

## Math 366 Chapter 10 Constructions, Congruence, and Similarity

### 10-1 Congruence Through Constructions

Similar,  $\sim$ , objects have the same shape but not necessarily the same size.

Congruent,  $\cong$ , objects have the same shape and size.

Are similar objects always congruent?

Are congruent objects always similar?

Note the difference between  $\angle A \cong \angle B$  and  $m\angle A = m\angle B$ .

Rep-tile – figure that is used to construct a larger similar figure

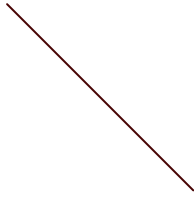
Circle – the set of all points in a plane equidistant (radius) from a given point (center). The radius is used to describe both the distance  $CA$  as well as the line segment  $\overline{CA}$ .

Congruent segments –  $\overline{AB} \cong \overline{CD}$  iff  $AB = CD$

Congruent angles –  $\angle ABC \cong \angle DEF$  iff  $m\angle ABC = m\angle DEF$

Constructions – use only a straight edge and compass (and writing instrument)

Construct a circle given its radius and center.



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Any circle is similar to any circle. A circle is congruent to another circle iff

Arc of a circle – any part of a circle that can be drawn without lifting a pencil (the entire circle could also be considered an arc)

Center of an arc – center of the circle containing the arc

Semi-circle – arc that is exactly half of a circle

Congruent triangles –  $\triangle ABC \cong \triangle DEF$  iff  $\angle A \cong \angle D$ ,  $\angle B \cong \angle E$ ,  $\angle C \cong \angle F$ ,  $\overline{AB} \cong \overline{DE}$ ,  $\overline{BC} \cong \overline{EF}$ , and  $\overline{AC} \cong \overline{DF}$ .

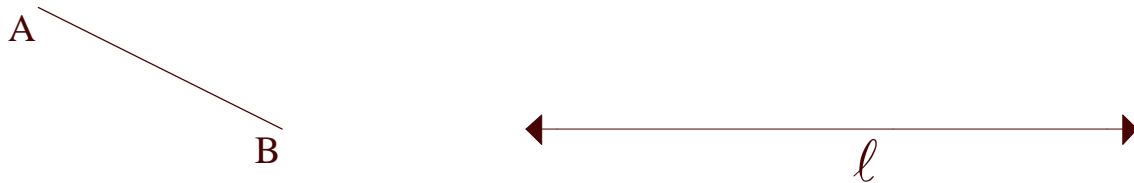
The tick marks indicate congruencies.

If  $\triangle PQR \cong \triangle STU$ , then  $\overline{PR} \cong$  .

CPCTC – corresponding parts of congruent triangles are congruent.

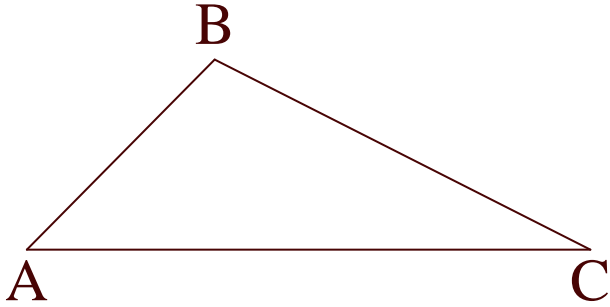
SSS Property – If three sides of one triangle are congruent, respectively, to three sides of a second triangle, then the triangles are congruent.

*Construction* – Given a segment  $\overline{AB}$ , construct a congruent segment on a line  $\ell$ .



1. Fix compass so ends are at A and B.
2. Place a point of compass on  $\ell$  and call this point P. Mark an arc with the pencil end through  $\ell$ , labeling the intersection point Q.
3.  $\overline{AB} \cong \overline{PQ}$  since they have the same length.

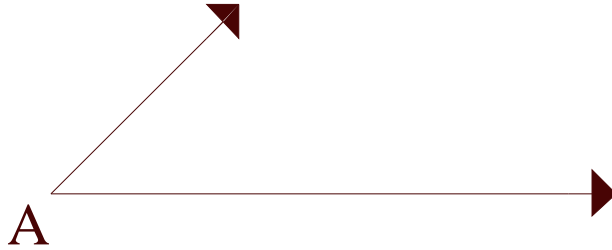
*Construction* – Given  $\triangle ABC$ , construct  $\triangle A'B'C'$  congruent to it.



1. Construct a segment  $\overline{A'C'}$  from  $\overline{AC}$ .
2. Construct a circle with center at A' and radius AB, and a circle with center at C' with radius BC.
3. Choose a point of intersection of the two circles and label it B'. Draw segments  $\overline{A'B'}$  and  $\overline{B'C'}$ .
4. By construction and SSS,  $\triangle ABC \cong \triangle A'B'C'$ .

Triangle Inequality Property – The sum of the measure of any two sides of a triangle must be greater than the measure of the third side.

*Construction – Copy an angle A.*

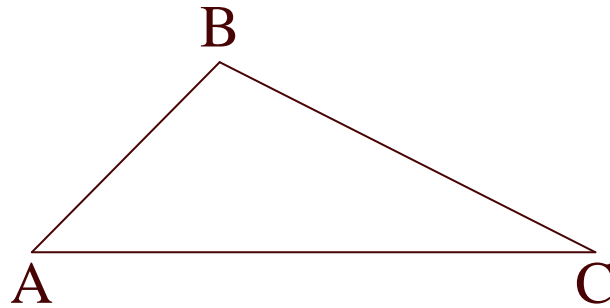


1. Make an arc  $\widehat{BC}$  to form isosceles triangle  $ABC$ . With points at  $A'$ , make an arc with the same radius. Label intersection of arc and  $\overline{A'X}$ , point  $C'$ .
2. So that  $\overline{BC} \cong \overline{B'C'}$  and with pointer at  $C$ , mark an arc that intersects the first arc at  $B$ . Keep the compass fixed and make an arc  $B'C'$  such that the arcs intersect; label the intersection point  $B'$ .
3. Draw  $\overline{A'B'}$ .
4.  $\angle A \cong \angle A'$  since  $\triangle ABC \cong \triangle A'B'C'$  by SSS.

How many different triangles can be constructed from two given segments?

SAS Property – If two sides and the included angle (angle between the two sides) of one triangle are congruent to those of another triangle, the two triangles are congruent.

*Construction* – Construct a triangle congruent to a given triangle using SAS.



1. Copy  $\overline{AC}$  to get  $\overline{A'C'}$
2. Copy  $\angle A$  at  $A'$  on  $\overline{A'C'}$
3. Copy  $\overline{AB}$  to get  $\overline{A'B'}$
4. Make  $\overline{B'C'}$
5. The triangles are congruent by SAS.

Perpendicular bisector – a line that is perpendicular,  $\perp$ , to a segment and bisects it

*Construction* – Construct a perpendicular bisector of a segment.

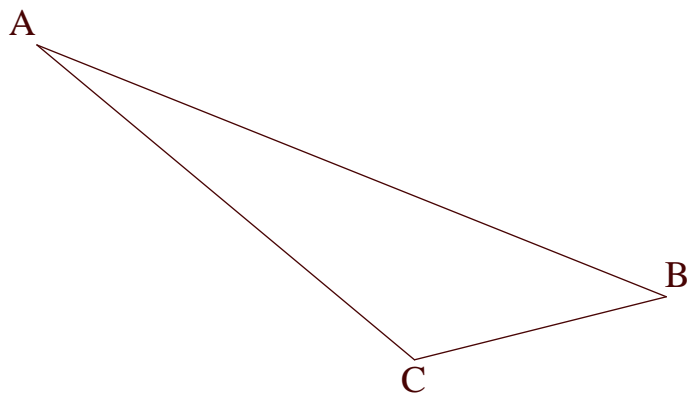


Put compass point on A and pencil point at any point past where you would guess the midpoint is. Fix the compass and draw circle (or arc between A and B). Keeping the compass fixed, put compass on point B and repeat. Draw the segment between the two circles' intersection points, labeled C and D.

Justification:  $\overline{CA} \cong \overline{CB}$ .  $\triangle CAD \cong \triangle CBD$  by SSS. Then  $\angle \alpha \cong \angle \beta$  by CPCTC.  $\overline{CM} \cong \overline{CM}$ . Now  $\triangle ACM \cong \triangle BCM$  by SAS. Then by CPCTC,  $\angle 1 \cong \angle 2$  and  $\overline{MA} \cong \overline{MB}$ . So  $\overline{CD}$  bisects  $\overline{AB}$ . Since  $\angle 1 \cong \angle 2$  and the angles form a straight angle,  $\angle 1$  and  $\angle 2$  are right angles. Therefore  $\overline{CD}$  is on the perpendicular bisector of  $\overline{AB}$ .



*Construction* – Construct a circle that circumscribes a triangle (every vertex of the triangle is on the circle)



1. Construct the perpendicular bisector of  $\overline{AB}$  and  $\overline{CA}$ .
2. The intersection D of the perpendicular bisectors is the center of the circle with radius  $\overline{DA}$ .

Circumcenter – center of circle circumscribed about the triangle

SSA is not a triangle congruence.

Problem Solving p. 572 – 573: Draw an arc on a sheet of paper and do paper folding.

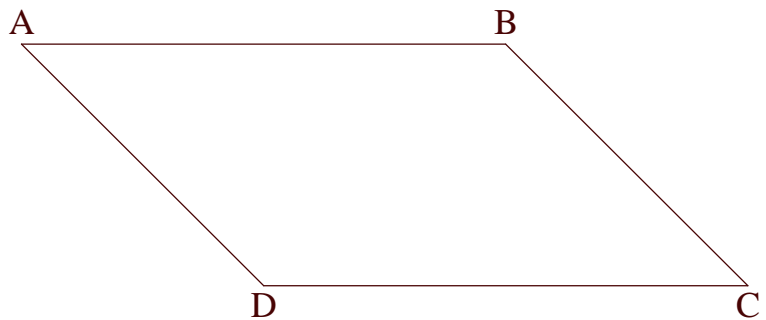
### 10-2 Other Congruence Properties

ASA Property – If two angles and the included side of one triangle are congruent to two angles and the included side of another triangle, respectively, then the triangles are congruent.

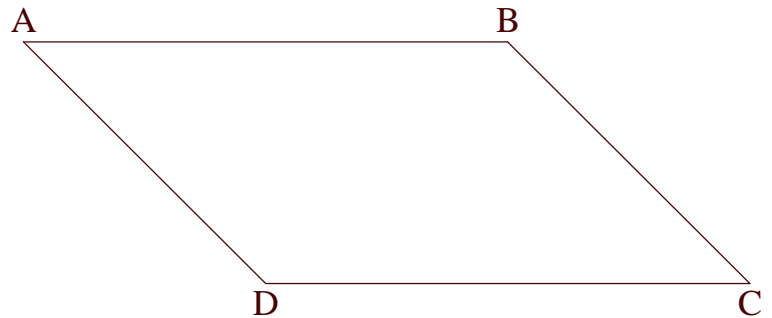
AAS Property – If two angles and a corresponding side of one triangle are congruent to two angles and a corresponding side of another triangle, respectively, then the two triangles are congruent.

\*\*\*\*\* Deeper Study of Quadrilaterals p. 579 – 580: Know these definitions and properties of quadrilaterals.

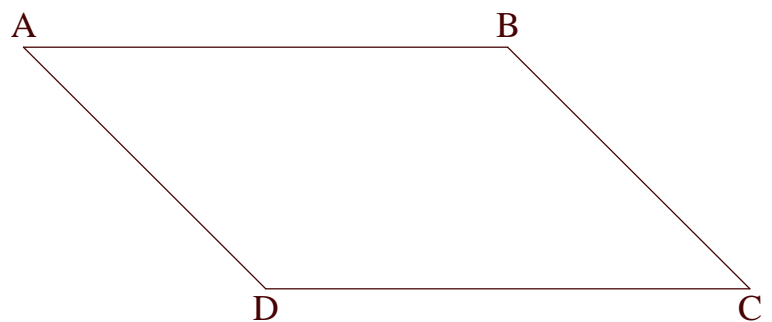
Prove the opposite sides of a parallelogram are congruent.



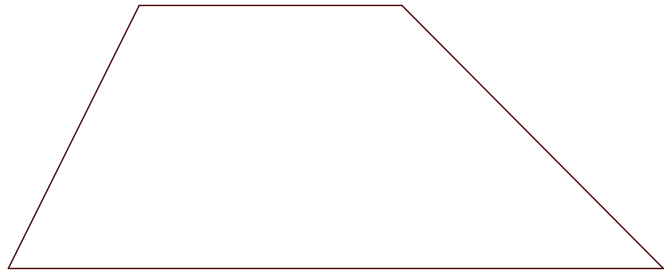
Prove the opposite angles of a parallelogram are congruent.



Prove the diagonals of a parallelogram bisect each other.



In a trapezoid, prove the consecutive angles between parallel sides are supplementary.



### 10-3 Other Constructions

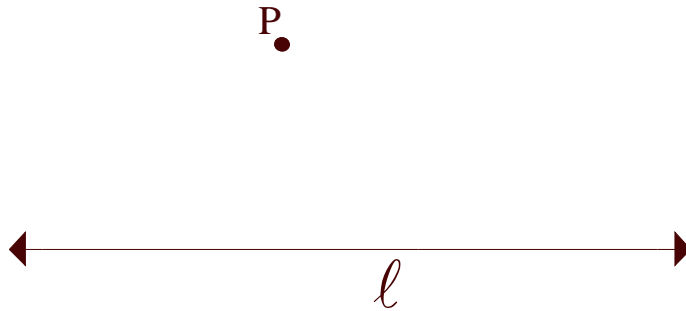
Rhombus – (definition) a parallelogram with all sides congruent

Rhombus properties

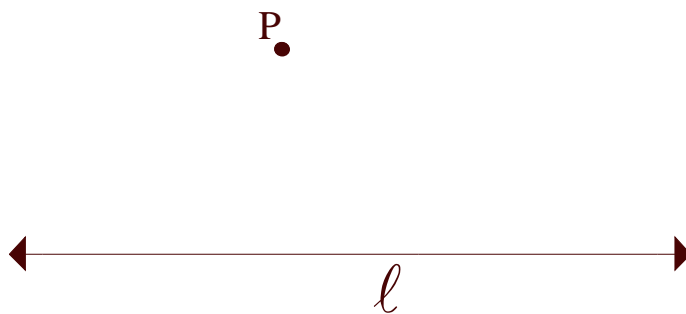
- a. A quadrilateral in which all the sides are congruent is a rhombus.
- b. Each diagonal of a rhombus bisects the opposite angles.
- c. The diagonals of a rhombus are perpendicular.
- d. The diagonals of a rhombus bisect each other.

*Construction* – Given a line  $\ell$  and a point P not on  $\ell$ , construct a line through P parallel to  $\ell$ .

- A. Rhombus method (construct rhombus APCB)

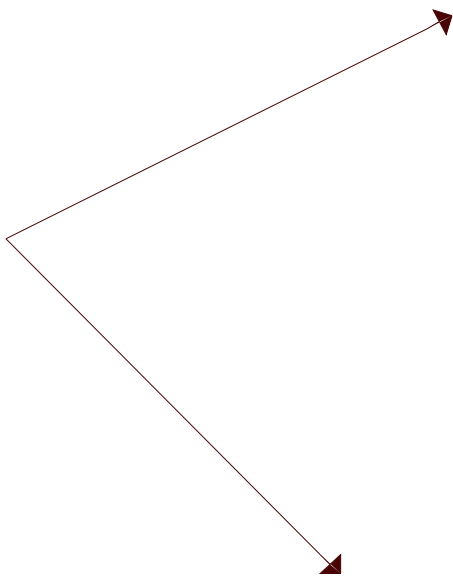


- B. Corresponding angle method (construct line parallel to  $\ell$ )

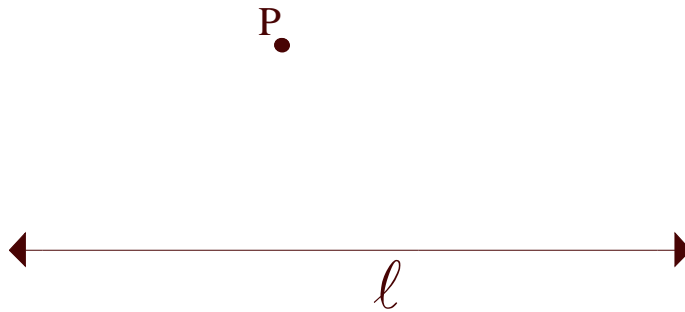


Angle bisector – a ray that separates an angle into two congruent angles

*Construction* – Bisect an angle (construct rhombus since its diagonals bisect the opposite angles)



*Construction* – Given line  $\ell$  and a point P not on  $\ell$ , construct a line through P that is perpendicular to  $\ell$  (construct a rhombus whose diagonals are perpendicular bisectors); note this method can be used to construct an altitude of a triangle)



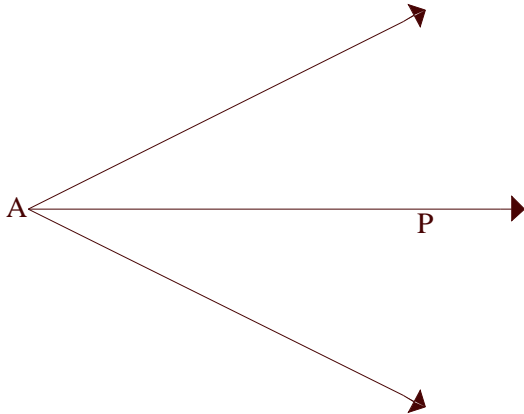
*Construction* – Given line  $\ell$  and point P on  $\ell$ , construct a perpendicular line to  $\ell$  through P (construct a rhombus whose diagonals are perpendicular bisectors)



Theorem 10-3:

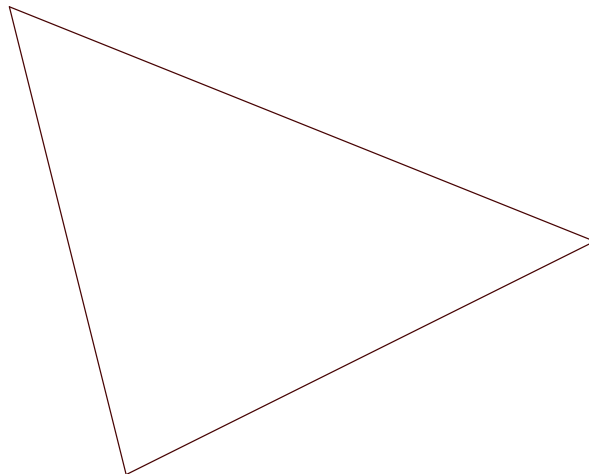
- a. Any point P on an angle bisector is equidistant from the sides of the angle.
- b. Any point that is equidistant from the sides of an angle is on the angle bisector of the angle

Proof of part a



How do you inscribe a circle in a triangle?

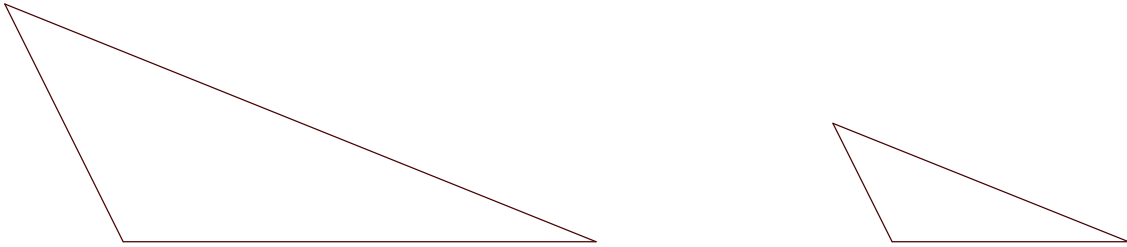
Bisect two or three angles of the triangle to find the center P of the circle, which is the intersection point P of the angle bisectors. For the radius construct a perpendicular from P to a side and use the length of that segment as the radius.



### 10-4 Similar Triangles and Similar Figures

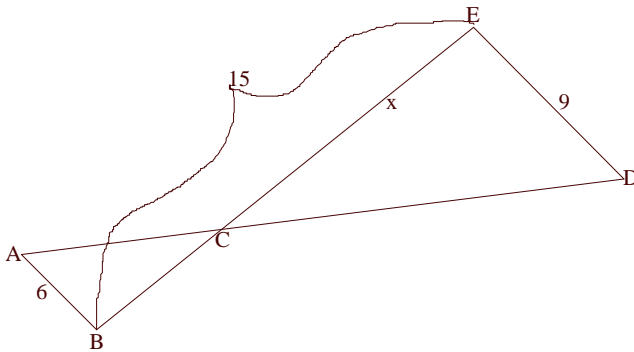
Triangle similarity -  $\triangle ABC \sim \triangle DEF$  iff

- $\angle A \cong \angle D$
- $\angle B \cong \angle E$
- $\angle C \cong \angle F$
- $\frac{AB}{DE} = \frac{AC}{DF} = \frac{BC}{EF}$  is the scale factor for the lengths of the sides (all sides are shrunk/enlarged by this scale factor)



Are all equilateral triangles similar?

AA~ Property – If two angles of one triangle are congruent, respectively, to two angles of a second triangle, then the triangles are similar.

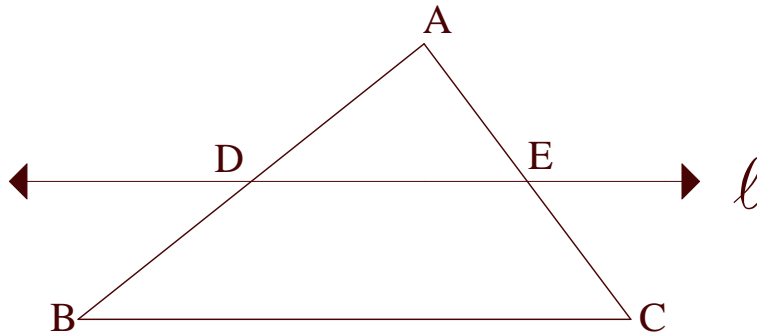


If angle A is congruent to angle D, then  $\triangle ABC \sim \triangle$  by

Solve for  $x$ .

*Parallel Line Problems*

Theorem 10-4: If a line parallel to one side of a triangle intersects the other sides, then it divides those sides into proportional segments.

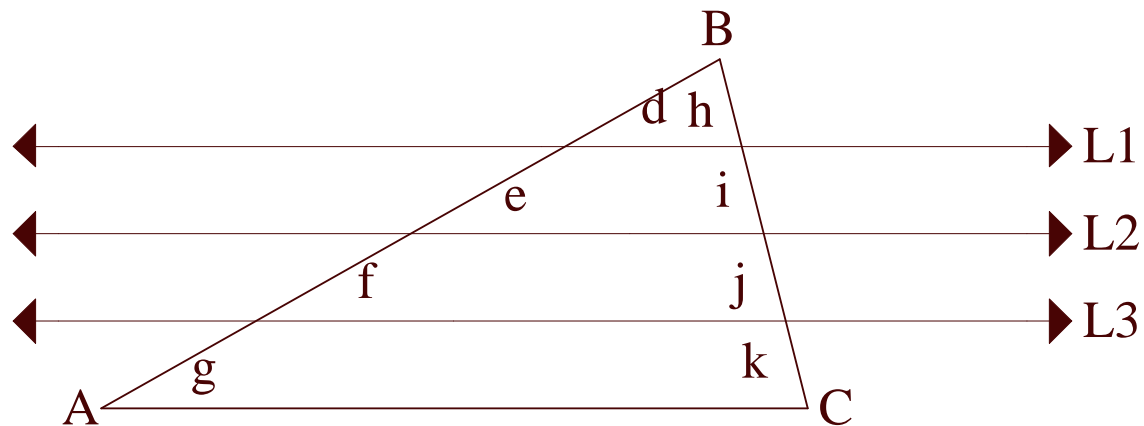


If line  $l \parallel \overline{BC}$ , then  $\frac{AD}{DB} = \frac{AE}{EC}$  and  $\frac{AD}{AB} = \frac{AE}{AC}$ . Note that  $\triangle ADE \sim \triangle ABC$  by AA~

Theorem 10-5 (converse of Theorem 10-4): If a line divides two sides of a triangle into proportional segments, then the line is parallel to the third side.

If  $\frac{AD}{DB} = \frac{AE}{EC}$  or  $\frac{AD}{AB} = \frac{AE}{AC}$ , then  $\overline{DE} \parallel \overline{BC}$

Theorem 10-6: If parallel lines cut off congruent segments on one transversal then they cut off congruent segments on any transversals.



If  $L1 \parallel L2 \parallel L3 \parallel \overline{AC}$  and if  $d = e = f = g$ , then  $h = i = j = k$ .

A segment can be separated into congruent parts by construction. In particular a line segment can be trisected by construction.

Important note: There is no construction to trisect an angle.

*Construction* – Separate a segment into congruent parts (for explanation, trisection will be done).

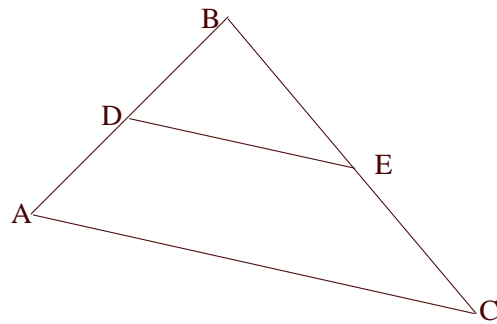


1. Draw a ray  $\overrightarrow{AC}$  such that A, B, and C are noncollinear.
2. Mark off the given number of congruent segments (any size) on  $\overrightarrow{AC}$ . Label these points  $A_1$ ,  $A_2$ , and  $A_3$ , since we are doing a trisection.
3. Draw a line segment from  $A_3$  to B.
4. Construct parallel lines to  $\overline{BA_3}$  through  $A_1$  and through  $A_2$ .
5. Thus  $\overline{AB_1} \cong \overline{B_1B_2} \cong \overline{B_2B}$ .

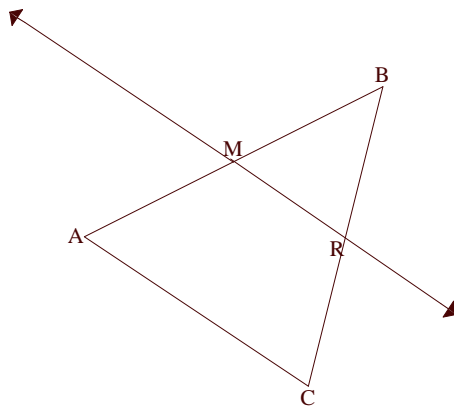
Can use this same method to bisect, trisect, 4-sect, n-sect a segment.

Midsegment – a segment that connects midpoints of two adjacent sides of a triangle or quadrilateral

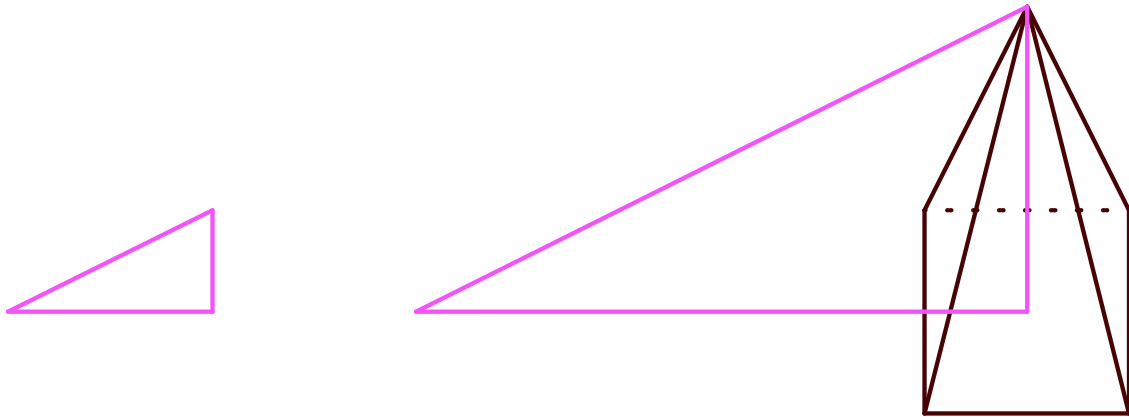
Theorem 10-7 Midsegment Theorem: The midsegment of a triangle is parallel to the third side and is half as long.



Theorem 10-8: If a line bisects one side of a triangle and is parallel to a second side, then it bisects the third side (thus, it is the midsegment).



Compute height of a pyramid.

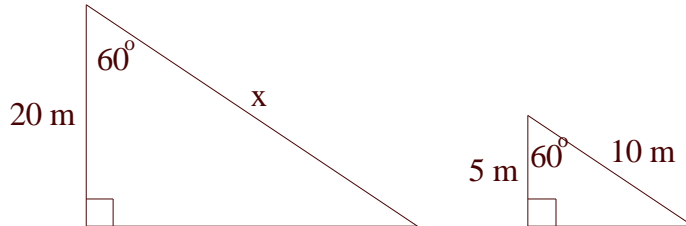


### 10-5 Trigonometry Ratios via Similarity

Trigonometry – triangle measure

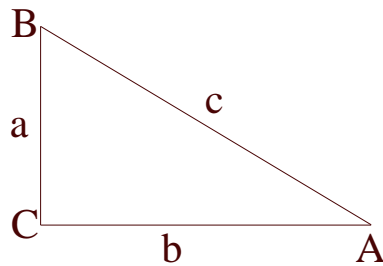
The trigonometry functions are based upon the properties of similar triangles.

Solve for x.



Note: In a  $30^\circ - 60^\circ - 90^\circ$  triangle, the length of the side opposite the  $30^\circ$  angle is half the length of the hypotenuse.

Trigonometric functions are defined as ratios of lengths of the sides of a right triangle.



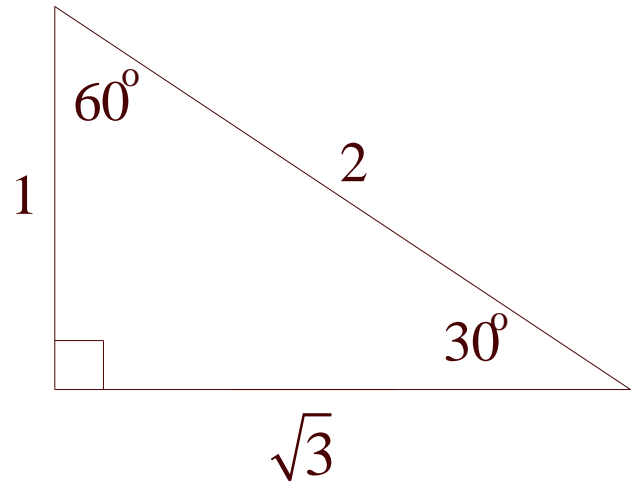
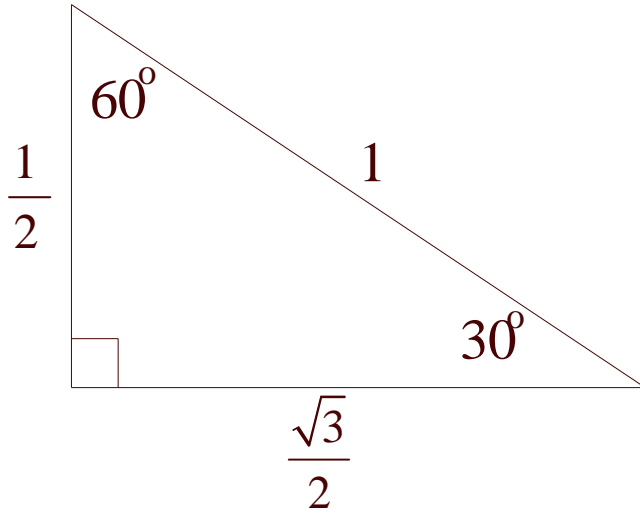
$$\sin m\angle A = \sin \angle A = \frac{\text{length of opposite side}}{\text{length of hypotenuse}} = \frac{opp}{hyp} = \frac{a}{c}$$

$$\cos m\angle A = \cos \angle A = \frac{\text{length of adjacent side}}{\text{length of hypotenuse}} = \frac{adj}{hyp} = \frac{b}{c}$$

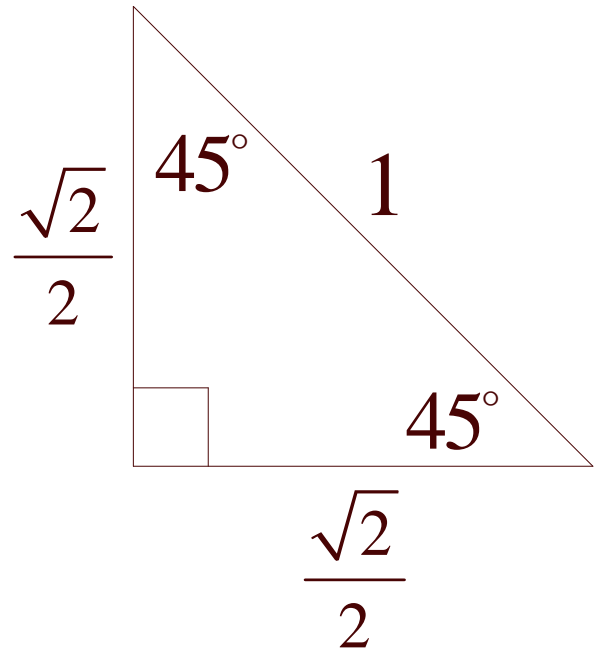
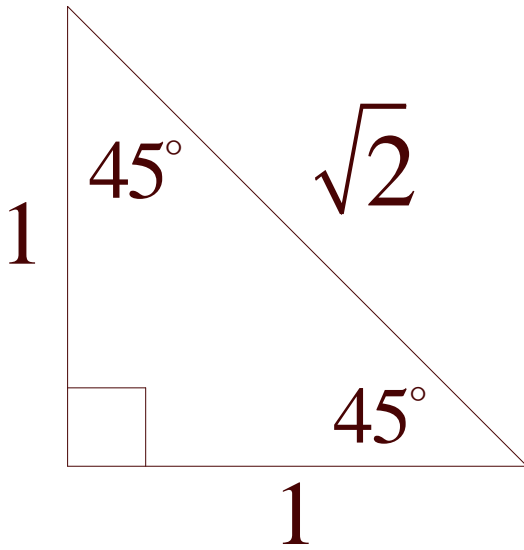
$$\tan m\angle A = \tan \angle A = \frac{\text{length of opposite side}}{\text{length of adjacent}} = \frac{opp}{adj} = \frac{a}{b}$$

Recall the Pythagorean Theorem: In a right triangle,  $a^2 + b^2 = c^2$ .

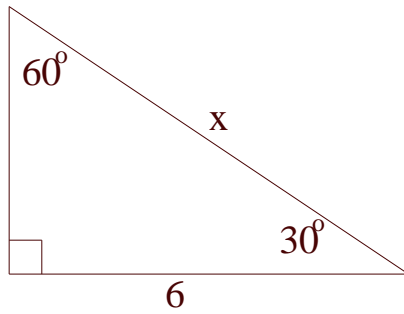
30° – 60° – 90° Triangle (use either one of these)



45° – 45° – 90° Triangle (use either one of these)



Solve for  $x$ .



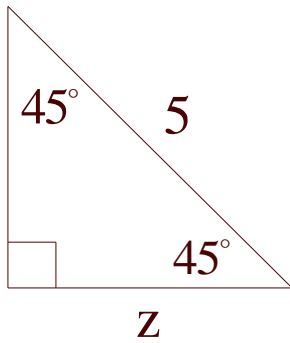
1. using ratios between the first  $30^\circ - 60^\circ - 90^\circ$  Triangle and the one above

2. using ratios between the second  $30^\circ - 60^\circ - 90^\circ$  Triangle and the one above

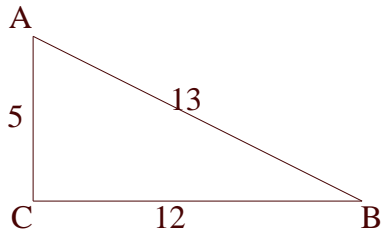
3. using the appropriate trigonometric function

4. using a calculator

Solve for  $z$ .



1. using ratios between the first  $45^\circ - 45^\circ - 90^\circ$  Triangle and the one above
2. using ratios between the second  $45^\circ - 45^\circ - 90^\circ$  Triangle and the one above
3. using the appropriate trigonometric function
4. using a calculator



1. Is triangle ABC a right triangle?

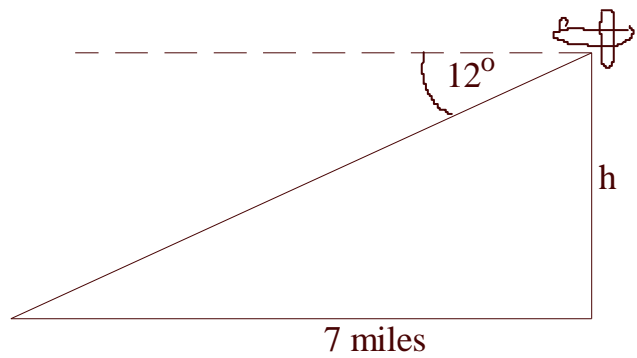
2. Find  $\angle B$ .

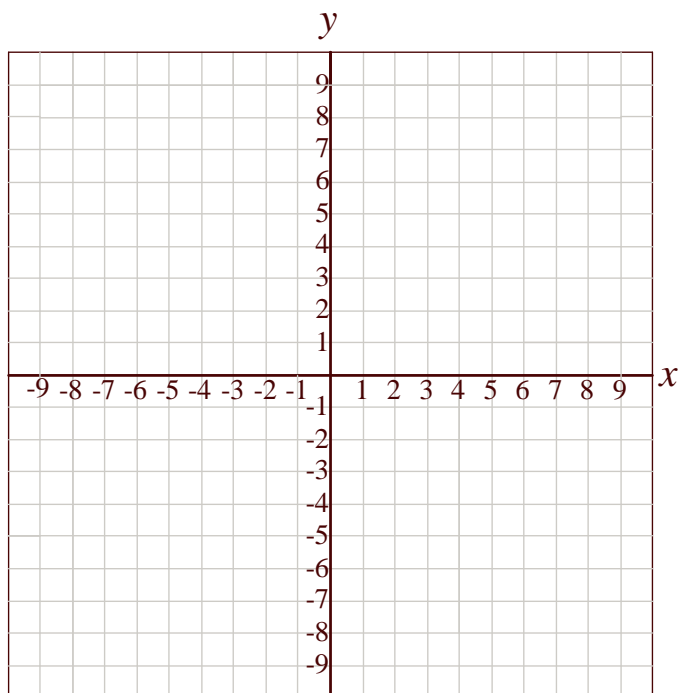
a.

b.

c.

Find the height of the plane.

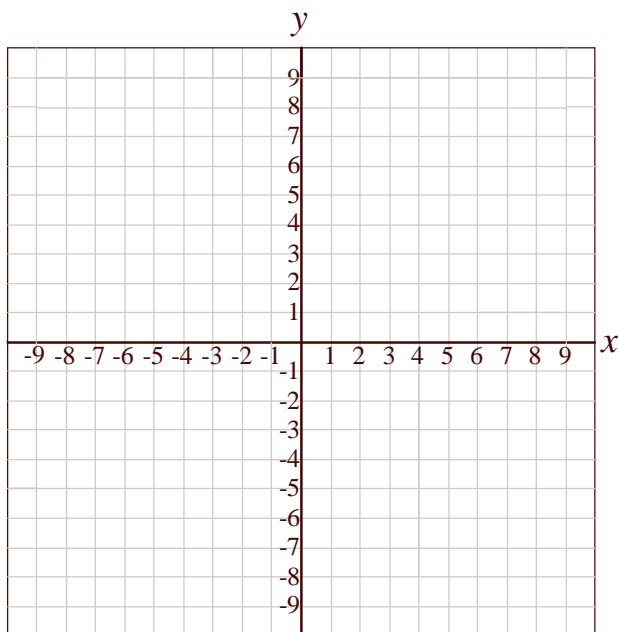


**10-6 Lines in a Cartesian Coordinate System**

Horizontal Lines:  $y = c$  where  $c$  is any real number

Vertical Lines:  $x = d$  where  $d$  is any real number

Examples: Graph  $x = 2$ ,  $y = -4$ ,  $x > 0$ , and  $y \leq -5$ .



Slope of non-vertical line through  $(x_1, y_1)$  and  $(x_2, y_2)$  is  $m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{\text{rise}}{\text{run}}$ .

Any two parallel lines have the same slope, or are vertical lines with undefined slope.

Find the slope of the line through  $(2, -3)$  and  $(-4, 5)$ .

Equation of a (non-vertical) line

Find the equation of the line through  $(8, -5)$  and  $(2, 7)$ .

Rearrange  $y - y_1 = m(x - x_1)$

Slope Intercept Form of a Line is  $y = mx + b$ .

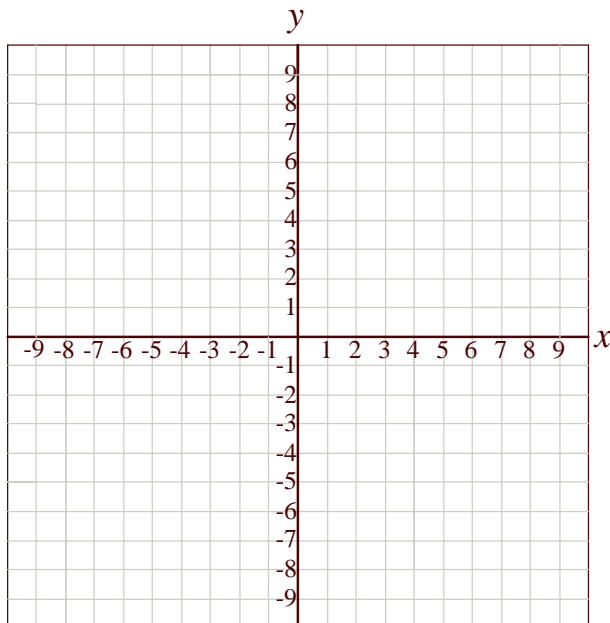
Every line (in the plane) has an equation of the form either  $y = mx + b$  or  $x = a$ , where  $m$  is the slope,  $b$  is the y-intercept, and  $a$  is any real number.

How do you find the x-intercept?

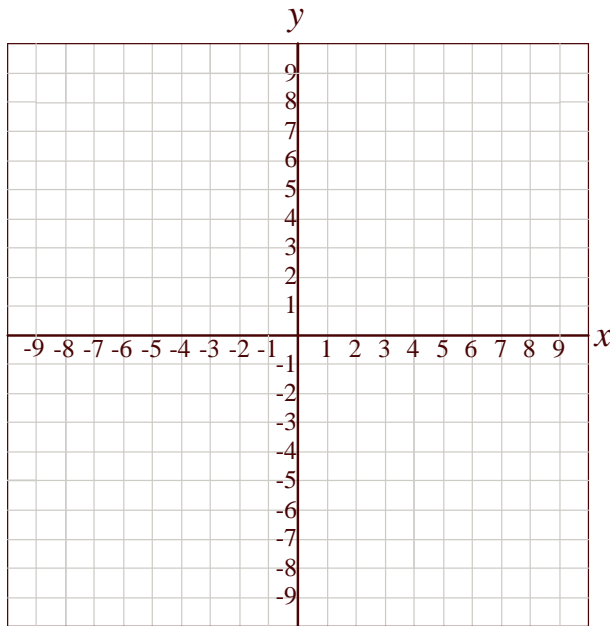
Find the line with slope of  $-3$  and y-intercept of  $1$ .

What is the x-intercept?

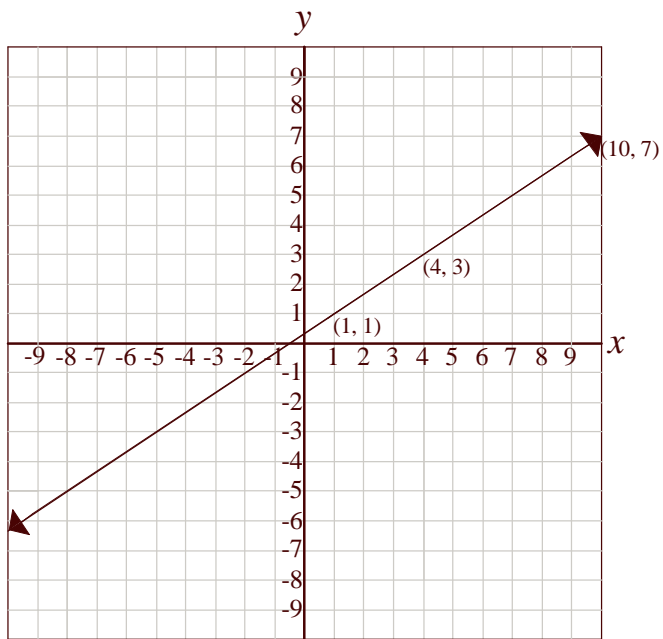
Graph  $y = 2x + 3$  using the t-chart (find at least 3 points).



Graph  $y - 4 = -1(x + 3)$  using its slope and y-intercept.



Use similar triangles to determine the slope of the line.



*Systems of Equations*

Intersections of Lines – solve systems of linear equations

Given two lines, what are the possibilities concerning their intersections?

Find the intersection point of the lines  $2x + 5y = 23$  and  $-3x + 2y = 13$ .

1. substitution method – solve one equation for one variable, plug it into the other equation, and solve for the remaining variable

2. elimination method – (simultaneous) add a non-zero multiple of one equation to the other to eliminate one variable

Solve the system of equations.

a.  $x + y = 3$   
 $x - y = 1$

b.  $2x - y = 8$   
 $-2x + y = -8$

c.  $x - y = 3$   
 $-x + y = 2$