							

The vertical asymptote is:

Find the vertical asymptote(s) of $r(x) = \dfrac{x+3}{x^2-9}$.

If $r(x) = \dfrac{p(x)}{q(x)}$ is in *lowest terms*, this rational function will have a vertical asymptote at all values of *x* such that $q(x) = 0$.

Find the domain, and vertical and horizontal asymptotes of $r(x) = \dfrac{3x^3 - 12x^2 - 21x + 30}{x^3 - 7x^2 + 7x + 15}$.

Find the domain, range, and vertical and horizontal asymptotes of $f(x) = \frac{-6}{x+4} - 6$.

Find the horizontal and vertical asymptotes of $f(x)=\dfrac{x^3-6x^2-18x}{3x^2-6x}$.

Find the horizontal and vertical asymptotes of $f(x)=\dfrac{(x-1)(x+4)(x-8)}{(x+1)(x+2)(x-6)(x-9)}$.

Find the horizontal and vertical asymptotes of $f(x)=\dfrac{(x+10)(15x-2)}{(2x+4)(3x-9)}$.

Math 150 Lecture Notes for Section 5C Exponential Functions

Review of Exponents

For any positive real number a and any real numbers x and y, such that each expression is defined,

- $a^x a^y = a^{x+y}$
- $a^{-x} = \frac{1}{a^x}$
- $\frac{a^x}{a^y} = a^{x-y} = \frac{1}{a^{y-x}}$
- $a^0 = 1$, if $a \neq 0$
- $(ab)^x = a^x b^x$
- $\left(\frac{a}{b}\right)^x = \frac{a^x}{b^x}$
- $\left(a^x\right)^y = a^{xy}$

$\left(\frac{64}{125}\right)^{\frac{-2}{3}} =$

$y_1 = 2^x$

Exponential Functions

The function $f(x) = a^x$, with $a > 0$ and x any real number, is an **exponential function with base *a***.

$y_1 = 2^x$

$y_2 = \left(\frac{1}{2}\right)^x = 2^{-x}$

What is the relationship between the graphs of these two functions?

If (3, 216) is on the graph of $f(x) = a^x$, what must a equal?

If $5^x = 625$, what must x equal?

Plot the functions $y = 5^x$ and $y = \left(\frac{1}{5}\right)^x$ on the same graph.

If $f(x) = a^x$, then

- If $a > 1$, then the graph is always increasing

- If $0 < a < 1$, then the graph is always decreasing

- If $a = 1$, then the graph is always constant (and is trivial)

Properties of Exponential Functions

- The domain of $f(x) = a^x$ is all real numbers
- The range of $f(x) = a^x$, for $a \neq 1$, is all positive real numbers. If $a = 1$, then the range is the set $\{1\}$.
- $a^x \neq 0$
- If $a > 1$, then a^x goes to infinity as x goes to infinity, and a^x goes to zero as x goes to negative infinity.
- If $0 < a < 1$, then a^x goes to zero as x goes to infinity, and a^x goes to infinity as x goes to negative infinity.
- If $a > 1$ and $x < y$, then $a^x < a^y$ and $f(x) = a^x$ is an increasing function.
- If $0 < a < 1$ and $x < y$, then $a^x > a^y$ and $f(x) = a^x$ is a decreasing function.

Which number is larger $\left(\frac{2}{7}\right)^8$ or $\left(\frac{2}{7}\right)^9$?

What does the graph of $y = 3 \cdot 3^x$ as compared with $y = 3^x$?

What does the graph of $y = 2.5^x$ and $y = \pi^x$ as compared with $y = 3^x$?

If $a < b$ and $x > 0$, then $a^x < b^x$.
If $a < b$ and $x < 0$, then $a^x > b^x$.

Which number is larger $4^{-\sqrt{5}}$ or $3^{-\sqrt{5}}$?

What does the graph of $y = 2^{x-3} - 5$ as compared with $y = 2^x$? What is the domain and range of $y = 2^{x-3} - 5$?

The Natural Exponential Function

The number ***e*** is defined as $\lim_{n\to\infty}\left(\left(1+\frac{1}{n}\right)^n\right)=e$. The number $e\approx 2.71828182845905$.

Not only is e an irrational number, it is also a **transcendental** number. A number c is **algebraic** is there is a polynomial $p(x)$ with integer coefficients such that $p(c)=0$. If the number c is not algebraic, it is **transcendental**.

Name some other transcendental numbers.

The most commonly used exponential function is $f(x)=e^x$. Since the number e lies between the numbers 2 and 3 then the graph of $y=e^x$ is between the graph of $y=2^x$ and $y=3^x$ except at their mutual intersection point (0, 1).

To five decimal places, where appropriate, calculate the following:

e^{-1}

e^0

e^1

e^4

Note that $e^x=\lim_{n\to\infty}\left(\left(1+\frac{x}{n}\right)^n\right)$ and that if $x=1$, we have the definition of e.

Use the formula $e^x=\lim_{n\to\infty}\left(\left(1+\frac{x}{n}\right)^n\right)$ to find an approximation to the number $e^{-1.5}$.

Math 150 Lecture Notes for Section 5D Logarithmic Functions

Definition of Logarithms

Definition of a logarithm to a base *a*: Let a be any positive number not equal to one. The logarithm of x to the base a is y iff $a^y = x$. The number $y = \log_a x$ is just an exponent. Note that $y = \log_a x$ and $a^y = x$ are inverse functions.

$\log_3 81 = 4$ iff $3^4 = 81$

$\log_5 25 =$

$\log_2 \frac{1}{8} =$

$\log_{10} \sqrt{10} =$

$\log_\pi \pi^9 =$

$\log_e e^{\frac{-5}{7}} =$

$\log_{\sqrt{2}} 1 =$

If $3.56^{10} = x$, what is a in $\log_a y = 10$?

If the logarithm of x to the base 2 is 32, then what does x equal?

If $\log_9 x = -2$, what does x equal?

What is the domain, range and graph of $y = \log_{10} x$?

If $y = \log_a x$ and $a^y = x$ are inverse functions, what is the domain and range of $y = \log_a x$?

Properties of Logarithms

Properties of Logarithms

- The domain of $y = \log_a x$ is all positive real numbers, and the range is all real numbers.
- $a^{\log_a x} = x$
- $\log_a a^x = x$
- $\log_a x = \log_a y$ iff $x = y$
- $\log_a 1 = 0$
- $\log_a xy = \log_a x + \log_a y$
- $\log_a x^y = y \log_a x$
- $\log_a \frac{x}{y} = \log_a x - \log_a y$

What is the domain and range of $y = \log_3 (x+4) - 5$?

What is the domain and range of $y = \log_5 (x-6) + 3$?

$4^{\log_4 7} =$

$\log_{\sqrt[5]{9}} 1 =$

$\log_5 8 + \log_5 y =$

$\log_e e^x =$

$2\log_2 3 - \log_2 36 =$

Expand $\log_{10} \dfrac{5x-15}{2y}$.

Solve $6 + 2\log_4 x = 12$ for x.

Solve $3^{5-2x} = 12$ for x.

If $a^{4.5} = 27.48$ what does $\log_a 27.48$ equal?

Natural Logarithm

The natural log function is the log function with base e, that is, $y = \log_e x = \ln x$. Note that $y = \ln x$ and $e^y = x$ are inverse functions. Note: $\ln e^x = x$ and $e^{\ln x} = x$.

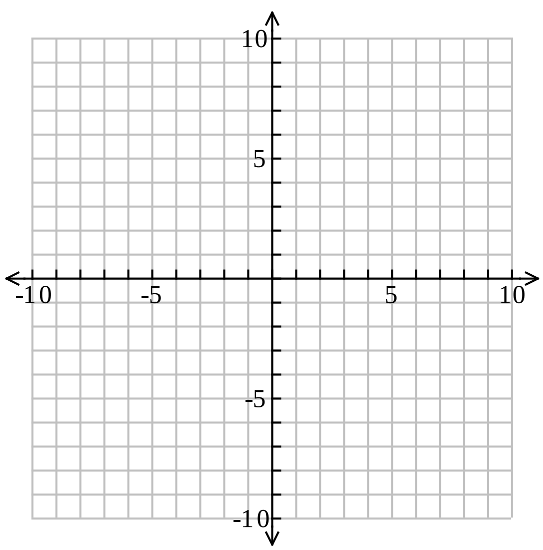
$\ln 1 =$

$\ln e =$

$\ln e^2 =$

To four decimal places, approximate $\ln 8$.

Graph $y = e^x$ and $y = \ln x$ on the same graph.



Change of Base Formula: $\log_a x = \dfrac{\ln x}{\ln a}$

To four decimal places, approximate the following:

$\log_{1.8} 5.37$

$\log_{\sqrt{5}} 25.98$

$\log_{10} 48$

A function has exponential growth if it has the form of $f(x) = a^{kx}$. Write this function in terms of the natural exponential function.

Solve the equation $5e^{4x-7} - 2 = 24$ for x.

Solve the equation $4e^{8-2x}+5=30$ for x.

Suppose $f(x)=ae^{kx}$ for some value of k and a such that $f(2)=4$ and $f(5)=6$. Solve for k and a.

If $f(x)=5e^{2x}-3$, find its inverse function.

In interval notation, what is the domain of the function $f(x)=\dfrac{\sqrt{x-4}}{\log_5(10-x)}$?