Math 365 Lecture Notes for Chapter 1 An Introduction to Problem Solving

Right notation: $\frac{1}{2}$

Poor notation: ½ because is 1/x+2 the same as $\frac{1}{x}+2$ or $\frac{1}{x+2}$?

0 is zero, a number; and $\varnothing$ is the empty set, a set

The natural numbers are N = {1, 2, 3, 4, 5, . . . }
The whole numbers are W = {0, 1, 2, 3, 4, 5, . . . }
The integers are Z = {. . . , -3, -2, -1, 0, 1, 2, 3, 4, 5, . . . }
The rational numbers are $Q = \left\{ \frac{a}{b} \middle| a, b \in Z; b \neq 0 \right\}$ which also include N, W, and Z.

Irrational Numbers R – Q or R\Q are non-repeating, non-terminating decimal number, and thus cannot be represented by a ratio of an integer and a non-zero integer.

Real numbers R are the set of all rational and irrational numbers.

The complex numbers are $C = \{a + bi | a, b \in R; i = \sqrt{-1}\}$. The complex numbers include all the real numbers.

$\therefore$ is the symbol for therefore.

"Iff" or "iff" stands for 'if, and only if.'

Text 1.1 Mathematics and Problem Solving

Read Polya's Four-Step Problem-Solving Process

1. Understand the problem

2. Devising a plan

3. Carrying out the plan

4. Looking back

Question: What is the sum of $1 + 2 + 3 + . . . + 98 + 99 + 100$?

Question: What is the sum of the first n natural numbers?

Arrange the numbers 21 through 29 into a 3 by 3 grid so that the sum of every row, column, and main diagonal is the same.

Text 1.2 Explorations with Patterns

inductive reasoning – the method of making generalizations based upon observations and patterns
Ex: The sun is in the sky. (because that is the way is has been for millions of years)

deductive reasoning – a conclusion is reached by going from established facts to particulars
ex: All triangles (in Euclidean geometry) have angle sum of 180 degrees. By deductive reasoning, all 'right' triangles (in Euclidean geometry) have angle sum of 180 degrees.

conjecture – a statement thought to be true, but not proven

counterexample – example that contradicts the conjecture, shows the conjecture false

Ex: Conjecture: If p is prime, then $2^p – 1$ is prime.

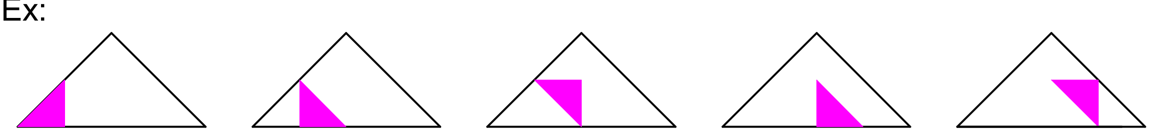
p	$2^p - 1$	
2	3	prime
3	7	prime
5	31	prime
7	127	prime
11	2047 = 23 * 89	not prime

Thus our conjecture is false since it is not true for all primes. That is, p = 11, thus, $2^p – 1 = 2047$ is our counterexample.

sequence – an ordered arrangement of numbers, figures, or objects

Ex: 2, 4, 6, 8, . . .

Ex:



<u>arithmetic sequence</u> – each successive term in a sequence is obtained from the previous term by adding a fixed number, the difference *d*.
Ex: $7, 4, 1, {}^{-}2, {}^{-}5, . . .$

The nth term of any **arithmetic sequence** with first term a_1 and difference d is
$\mathbf{a_n = a_1 + d(n – 1)}$. Note $n \in N$.

Ex: So for the arithmetic sequence, $7, 4, 1, {}^{-}2, {}^{-}5, . . .$, $a_1 = 7$ and $d = {}^{-}3$. Thus for this sequence $a_n = 7 + {}^{-}3(n – 1)$. To find a_6 of this sequence we can just add ${}^{-}3$ to ${}^{-}5$ to get ${}^{-}8$ (using the definition of arithmetic sequence) or we can use the formula $a_n = 7 + {}^{-}3(n – 1)$ for this sequence. Thus
$a_6 = 7+ {}^{-}3(6 – 1) = 7 + {}^{-}3(5) = {}^{-}8$.

Question: Given $a_1 = 5$ and given that the 20^{th} term of an arithmetic sequence is 81, what is the nth term of this sequence?

Answer:

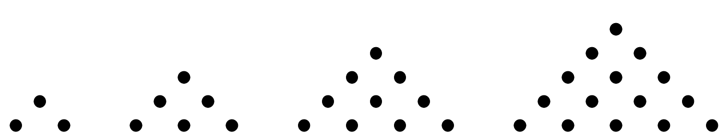
<u>geometric sequence</u> – each successive term in a sequence is obtained from its predecessor by multiplying it by a fixed ratio r.
Ex: 3, 6, 12, 24, . . .

The nth term of any **geometric sequence** with first term a_1 and ratio r is
$\mathbf{a_n = a_1 * r^{n-1}}$. Note $n \in N$.

Ex: So for the geometric sequence, 3, 6, 12, 24, . . ., $a_1 = 3$ and $r = 2$. To find a_5 of this sequence we can just multiply 24 by 2 to get 48 (using the definition of geometric sequence) or we can use the formula $a_n = 3 * 2^{n-1}$ for this sequence.
Thus $a_5 = 3 * 2^{5-1} = 3 * 2^4 = 3 * 16 = 48$.

BEWARE: $3 * 2^{n-1} \neq 6^{n-1}$.

<u>recursive sequence</u> – after one or more consecutive terms are given to start, each successive term of the sequence is obtained from the previous terms.
Warning: Since both geometric and arithmetic sequences can be also be written as recursive sequences, please name them as geometric or arithmetic and not as recursive.

Ex: 

Ex: $a_1 = 2$, $a_2 = 3$, $a_n = a_{n-2} + a_{n-1}$ for $n > 2$ [after the first two terms are given, the next term is obtained from the sum of the previous two terms] so this sequence is 2, 3, 5, 8, 13, . . .

<u>other sequences</u> examples:
SMTWT (what are the next two terms?)

1, 4, 9, 16, 25, . . . [here $a_n = n^2$ and these are the perfect squares]

Question: Find the sum of the first 35 terms of an arithmetic sequence in which the 5th term is 41 and the 42nd term is 300.

Text 1.3 Algebraic Thinking

Variable – unknown quantity or quantities or changing quantities

Properties

- Addition Property of Equality: If a = b, then a + c = b + c
- Multiplication Property of Equality: If a = b, then ac = bc.
- Cancellation Property of Equality for Addition: If a + c = b + c, then a = b.
- Cancellation Property of Equality for Multiplication: If c $\neq$ 0 and if ac = bc, then a = b.

How many milliliters of a 15% alcohol solution are added to a 40% alcohol solution to produce 50 milliliters of a 30% alcohol solution?

The sum of three consecutive odd numbers is –111. What is the smallest of the three numbers?

Text 1.4 Logic: An Introduction

In logic, a statement is a sentence that is either true or false, but not both.

Examples that are statements:
5 + 4 = 9 (true)
8 – 3 = 9 (false)
If today is Saturday, we do not have class (true)
All dogs are whippets (false)

Examples that are NOT statements:
5 + 8
Wow!
How are you?
I am a liar.

The negation of a statement is a statement with the opposite truth value of the given statement.

Statement: My name is Sherry.
Negation: My name is not Sherry.

Statement: We are not in Hawaii.
Negation: We are in Hawaii.

Be careful with quantifiers: all, every, no, some, at least one, there exists, each, etc.

Statement: All mammals are animals.
Negation: Some mammals are not animals. [There exists one mammal that is not an animal.]

Statement: Some animals are mammals.
Negation: No animals are mammals.

Statement: No mammal is an animal.
Negation: Some mammals are animals. [There exists one mammal that is an animal.]

Statement: Some animals are not mammals.
Negation: All animals are mammals.

See text for statement and negation columns, and know how to negate a statement.

Truth tables show all possible true/false (T/F) patterns for statement

Let p and q represent statements.

~p is the negation of p

$p \wedge q$ means **p and q**, is a conjunction, is true iff p and q are both true

$p \vee q$ means **p or q**, is a disjunction, is false iff p and q are both false

$p \equiv q$ means p and q are logically equivalent. Two statements are logically equivalent iff they have the same truth values.

$p \rightarrow q$ or $q \leftarrow p$ is a conditional statement, means "if p, then q," means "p implies q," means "q if p," and where p is the hypothesis and q is the conclusion.

Know all the truth tables from your text.

conditional statement $p \rightarrow q$
its converse $q \rightarrow p$
its inverse $\sim p \rightarrow \sim q$
its contrapositive $\sim q \rightarrow \sim p$

Conditional Statement: If it is a square, it is a polygon. [If it is a square, then it is a polygon.] [It is a polygon, if it is a square.]

Here p, the hypothesis, is "it is a square," and q, the conclusion, is "it is a polygon." Note: The hypothesis does not include the word "if" and the conclusion does not include the word "then."

Converse: If it is a polygon, it is a square. [If it is a polygon, then it is a square.] [It is a square if it is a polygon.]

Inverse: If it is not a square, it is not a polygon. [If it is not a square, then it is not a polygon.] [It is not a polygon, if it is not a square.]

Contrapositive: If it is not a polygon, it is not a square. [If it is not a polygon, then it is not a square.] [It is not a square if it is not a polygon.]

Show that a conditional statement ($p \rightarrow q$) is logically equivalent to its contrapositive ($\sim q \rightarrow \sim p$). Hint: Use a truth table.

$\therefore \ (p \rightarrow q) \equiv (\sim q \rightarrow \sim p)$

Negate, "This course is required and I will study." [Hint: Conjecture what might be equivalent to $\sim(p \wedge q)$.]

$p \leftrightarrow q$ is a biconditional and means p iff q, which is the same as
$p \rightarrow q$ AND $q \rightarrow p$.

Ex: A number is the multiplicative identity iff it is the number one.
This statement is equivalent to "If a number is the multiplicative identity then it is the number one, and if it is the number one, it is the multiplicative identity."

Reasoning is valid if the conclusion follows unavoidably from a true hypothesis.

Hypothesis: We are in College Station. College Station is in Texas.
Conclusion: We are in Texas.

"All mammals are animals" can be written as the conditional statement, "If a living creature is a mammal, then it is an animal."

Example of Euler Diagram:

Direct reasoning, law of detachment, modus ponens:
If "$p \rightarrow q$" is true and if "p" is true, then "q" must be true.

Assume true: If it is raining, we will use an umbrella. It is raining.
Conclude:

Indirect reasoning, modus tollens:
If "$p \rightarrow q$" is true and if "q" is false, then "p" must be false.

Assume true: A gardenia is an evergreen.
Know: A bald cypress is not an evergreen.
Conclude:

Chain rule: If "$p \to q$" is true and "$q \to r$" is true, then "$p \to r$" is true.

Assume true: If today is Thursday, I have dance lessons. If I have dance lessons, I get to sleep late.

Conclude:

False reasoning – invalid reasoning

Assume true: Smart people use computers.

False conclusion: If a person uses a computer, they are smart.

False conclusion: If a person is not smart, then they do not use a computer.