Math 365 Lecture Notes for Chapter 2 Sets, Whole Numbers, and Functions

Text 2.1 Describing Sets

Set – understood to be any collection of objects, must be well-defined

Element or member – individual objects of a set

Set A = {m, a, t, h}
$m \in A$
$p \notin A$

set-builder notation: $D = \{x \mid x \in N \text{ and } x < 6\}$

roster notation: B = {9, c, y}

Two sets are equal iff they contain exactly the same elements. The order of the elements does not matter.

E = {1, 2, 3, 4, 5}
F = {2, 4, 1, 3, 5}
G = {1, a}

D = E = F
$F \neq G$

Sets P and Q are in one-to-one correspondence, 1 – 1 correspondence, if the elements of P and Q can be paired so that for each element of P there is exactly one element of Q, and for each element of Q there is exactly one element of P.

Ex: There is a 1 – 1 correspondence between students in this class and their student id number.

Ex: Show a 1 – 1 correspondence between P = {a, b, c} and Q = {1, 2, 3}.

How many ways can we have a 1 – 1 correspondence between P = {a, b, c} and Q = {1, 2, 3}?

i.

ii.

iii.

Fundamental Counting Principle: If event M can occur in m ways, and after it has occurred, event N can occur in n ways, then event M followed by event N can occur in mn ways.

Ex: There are three types of pizza crust, two choices for pizza sauce (with or without), and five different toppings. How many different 1-topping pizzas can you have?

Two sets A and B are equivalent, A ~ B, iff there exists a 1 – 1 correspondence between the two sets.

Ex: If P = {a, b, c} and Q = {1, 2, 3}, then P ~ Q.

The cardinal number of a set A, n(A), indicates the number of elements in set A.

Ex: n(P) =

A set is finite if its cardinal number is a whole number.

Ex: Set C = {7, n, *, H} is finite since n(C) = 4.

An infinite set is a set that is not finite.

Ex:

A set that contains no elements has cardinal number 0 and is called the empty set or null set, designated { } or $\varnothing$.

Ex: B = {x | x is a person who lives on the moon}

n(B) =

n({ }) =

n($\varnothing$) =

n({$\varnothing$}) =

The universal set U is the set that contains all elements being considered in the discussion.

The <u>complement of a set</u> A, written $\overline{A}$, is the set of all elements in the universal set U that are not in A. $\overline{A} = \{x \mid x \in U, x \notin A\}$

Venn Diagram:

Ex: U = {a, b, c, d, e}

A = {a, c, e}

$\overline{A}$ =

Set B is a <u>subset</u> of A, B$\subseteq$ A, iff every element of B is an element of A.

Ex:

Venn Diagram:

Set B is a <u>proper subset</u> of A, B $\subset$ A, iff A $\neq$ B and B$\subseteq$ A. That is, every element of B is an element of A, and A has at least one more element not in B.

Ex:

Venn Diagram:

The empty set is a subset of itself and a proper subset of any other set. Since the empty set contains no elements there cannot be an element in the empty set that is not in a set.

Ex: Find all the subsets of E = {5, h}.

How many subsets does a finite set have?

Number of Elements	Number of Subsets	Example Set	List of its Subsets	Number of Proper Subsets
0				
1				
2				
3				

Therefore

Ex: E = {5, h} has how many subsets (using the formula)? Is that what we had?

For finite set A, how many proper subsets does it have? (Hint: Add a column "Number of Proper Subsets" to the above table.)

"less than" using sets: If A and B are finite sets then n(A) is <u>less than</u> n(B), written n(A) < n(B), if A is equivalent to a proper subset of B. So if n(A) = a and n(B) = b, then a < b. Similarly we define greater than: n(A) > n(B) or a > b, which is n(B) < n(A) or b < a, respectively.

Ex:
A = {1, g, *}

Represents a 1 – 1 correspondence (equivalence)

B = {5, 8, n, p, r}

Therefore n(A) < n(B) or 3 < 5 since A is equivalent to a proper subset of B.

If both sets are infinite, where A $\subset$ B, the sets can still be equivalent.

Ex: Let E be the even natural numbers.
N = {1, 2, 3, 4, . . . , n, . . . }

Represents a 1 – 1 correspondence (equivalence)

E = {2, 4, 6, 8, . . . , 2n, . . . }

n $\in$ N corresponds to 2n $\in$ E.

Therefore N ~ E

Text 2.2 Other Set Operations and Their Properties

Set intersection: $A \cap B = \{x \mid x \in A \quad and \quad x \in B\}$

Two sets A and B are disjoint iff $A \cap B = \varnothing$.

Set union: $A \cup B = \{x \mid x \in A \quad or \quad x \in B\}$

Set complement of A relative to B: $B - A = \{x \mid x \in B \quad and \quad x \notin A\}$

Ex: A = {x, y, z}, B = {x, y, 2, 4}, C = {2, 6, 9}

$A \cap B$ =

$A \cup B$ =

$A \cap C$ =

$A \cup C$ =

A – B =

B – A =

Commutative property of set union: $A \cup B = B \cup A$
Commutative property of set intersection: $A \cap B = B \cap A$
Distributive property of set intersection over set union: $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

Venn Diagram Examples where A, B, C $\subset$ U

$B \cap \overline{C}$ $A \cup \overline{(B \cup C)}$

Cartesian Product of sets A and B: $A \times B = \{(a,b) \mid a \in A, b \in B\}$
Note: (a, b) is an "ordered" pair, "A x B" is read "A cross B"

Ex: The Cartesian plane (coordinate plane, x-y plane) is R x R.

Ex: A = {5, 8, n}, B = {8, v}

A x B =

B x A =

Text 2.3 Addition and Subtraction of Whole Numbers

Definition of Addition of Whole Numbers: Let A and B be *disjoint finite* sets. If n(A) = a and n(B) = b, then a + b = n(A $\cup$ B).

a and b are the addends, and (a + b) is the sum.

Set Model Example (sets must be disjoint and finite!):

The problem with the set definition of addition is that not all sets are disjoint.

Number Line Model: If you have 3 meters of copper wire and 2 meters of aluminum wire, how many meters of wire do you have all together?

Definition of <u>Less Than</u>: For any a, b $\in$ W, a is less than b, written $a < b$, iff there exists a k $\in$ **N** such that $a + k = b$.

$a \leq b$ means $a < b$ or $a = b$; also have $a > b$ and $a \geq b$

Ex: Use the definition of less to show $5 < 7$.

Whole Number Addition Properties

- Closure: If m, n $\in$ W, then m + n $\in$ W
- Commutative: $a + b = b + a$
- Associative: $(a + b) + c = a + (b + c)$
- Unique Identity 0: $a + 0 = 0 + a = a$

Examples:

Basic Addition Strategies

- Counting on: 3 + 2 is 3, 4, 5 so $3 + 2 = 5$
- Doubles: $2 + 3 = 2 + 2 + 1 = 4 + 1 = 5$
- Making Tens: $7 + 5 = 7 + (3 + 2) = (7 + 3) + 2 = 10 + 2 = 12$

Definition of <u>Subtraction of Whole Numbers</u>: For any a, b $\in$ W, such that $a \geq b$, $a – b$ is a unique c $\in$ W such that $a = b + c$

So subtraction is defined in terms of what?

Are the whole numbers closed under subtraction?

If a, b $\in$ W such that $a < b$, then $a – b$ is not defined in the whole numbers.

Set Take Away Model: 5 – 3 =

Missing Addend Model: 5 – 3 =

Comparison Model: If you have 5 blocks and 3 balls, how many more blocks than balls?

Number Line Model: 7 – 4

[The Number Line Model – Adding and Subtracting

- Start at zero facing the positive direction
- Add means stay facing the same direction
- Subtract means turn around
- Positive number means go forwards
- Negative number means go backwards]

Text 2.4 Multiplication and Division of Whole Numbers

Definition of Multiplication of Whole Numbers: For any $a \in W$ and $n \neq 0$,
$n \cdot a = a + a + a + \ldots + a$ where there are n terms.

a and b are factors of $a \cdot b = ab$, and $a \cdot b$ or ab is the product

So multiplication is defined in terms of what?

Repeated Addition Model using Colored Rods: $4 \cdot 3$

Repeated Addition Model Using a Number Line: $4 \cdot 3$

Array Model Using Intersections: $4 \cdot 3$

Array Model Using Unit Squares: $4 \cdot 3$

Cartesian Product Model: $2 \cdot 3$
Crust = {thin, thick}, Toppings = {pepperoni, cheese, mushroom}
Tree Diagram

By the Fundamental Counting Principle, the number of ordered pairs in Crust x Toppings is $2 \cdot 3$.

Alternate Definition of Multiplication of Whole Numbers: For finite sets A and B where n(A) = a and n(B) = b, then $a \cdot b = n(A \times B)$.

Whole Numbers Multiplication Properties:

- Closure: If $a, b \in W$, then $ab \in W$
- Commutative: $ab = ba$
  Block illustration:

- Associative: $a(bc) = (ab)c$
  Block illustration:

- Unique Identity 1: $a \cdot 1 = 1 \cdot a = 1a = a$
- Zero: $a \cdot 0 = 0 \cdot a = 0$

Distributive Property of Multiplication Over Addition of Whole Numbers:

a(b + c) = ab + ac

Block illustration:

Note: ab + ac = a(b + c) is also known as factoring, but it is still the distributive property of multiplication over addition.

Algebra Tile Model: (x + y) (z + w)

Order of Operations:
Please Excuse | My Dear | Aunt Sally |
Parenthesis Exponents | Multiplication and Division in order from left to right | Addition and Subtraction in order from left to right

$-5^2 - 60 \div 5 \cdot 4 - 8 + 3 - 2(5-7)^2 + 3 \cdot 2^3 =$

Definition of Division of Whole Number: For any a, b $\in$ W, b $\neq$ 0, a $\div$ b = c iff c is the unique whole number such that b $\cdot$ c = a.

a is the dividend, b is the divisor, c is the quotient

So division is defined in terms of what?

Set Model (dealing out): Have 12 cookies and want each of 4 people to get the same amount so have 12 $\div$ 4.

Set Model (partition): Have 12 cookies and want each of 4 people to get the same amount so have 12 $\div$ 4.

Missing Factor Model: $12 \div 4 =$
So what times 4 equals 12? Since $4 \cdot 3 = 12$, then $12 \div 4 = 3$.

Repeated Subtraction Model: Division can also be thought of as repeated subtraction.
$12 \div 4 = 12 - 4 - 4 - 4 = 0$ so $12 \div 4 = 3$ since it took subtracting 3 fours to get to zero.

Now $25 \div 6$ has no meaning in the whole numbers since $25 \div 6 \notin W$. However, if I had 25 M&Ms to divide between 6 kids, I would give each of them 4 M&Ms and have 1 left for me. Thus we have the need for the Division Algorithm.

<u>Division Algorithm</u>: Given a, b $\in$ W, such that $b \neq 0$, there exists a unique whole number q (think quotient) and r (think remainder) such that a = bq + r with $0 \leq r < b$.

If $a \div b$ has no remainder, then a is <u>divisible</u> by b.

So using the Division Algorithm on $25 \div 6$, we have $25 = 6 \cdot 4 + 1$ where a = 25, b = 6, q = 4 and r = 1.

$153 \div 7 \notin W$ but we can use the Division Algorithm:

Special Cases:

- Case: $0 \div n$, $n \neq 0$, $n \in W$
  Proof: By the definition of division of whole numbers, $0 \div n = x$ iff x is a unique whole number such that 0 = nx. Since $n \neq 0$ and since 0 = nx, then by the zero multiplication property of whole numbers, x = 0. Therefore $0 \div n = 0$ for $n \neq 0$.

- Case: $n \div 1$, $n \in W$
  Proof: By the definition of division of whole numbers, $n \div 1 = x$ iff x is a unique whole number such that n = 1x. By the multiplicative identity of whole numbers 1x = x, so n = 1x = x, that is n = x. Therefore $n \div 1 = n$.

Why Can't We Divide by Zero?

Explanation:

Proof:

- Case: $n \div 0$, $n \neq 0$, $n \in W$
  Proof: By the definition of division of whole numbers, $n \div 0 = x$ iff x is a unique whole number such that $n = 0x$. By the zero property of multiplication of whole numbers $0x = 0$, so $n = 0x = 0$. But $n = 0$ contradicts our assumption that $n \neq 0$. Therefore for $n \in N$ (that is, $n \neq 0$, $n \in W$), $n \div 0$ is undefined.

- Case: $0 \div 0$
  Proof: By the definition of division of whole numbers, $0 \div 0 = x$ iff x is a "unique" whole number such that $0 = 0x$. By the zero multiplication property of whole numbers, $0 = 0x$ for all x, so x is not unique. Therefore $0 \div 0$ is undefined.

Explain the relationships among the operations of addition, subtraction, multiplication and division.

Text 2.5 Functions

A <u>function</u> from set A to set B is a correspondence from A to B in which each element of A is paired with one, and only one, element of B.

<u>Domain</u> – set A, the set of all inputs
<u>Range</u> – subset of set B, set of all outputs

Functions can be represented by

A sequence can be represented by a function. Give the function that represents the geometric sequence that has 2 as its first term and whose ratio is 5.

Make a T-chart showing the first 4 terms of this function.

The composition of two functions f and g is $(f \circ g)(x) = f(g(x))$. The domain of $f \circ g$ is the set of all x in the domain of g such that g(x) is in the domain of x.

Figure:

Let $f(x) = 5x + 2$ and $g(x) = 3x$.

What is the domain of f?

What is the range of f?

What is the domain of g?

What is the range of g?

$(f \circ g)(x) =$

What is the domain of $f \circ g$?

$(g \circ f)(x) =$

What is the domain of $g \circ f$?

Is function composition commutative?

$(f \circ f)(x) =$

Let $h(x)=\frac{1}{x^2}$ and $F(x)=\sqrt{x-5}$.

What is the domain of h?

What is the domain of F?

$(h\circ F)(x)=$

What is the domain of $h\circ F$?

$(F\circ h)(x)=$