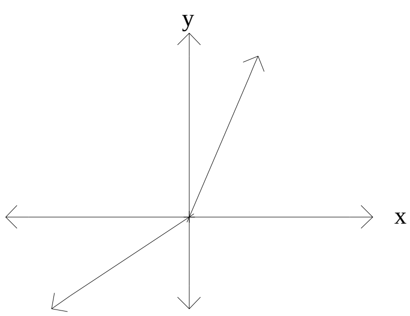
Math 150 Lecture Notes for Chapter 8 Trigonometry

Math 150 Lecture Notes for Section 8A Angles and Circles

Angles



Measures of Angle

Angles are measured in revolutions (rev), in degrees (^{o}), or in radians (rad). If no units are given it is understood to be in radians. Radians are the unit of choice in calculus!

1 rev = 360^o = 2π rad = 2π

$\frac{1}{2}$ rev = 180^o = π rad = π

Convert $\frac{\pi}{6}$ to revolutions and to degrees. Illustrate $\frac{\pi}{6}$ as an angle in standard position.

Convert 135^o to revolutions and to radians. Illustrate 135^o as an angle in standard position.

Convert $\frac{-1}{3}$ rev to degrees and to radians. Illustrate $\frac{-1}{3}$ rev as an angle in standard position.

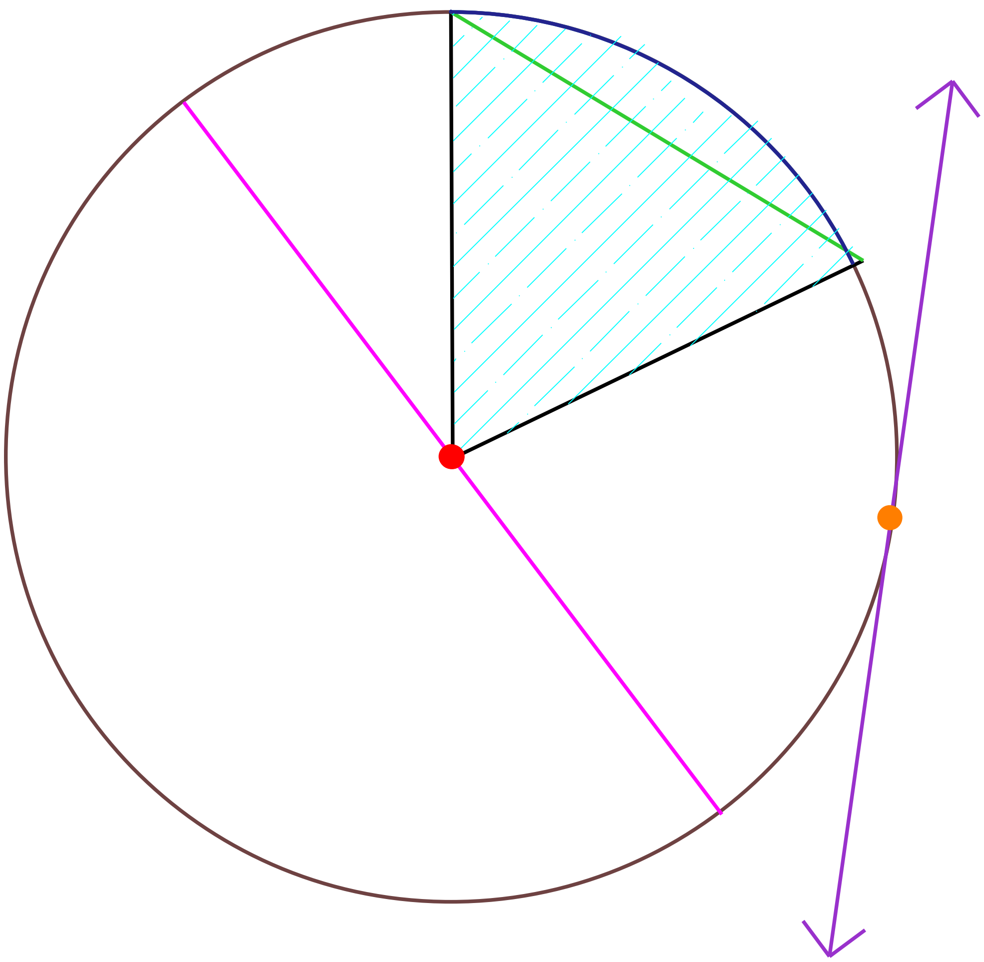
An **acute** angle has measure between 0 and 90 degrees or between 0 and $\frac{\pi}{2}$. An **obtuse** angle has measure between 90 and 180 degrees or between $\frac{\pi}{2}$ and π. The **complement** of an angle whose measure is θ is an angle whose measure is $\frac{\pi}{2}-\theta$. The **supplement** of an angle whose measure is θ is an angle whose measure is $\pi-\theta$.

Sketch and label an acute angle θ and its supplement α.

Parts of a Circle

The set of all points $P(x,y)$ in the plane that are a fixed distance (**radius**, r) from a fixed point (**center**, $C(h,k)$) is a **circle**, whose equation can be put in the standard form of $(x-h)^2+(y-k)^2=r^2$. The equation of a circle centered at the origin is $x^2+y^2=r^2$.

The **radius** is any line segment or the length r of this line segment from the center of the circle to any point on the circle. The **diameter** is any line segment or the length d of this line segment that passes through the origin of the circle and whose endpoints are on the circle. The **circumference**, $C=\pi d=2\pi r$, of a circle is the distance around the circle. The **area of a circle** is $A=\pi r^2$. A **disk** is the region inside the circle. An **arc** is any piece of the circle between two points on the circle. A **chord** of a circle is a line segment whose endpoints lie on the circle. A **central angle** is an angle whose vertex is at the center of a circle. A **secant line** is a line that intersects the circle twice or is a line that contains a chord of the circle. A **tangent line** is a line which intersects the circle at exactly one point, which is called the **point of tangency**.



$C =$ Circumference of the circle

$L =$ Length of an arc of the circle

$A_{\bigcirc} =$ Area of the whole circle

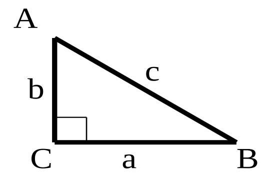
$A_{\measuredangle} =$ Area of a sector of the circle

$$L = \begin{cases} \dfrac{\alpha}{2\pi} C, & \text{for } \alpha \text{ in radians} \\ \dfrac{\beta}{360^o} C, & \text{for } \beta \text{ in degrees} \end{cases} \qquad A_{\measuredangle} = \begin{cases} \dfrac{\alpha}{2\pi} A_{\bigcirc}, & \text{for } \alpha \text{ in radians} \\ \dfrac{\beta}{360^o} A_{\bigcirc}, & \text{for } \beta \text{ in degrees} \end{cases}$$

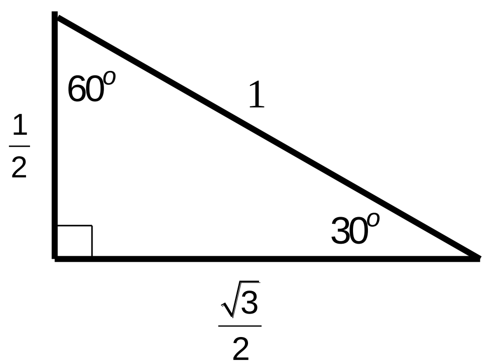
If a circle has a 20 mm diameter, find the arc length and sector area subtended by a central angle of $\frac{\pi}{3}$.

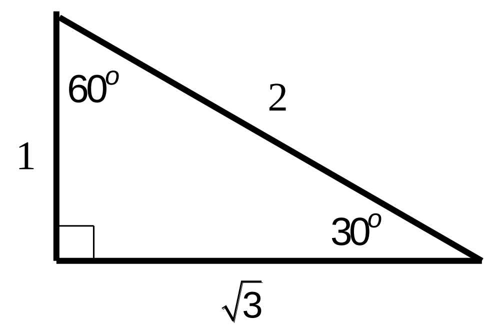
Math 150 Lecture Notes for Section 8B Trigonometric Functions

Pythagorean Theorem: If a right triangle has legs of length *a* and *b* and hypotenuse of length *c*, then $c^2 = a^2 + b^2$.

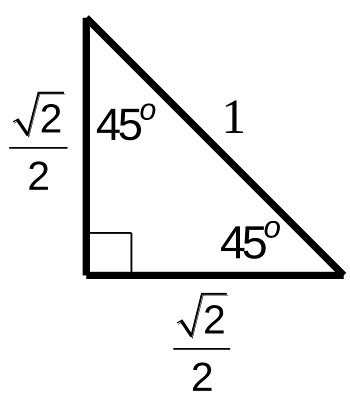


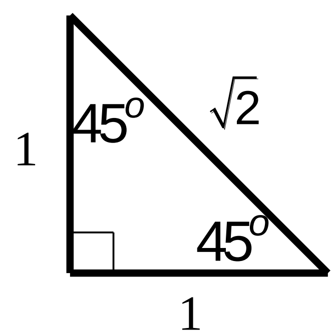
In a <u>30 – 60 – 90 degree triangle</u> $\left(\frac{\pi}{6}, \frac{\pi}{3}, \frac{\pi}{2} \quad triangle\right)$, the hypotenuse is twice as long as the length of the leg across from the 30-degree angle.





In a <u>45 – 45 – 90 degree triangle</u> $\left(\frac{\pi}{4}, \frac{\pi}{4}, \frac{\pi}{2} \quad triangle\right)$, the length of the legs are equal.





Definitions of Trig Functions (using Triangle)

$\sin\theta = \frac{opp}{hyp}$ $\csc\theta = \frac{hyp}{opp}$

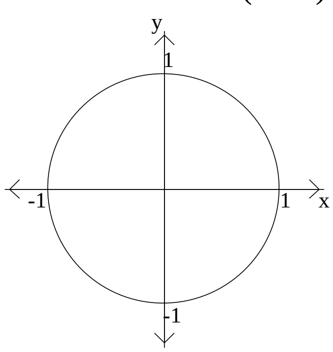
$\cos\theta = \frac{adj}{hyp}$ $\sec\theta = \frac{hyp}{adj}$

$\tan\theta = \frac{opp}{adj}$ $\cot\theta = \frac{adj}{opp}$

"sohcahtoa"

Definitions of Trig Functions (using Circle)

Due to similar triangles, the trig functions do not depend upon the radius of a circle, so we will use the unit circle (r = 1).



The coordinates of a point on the unit circle are $(x, y) = (\cos\theta, \sin\theta)$

$\sin\theta = y$ $\csc\theta = \frac{1}{y}$

$\cos\theta = x$ $\sec\theta = \frac{1}{x}$

$\tan\theta = \frac{y}{x}$ $\cot\theta = \frac{x}{y}$

All **S**tudents **T**ake **C**alculus (where trig functions are positive)

$\sin\frac{\pi}{3} =$

$\cos\frac{5\pi}{6} =$

$\tan\frac{5\pi}{4} =$

$\sec\frac{3\pi}{4} =$

$\cot\frac{23\pi}{6} =$

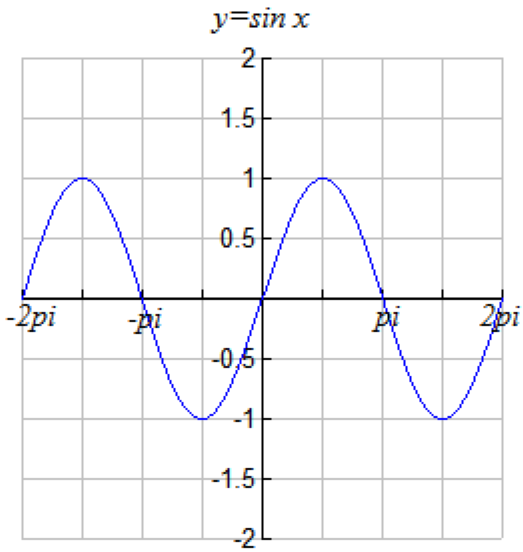
$\csc\frac{\pi}{2} =$

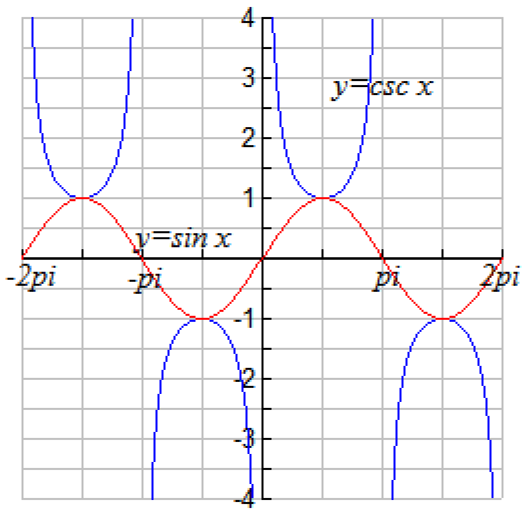
$\sec\frac{3\pi}{2} =$

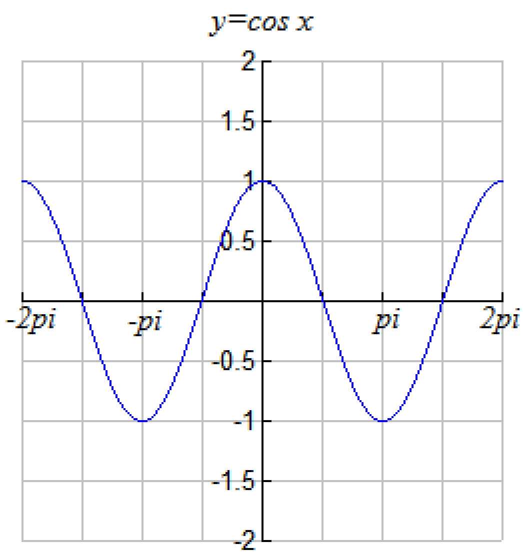
What quadrant does x lie if $\sin x < 0$ and $\sec x > 0$?

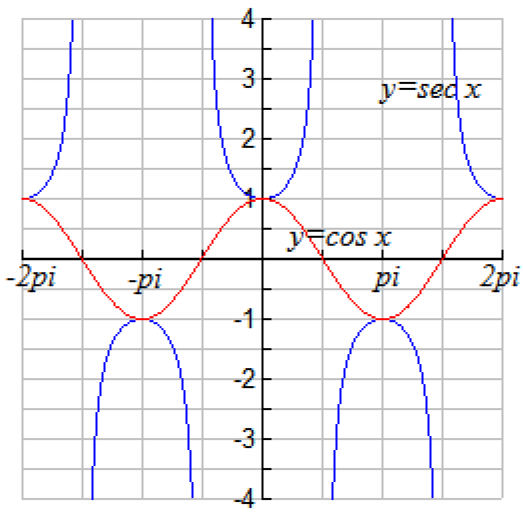
If $\tan x = \frac{6}{7}$ and x is in Quadrant III, find the values of all the trig functions.

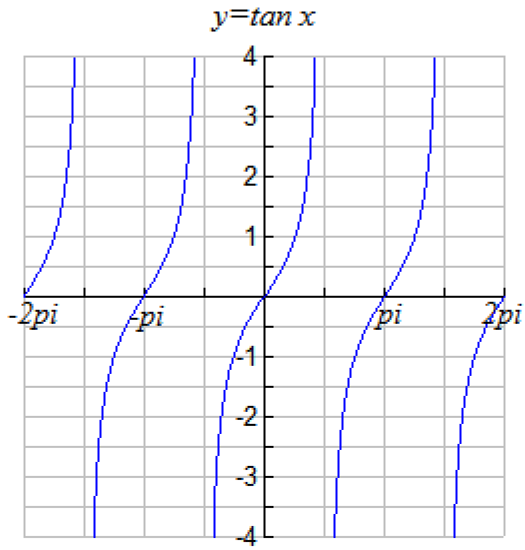
Math 150 Lecture Notes for Section 8C Graphs of the Trig Functions

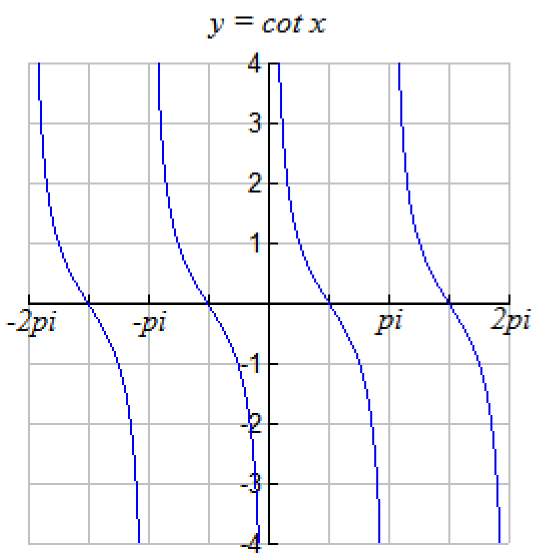





y=cos x
2
1.5
1
0.5
-0.5
-1
-1.5
-2
-2pi
-pi
pi
2pi


y=sec x
y=cos x
4
3
2
1
-1
-2
-3
-4
-2pi
-pi
pi
2pi


y=tan x
4
3
2
1
-1
-2
-3
-4
-2pi
-pi
pi
2pi

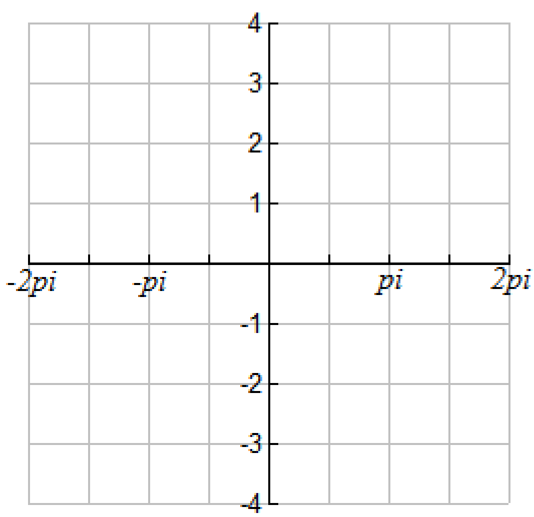

y = cot x
4
3
2
1
-1
-2
-3
-4
-2pi
-pi
pi
2pi

Mathematicians almost always compute and graph the trig functions in radians. This makes derivatives and integrals in calculus much easier. Since the coordinates of a point on a circle repeat with each revolution, the graphs of the basic trig functions repeat every 2π. Thus the trig functions are periodic and have a period of 2π (or π in the case of $y = \tan x$ and $y = \cot x$). Caution, the x and y in the formula such as $y = \sin x$, where x is the independent variable and y is the dependent variable have absolutely nothing to do with x and y in the circle definition of the trig functions.

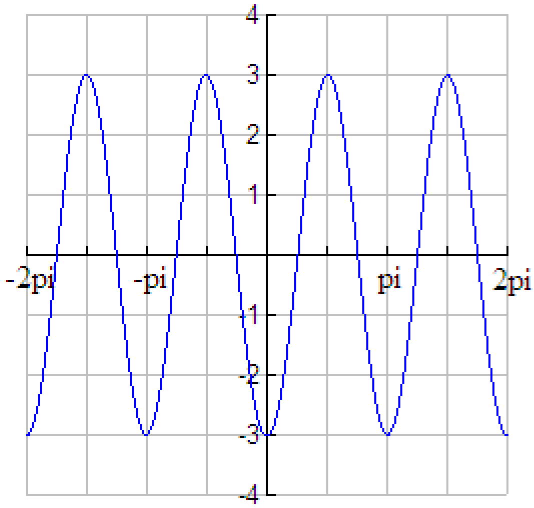
The curves of $y = a \sin k(x - b) + c$ and $y = a \cos k(x - b) + c$ for $k > 0$ have amplitude $|a|$, period $\frac{2\pi}{k}$, phase shift b (right or left) and vertical shift $|c|$ (up or down).

If $y = -3\cos(4x - 2) + 5$ find the amplitude, period, reflection, phase shift, and vertical shift. Graph it and the cosine function on the same window on your calculator and compare.

Graph $y = 4\sin(2x)$.



Write a function of the form $f(x) = a\sin k(x-b)$, whose graph is shown below, where a, k, and b are positive.



Math 150 Lecture Notes for Section 8D Trigonometric Identities

Negative Angle Identities (cosine and secant are even functions and the others are odd)

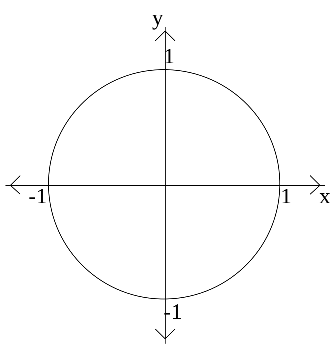
$\sin(-\theta) = -\sin\theta$ $\cos(-\theta) = \cos\theta$ $\tan(-\theta) = -\tan\theta$

$\csc(-\theta) = -\csc\theta$ $\sec(-\theta) = \sec\theta$ $\cot(-\theta) = -\cot\theta$

Complementary Angle Identities

$\sin\left(\frac{\pi}{2}-\theta\right) = \cos\theta$ $\tan\left(\frac{\pi}{2}-\theta\right) = \cot\theta$ $\sec\left(\frac{\pi}{2}-\theta\right) = \csc\theta$

$\cos\left(\frac{\pi}{2}-\theta\right) = \sin\theta$ $\cot\left(\frac{\pi}{2}-\theta\right) = \tan\theta$ $\csc\left(\frac{\pi}{2}-\theta\right) = \sec\theta$



There are also *supplementary angle identities* that are easy to prove from the corresponding complementary angle identities.

Pythagorean Identities

$\sin^2\theta + \cos^2\theta = 1$ $\tan^2\theta + 1 = \sec^2\theta$ $\cot^2\theta + 1 = \csc^2\theta$

Sum of Two Angles Formulas

$\sin(x+y) = \sin x\cos y + \cos x\sin y$

$\cos(x+y) = \cos x\cos y - \sin x\sin y$

$\tan(x+y) = \dfrac{\tan x + \tan y}{1 - \tan x\tan y}$

The *difference of two angles formulas* follow easily from the sum of two angles formulas and negative angle identities.

Double Angle Formulas

$\sin 2\alpha = 2\sin\alpha\cos\alpha$

$\cos 2\alpha = \cos^2\alpha - \sin^2\alpha = 2\cos^2\alpha - 1 = 1 - 2\sin^2\alpha$

$\tan 2\alpha = \dfrac{2\tan\alpha}{1-\tan^2\alpha}$

There are also square formulas, half-angle formulas, product-to-sum formulas, and sum-to-product formulas.

Guidelines for Proving Trigonometric Identities

1. Start with one side, preferably the more complicated side.
2. Use algebra and trig identities to simplify expressions.
3. If needed, try rewriting all expressions in terms of sine and cosine.

Prove $\cot^2\theta + 1 = \csc^2\theta$.

Verify $\dfrac{1-\sin^2 x}{\cos x} = \cos x$.

Verify $\tan x \sin x = \dfrac{\csc x}{\cot x + \cot^3 x}$.

Verify $\cos 2x = \cos^2 x - \sin^2 x$.

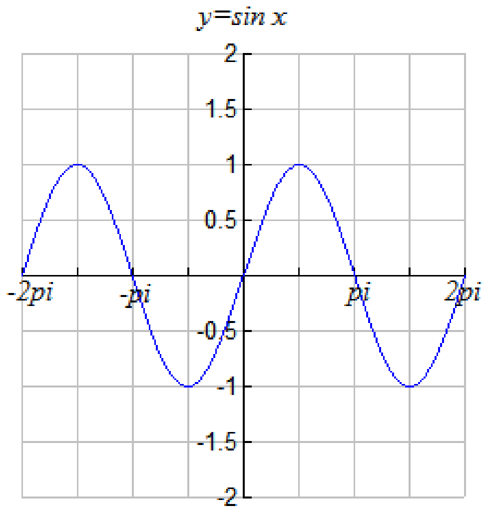
Exactly evaluate $\sin \frac{7\pi}{12}$.

Exactly evaluate $\sin \frac{2\pi}{3} \sin \frac{\pi}{12} - \cos \frac{2\pi}{3} \cos \frac{\pi}{12}$.

Given $\cot \theta = \frac{-15}{8}$ and θ is in quadrant IV, exactly evaluate $\sin 2\theta$.

Simplify $2\sin \frac{x}{4} \cos \frac{x}{4}$.

Math 150 Lecture Notes for Section 8E Inverse Sine Function



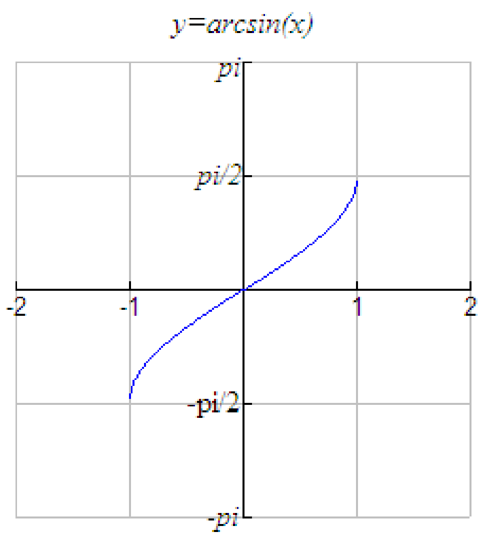
The sine curve is not one-to-one, but if we restrict its domain to $\left[\frac{-\pi}{2}, \frac{\pi}{2}\right]$, then it would be one-to-one and have an inverse function.

$$f(x) = \sin x,\ \frac{-\pi}{2} \le x \le \frac{\pi}{2} \qquad\qquad f^{-1}(x) = \sin^{-1} x = \arcsin x$$

Domain

Range Quad

$y = \sin^{-1} x = \arcsin x$ iff $\sin y = x$ where $y \in \left[\frac{-\pi}{2}, \frac{\pi}{2}\right]$



Note that $(\sin x)^{-1} = \left(\dfrac{1}{\sin x}\right) = \csc x \neq \sin^{-1} x = \arcsin x$

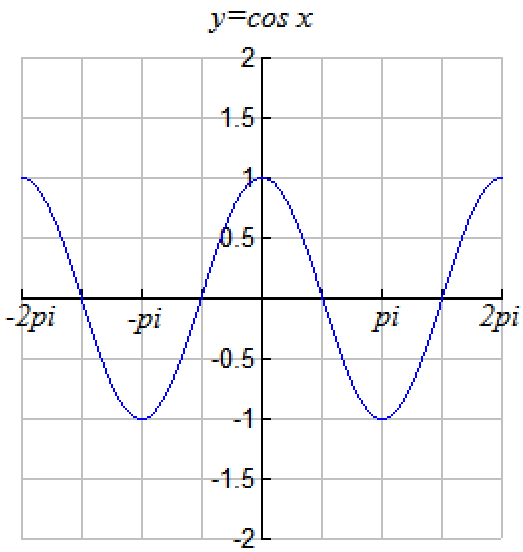
If $\sin\dfrac{\pi}{4} = \dfrac{\sqrt{2}}{2}$, then $\sin^{-1}\left(\dfrac{\sqrt{2}}{2}\right) =$

$\sin^{-1}\left(\dfrac{\sqrt{3}}{2}\right) =$

$\sin^{-1}\left(\sin\dfrac{5\pi}{3}\right) =$

$\tan\left(\sin^{-1}\left(\dfrac{-3}{4}\right)\right) =$

Math 150 Lecture Notes for Section 8F Inverse Trig Functions



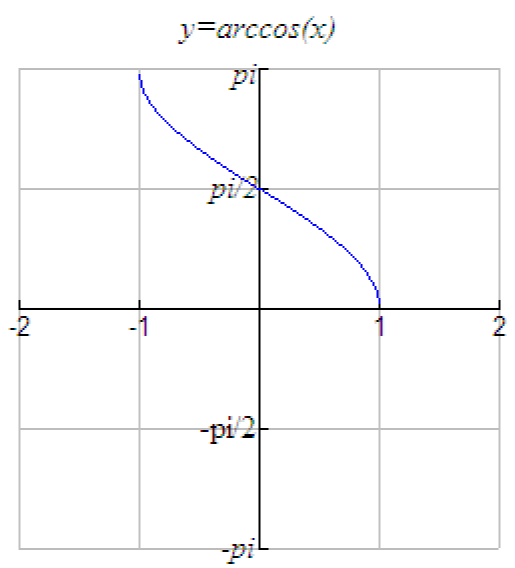
The cosine curve is not one-to-one, but if we restrict its domain to $[0,\pi]$, then it would be one-to-one and have an inverse function.

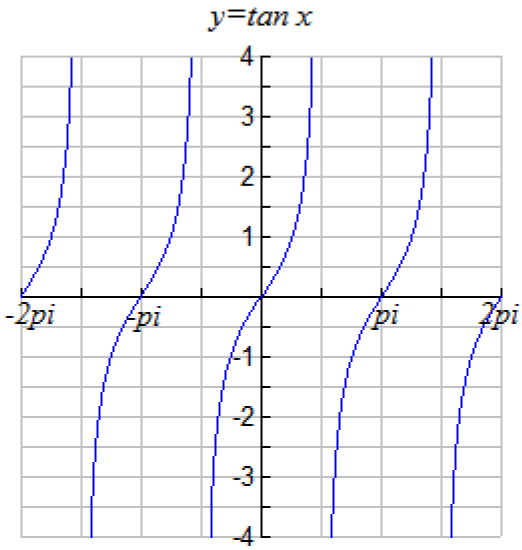
$f(x) = \cos x,\ 0 \le x \le \pi$ $\qquad f^{-1}(x) = \cos^{-1} x = \arccos x$

Domain

Range Quad

$y = \cos^{-1} x = \arccos x$ iff $\cos y = x$ where $y \in [0,\pi]$





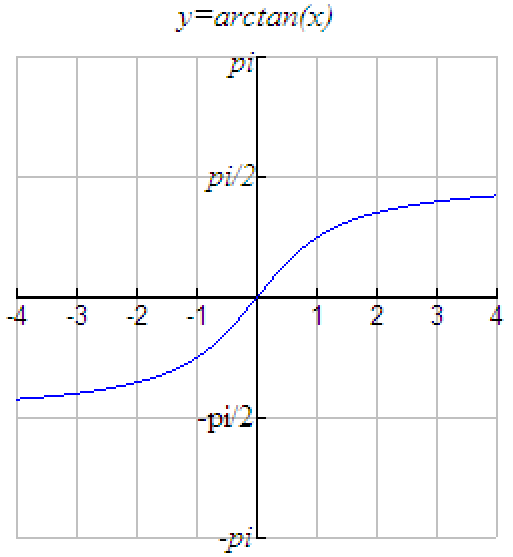
The tangent curve is not one-to-one, but if we restrict its domain to $\left(\frac{-\pi}{2}, \frac{\pi}{2}\right)$, then it would be one-to-one and have an inverse function.

$$f(x) = \tan x, \; -\frac{\pi}{2} < x < \frac{\pi}{2} \qquad\qquad f^{-1}(x) = \tan^{-1} x = \arctan x$$

Domain

Range Quad

$y = \tan^{-1} x = \arctan x$ iff $\tan y = x$ where $y \in \left(\frac{-\pi}{2}, \frac{\pi}{2}\right)$



$\cos^{-1}(-1) =$

$\tan\left(\tan^{-1}(3)\right) =$

$\cos\left(\cos^{-1}(\pi)\right) =$

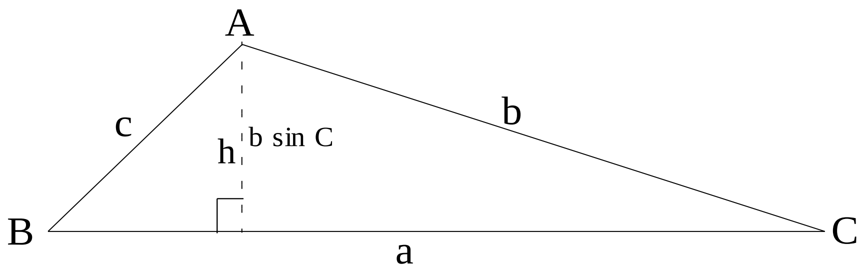
$\sec\left(\tan^{-1}\left(\frac{-3}{5}\right)\right) =$

Exactly evaluate $\sin\left(\cos^{-1}(3x)\right)$ if $0 \le x \le \frac{1}{3}$.

Math 150 Lecture Notes for Section 8G Law of Sines and Cosines

Law of Sines

In triangle *ABC*, $\dfrac{\sin A}{a} = \dfrac{\sin B}{b} = \dfrac{\sin C}{c}$



How do we deal with *oblique triangles*, triangles without a right angle?

From geometry SSS, SAS, ASA, and SAA uniquely determine a triangle.

Ambiguous Case
WARNING: There is not a SSA rule of triangles, that is, SSA does not uniquely determine a triangle.

Law of Cosines

In any triangle *ABC*,
$a^2 = b^2 + c^2 - 2bc\cos A$
$b^2 = a^2 + c^2 - 2ac\cos B$
$c^2 = a^2 + b^2 - 2ab\cos C$

If *ABC* is a right triangle with right angle *A* (thus having hypotenuse *a*), then $a^2 = b^2 + c^2 - 2bc\cos A$ becomes $a^2 = b^2 + c^2$ since cos $90^o = 0$. Thus the Pythagorean Theorem is a special case of the Law of Cosines!

Note: In any triangle, the longest side is across from the largest angle (and vice versa), and the shortest side is across from the smallest angle (and vice versa).

A triangle has six parts, three sides and three angles. If you know
Case 1: one side and two angles (ASA or SAA), use Law of Sines
Case 2: two sides and the angle opposite one of those sides (SSA), use Law of Sines;
WARNING
Case 3: two sides and the included angle (SAS), use Law of Cosines (next section)
Case 4: three sides (SSS) use Law of Cosines.

Given $a = 14$, $A = 21^o$ and $B = 35^o$, solve the triangle.

If $A = 30^{o}$, $b = 18\sqrt{2}$, and $a = 18$, solve the triangle.

Find the three angles of the triangle with sides of length 5.2 cm, 3.7 cm and 7.1 cm.

Given $a = 1$, $b = 2$, and $A = 45^{o}$, solve the triangle.

Math 150 Lecture Notes for Section 8H Solving Trig Equations

Solve $2\sin x - \sqrt{3} = 0$.

Solve $\sin x \tan x = \sin x$.

Solve $2\cos x + \sin 2x = 0$ on the interval $\left[0, 2\pi\right)$.