Math 365 Lecture Notes for Chapter 5 Rational Numbers as Fractions

How do we solve 2x = 1 or x + x = 1? We need a 'new' kind of number to solve this. Numbers like $\frac{1}{2}$.

Text 5.1 The Set of Rational Numbers

The **rational numbers** are $Q = \left\{ \frac{a}{b} | a, b \in Z; b \neq 0 \right\}$. That is, a rational number can be written as an integer over a nonzero integer.

Fractions are of the form $\frac{a}{b}$, b $\neq$ 0, but a and b are not necessarily integers. Some examples of fractions are

Fundamental Law of Fractions

$\frac{a}{b} = \frac{an}{bn}$ where a, b, and n can be any number such that b $\neq$ 0 and n $\neq$ 0. The value of a fraction does not change if its numerator and denominator are multiplied by the same nonzero number because you are really multiplying the fraction by one.

Example:

If $\frac{a}{b}$ or a $\div$ b, then a is the **numerator** and b is the **denominator**.

Do not write "½ " with a slash because then students learn bad habits, especially as the fractions get more complicated. When a student writes "1/x+2," is it $\frac{1}{x}+2$ or $\frac{1}{x+2}$? Write all fractions with a horizontal fraction bar, example: $\frac{3}{5}$.

Use rational numbers in multiplication and division problems, to describe a part of a whole, as ratios, and as probabilities.

Models of $\frac{1}{4}$:

Proper fraction: $\frac{a}{b}$ where $0 \leq |a| < |b|$.

Examples of proper fractions:

Improper fraction: $\frac{a}{b}$ where $|a| \geq |b| > 0$.

Examples of improper fractions:

Equivalent or **equal fractions** represent equal amounts.

Examples of equivalent or equal fractions:

A rational number $\frac{a}{b}$ is in **simplest form** or **lowest terms** if a and b have no common factor greater than 1, that is, if a and b are relatively prime.

To write $\frac{a}{b}$ in simplest form divide both a and b by gcd (a, b). Why are we allowed to do this?

Write in simplest form:

$$\frac{6}{21} =$$

$$\frac{6}{21} =$$

$$\frac{b^2 - a^2}{5a - 5b} =$$

Equality of Fractions

1. Reduce both fractions, $\frac{27}{33}$ and $\frac{45}{55}$, to simplest form.

2. Rewrite both fractions, $\frac{27}{33}$ and $\frac{45}{55}$, with the same least common denominator.

3. Rewrite both, $\frac{27}{33}$ and $\frac{45}{55}$, with a common denominator (not necessarily the least).

Property: Two fractions $\frac{a}{b}$ and $\frac{c}{d}$ are equal iff ad = bc.

Theorem: If a, b, and c are integers and b > 0, then $\frac{a}{b} > \frac{c}{b}$ iff a > c.

Theorem: If a, b, c, and d are integers, and b > 0, d > 0, $\frac{a}{b} > \frac{c}{d}$ iff ad > bc.

Denseness of Rational Numbers: If $\frac{a}{b}$ and $\frac{c}{d}$ are two distinct rational numbers, then there is another rational number between these two numbers. [Actually there are a countably-infinite number of rational numbers between any two distinct rational numbers.]

Find three rational numbers between $\frac{1}{2}$ and $\frac{1}{3}$.

Theorem: Let $\frac{a}{b}$ and $\frac{c}{d}$ be any rational numbers with positive denominators where $\frac{a}{b} < \frac{c}{d}$. Then $\frac{a}{b} < \frac{a+c}{b+d} < \frac{c}{d}$.

Proof:

Text 5.2 Addition and Subtraction of Rational Numbers

Use Polya's 4-Step Problem Solving Process to find $\frac{2}{5}+\frac{1}{2}$.

a. Understanding the problem

b. Devising a plan (by solving a simpler problem)

c. Carrying out a plan

d. Looking back

Definition of ***Addition of Rational Numbers***: If $\frac{a}{b}$ and $\frac{c}{b}$ are rational numbers, then $\frac{a}{b}+\frac{c}{b}=\frac{a+c}{b}$.

Property: If $\frac{a}{b}$ and $\frac{c}{d}$ are rational numbers, then $\frac{a}{b}+\frac{c}{d}=\frac{ad+bc}{bd}$.

Proof:

Mixed numbers are made up of integers and a fractional part of an integer.

Example:

Since mixed numbers are rational, they can be written as a ratio of an integer and a nonzero integer. Writing a rational number as a ratio of an integer and a nonzero integer is considered in simplest form and is mostly preferred over writing it as a mixed number.

Write $7\frac{2}{5}$ as a ratio of an integer and a nonzero integer.

Write $\frac{29}{3}$ as a mixed number.

By the division algorithm, 29 =

Calculator: 29 $a\frac{b}{c}$ 3 enter will yield 9_2_| 3

Rational numbers have the following addition properties: closure, commutative, associative, additive identity, and additive inverse.

Additive Inverse Property of Rational Numbers: For any rational number $\frac{a}{b}$, there exists a unique rational number $-\frac{a}{b}$, the additive inverse of $\frac{a}{b}$, such that $\frac{a}{b}+\left(-\frac{a}{b}\right)=0=\left(-\frac{a}{b}\right)+\frac{a}{b}$.

Note $-\frac{a}{b}$ = $\frac{-a}{b}$.

Property Analogies

Integers	Rational Numbers
$-(-a) = a$	$-\left(-\frac{a}{b}\right) = \frac{a}{b}$
$-(a + b) = -a + -b$	$-\left(\frac{a}{b} + \frac{c}{d}\right) = \frac{-a}{b} + \frac{-c}{d}$

Addition Properties of Equality of Rational Numbers

If $\frac{a}{b} = \frac{c}{d}$, then $\frac{a}{b} + \frac{e}{f} = \frac{c}{d} + \frac{e}{f}$ where $\frac{a}{b}$, $\frac{c}{d}$, and $\frac{e}{f}$ are rational numbers.

Definition of Subtraction of Rational Numbers in Terms of Addition

If $\frac{a}{b}$ and $\frac{c}{d}$ are any rational numbers, then $\frac{a}{b} - \frac{c}{d}$ is the unique rational number $\frac{e}{f}$ such that $\frac{a}{b} = \frac{c}{d} + \frac{e}{f}$.

Theorem: If $\frac{a}{b}$ and $\frac{c}{d}$ are any rational numbers, then $\frac{a}{b} - \frac{c}{d} = \frac{a}{b} + \frac{-c}{d}$

Example:

Theorem: If $\frac{a}{b}$ and $\frac{c}{d}$ are any rational numbers, then $\frac{a}{b} - \frac{c}{d} = \frac{ad - bc}{bd}$.

Proof:

Estimate $\frac{55}{26} + \frac{15}{16}$.

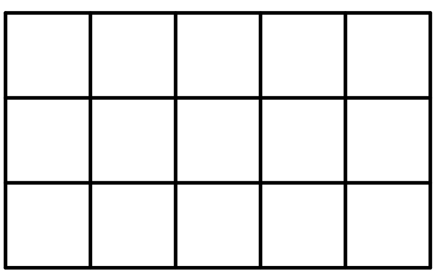
Text 5.3 Multiplication and Division of Rational Numbers

Multiplication – repeated addition

$2\left(\frac{3}{4}\right) =$

But $\frac{1}{3} \cdot \frac{2}{5}$ cannot be easily considered as repeated addition.

$\frac{1}{3} \cdot \frac{2}{5} =$



Definition of Multiplication of Rational Numbers

If $\frac{a}{b}$ and $\frac{c}{d}$ are any rational numbers, then $\frac{a}{b} \cdot \frac{c}{d} = \frac{ac}{bd}$.

Is $\frac{\frac{1}{3}}{\frac{5}{7}}$ a rational number?

Rational numbers have the following multiplication properties: closure, commutative, associative, multiplicative identity, and multiplicative inverse.

Multiplicative Identity of Rational Numbers:

The number 1 is the unique number such that for every rational number $\frac{a}{b}$,

$1 \cdot \frac{a}{b} = \frac{a}{b} = \frac{a}{b} \cdot 1$,

Multiplicative Inverse of Rational Numbers: For any nonzero rational number $\frac{a}{b}$, $\frac{b}{a}$ is the unique rational number such that $\frac{a}{b} \cdot \frac{b}{a} = 1 = \frac{b}{a} \cdot \frac{a}{b}$. The multiplicative inverse of $\frac{a}{b}$ is also called the **reciprocal** of $\frac{a}{b}$.

The multiplicative inverse of $\frac{-2}{5}$ is

The reciprocal of 8 is

The multiplicative inverse of $2\frac{3}{5}$ is

Distributive Property of Multiplication over Addition for Rational Numbers

If $\frac{a}{b}$, $\frac{c}{d}$, and $\frac{e}{f}$ are any rational numbers, then $\frac{a}{b}\left(\frac{c}{d} + \frac{e}{f}\right) = \frac{ac}{bd} + \frac{ae}{bf}$.

Example:

Multiplication Property of Equality for Rational Numbers

If $\frac{a}{b}$ and $\frac{c}{d}$ are any rational numbers such that $\frac{a}{b} = \frac{c}{d}$, and $\frac{e}{f}$ is any rational number, then $\frac{a}{b} \cdot \frac{e}{f} = \frac{c}{d} \cdot \frac{e}{f}$.

Example:

Multiplication Property of Zero for Rational Numbers

If $\frac{a}{b}$ is any rational number, then $\frac{a}{b} \cdot 0 = 0 = 0 \cdot \frac{a}{b}$.

Example:

Multiply $6\frac{2}{3} \cdot 5\frac{1}{4}$ by using

a. improper fractions

b. the distributive property of multiplication over addition of rational numbers

Definition of Division of Rational Numbers

If $\frac{a}{b}$ and $\frac{c}{d}$ are any rational numbers and $\frac{c}{d} \neq 0$, then $\frac{a}{b} \div \frac{c}{d} = \frac{e}{f}$ iff $\frac{e}{f}$ is the unique rational number such that $\frac{c}{d} \cdot \frac{e}{f} = \frac{a}{b}$.

Note: $\frac{c}{d} \neq 0$ implies $c \neq 0$.

Again, division is defined in terms of multiplication.

$10 \div 2$ means "how many times will 2 go into 10?"

$6 \div \frac{1}{3}$ means "how many times will $\frac{1}{3}$ go into 6?"

$\frac{2}{3} \div \frac{1}{6}$ means "how many times will $\frac{1}{6}$ go into $\frac{2}{3}$?"

Model $\frac{2}{5} \div \frac{3}{4}$.

Model $\frac{3}{4} \div \frac{2}{5}$.

Model $\frac{5}{6} \div \frac{2}{3}$.

Algorithm for Fraction Division

$\frac{a}{b} \div \frac{c}{d} = \frac{ad}{bc}$ where $\frac{c}{d} \neq 0$.

Proof:

Mental Math

Use mental math to find $\frac{2}{3} \cdot 12 \cdot \frac{5}{16}$.

Estimate

Estimate $35\frac{2}{3} \div \frac{7}{18}$.

Definition of a^n

$a^n = \underbrace{a \cdot a \cdot a \cdot \ldots \cdot a}_{n \text{ factors}}$ where a is any rational number and n is any natural number.

Laws of Exponents (such that you never have 0^0 which is an indeterminate (undefined))

- $a^0 = 1, \quad a \neq 0$
- $a^{-n} = \dfrac{1}{a^n}, \quad a \neq 0$
- $a^m a^n = a^{m+n}$
- $\dfrac{a^m}{a^n} = a^{m-n}, \quad a \neq 0$
- $\left(a^m\right)^n = a^{mn}$
- $(ab)^n = a^n b^n$
- $\left(\dfrac{a}{b}\right)^n = \dfrac{a^n}{b^n}, \quad b \neq 0$
- $\left(\dfrac{a}{b}\right)^{-n} = \dfrac{b^n}{a^n} = \left(\dfrac{b}{a}\right)^n, \quad a, b \neq 0$
- $\dfrac{a^{-n}}{b^{-m}} = \dfrac{b^m}{a^n}, \quad a, b \neq 0$

Text 5.4 Proportional Reasoning

Ratio $\frac{a}{b}$ or a:b where b $\neq$ 0.

There were 10 horses and 9 cows on the small ranch.

a. Express the number of horses to the number of cows as a ratio.

b. Express the number of cows to the number of horses as a ratio.

c. Express the number of horses to the total number of large animals (horses and cows) on the ranch as a ratio.

Two ratios are **proportional** iff the fractions representing the ratios are equal.

Solve $\frac{51}{x} = \frac{3}{5}$.

Cross Multiplication Property
If a, b, c, and d are real numbers such that b $\neq$ 0 and d $\neq$ 0, then
$\frac{a}{b} = \frac{c}{d}$ iff ad = bc.

Example:

Property
If $\frac{a}{b}$ and $\frac{c}{d}$ are rational numbers such that a $\neq$ 0 and c $\neq$ 0, then
$\frac{a}{b} = \frac{c}{d}$ iff $\frac{b}{a} = \frac{d}{c}$.

Example:

Property

If $\frac{a}{b}$ and $\frac{c}{d}$ are rational numbers such that a $\neq$ 0, b $\neq$ 0 and c $\neq$ 0, then $\frac{a}{b}=\frac{c}{d}$ iff $\frac{a}{c}=\frac{b}{d}$.

Example:

Scale Drawings – The **scale** is the ratio of the size of the drawing to the actual size of the object.

Example:

In a photograph of an adult and young child, the child's height is 2.4 cm and the adult's height is 5.6 cm. If the adult is actually 182 cm tall, how tall is the child?

Given $\frac{a}{b}=\frac{c}{d}$, prove $\frac{a}{a+b}=\frac{c}{c+d}$.