Math 366 Chapter 12 Motion Geometry and Tessellations

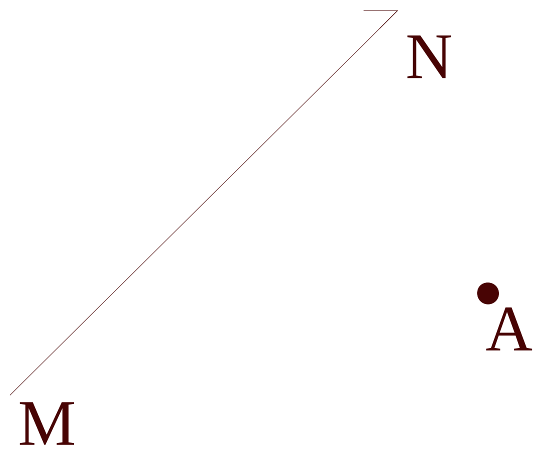
For a blank 'geoboard,' a coordinate plane and a grid, see the last three pages of these notes.

12-1 Translations and Rotations

Translation – is a motion of a plane that moves every point of the plane in a specified distance in a specified direction along a straight line (which can be shown by a slide-arrow or vector). Translations yield congruent figures and slopes of segments are maintained.

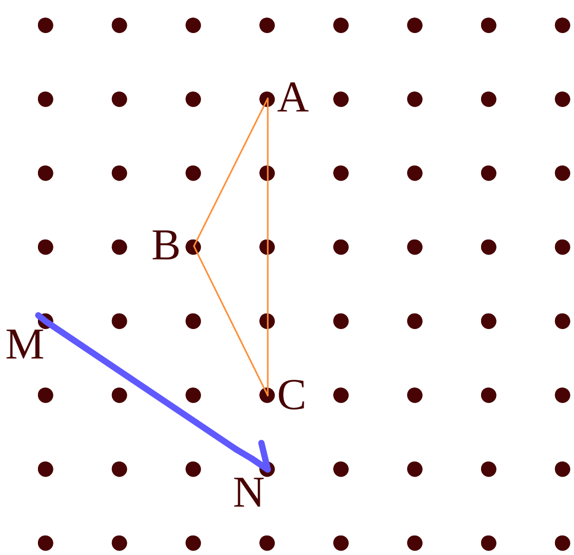
Isometry or rigid motion– is any motion that preserves length or distance; an example is a translation

Construction – Construct the image of a point. Given point A and vector $\overrightarrow{MN}$, construct the image A' of A.

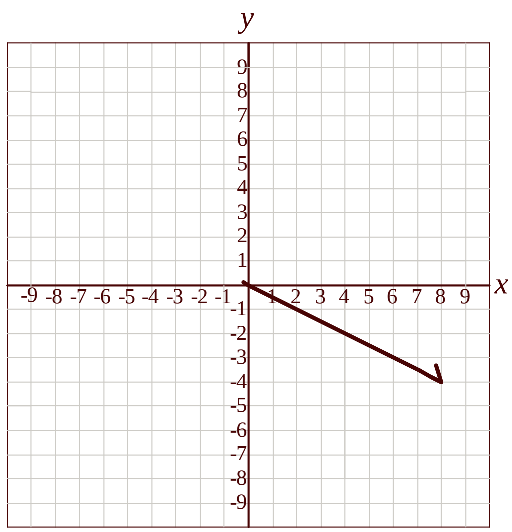


Create ray $\overrightarrow{MA}$ and then copy $\angle NMA$ such that the vertex is at A where $\overline{MN} \parallel \overline{AP}$; and copy angle $\angle NMA$ such that the vertex is at N where $\overline{MA} \parallel \overline{NA'}$, $\overline{MN} \parallel \overline{AA'}$, and $\overline{MN} \cong \overline{AA'}$. A' is the image.

You can point-wise find an image of a set of points.



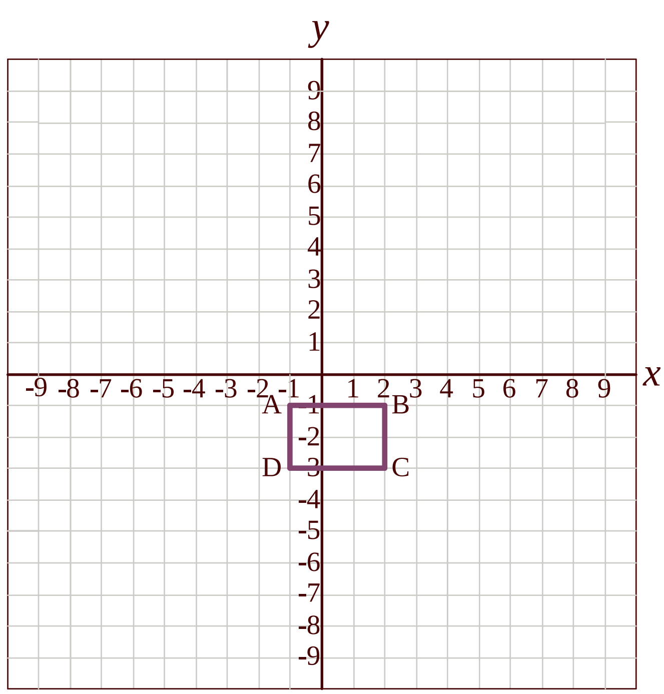
Formulas can be used when the translation is done in the rectangular coordinate system.



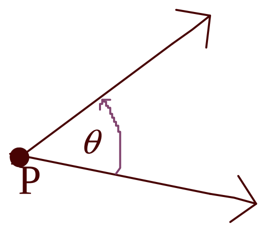
The notation for the above translation is

A translation in a coordinate system is a function from the plane to the plane such that to every point (x, y) corresponds the point (x + *a*, y + *b*) where *a* and *b* are real numbers.

Given rectangle ABCD, find it image after the translation (x, y) → (x – 1, y + 3).

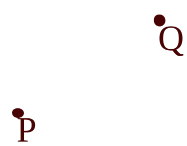


Rotation – is a transformation of the plane determined by rotating the plane about a fixed point, the center, by a certain amount in a certain direction. Usually a positive measure is a counterclockwise turn and a negative measure is a clockwise turn. (Sometimes tracing paper and a protractor can be used to assist in rotations.)



Rotate the below object 90 degrees about P.

Construction – Construct the image of Q under a 45-degree rotation about P.



First construct a right angle and then bisect it to get a 45-degree angle. Then copy this angle such that $\overline{PQ} \cong \overline{PQ'}$ and $m\angle Q'PQ = 45^o$.

Rotation preserves congruence and 'orderings' of labeled vertices, but not slopes. A rotation of 360 degrees about a point is an <u>identity rotation</u>. A rotation of 180 degrees about a point is a <u>half-turn</u>.

Theorem – If $y = m_1x + b_1$ and $y = m_2x + b_2$ are two distinct lines, then

a. $m_1 = m_2$ iff the lines are parallel
b. $m_1m_2 = -1$ iff the lines are perpendicular.

Note this theorem excludes vertical lines. Two distinct vertical lines are parallel. A vertical line (x = a) is perpendicular to a horizontal line (m = 0, that is, y = b).

Proof of part b:

i. Given non-vertical lines l_1 and l_2 are perpendicular; prove $m_1m_2 = -1$.

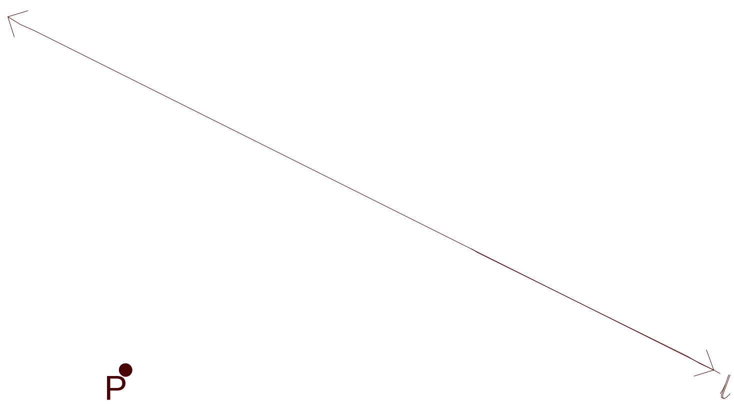
ii. Given $m_1m_2 = -1$, show l_1 and l_2 are perpendicular.

$\therefore$ If $y = m_1x + b_1$ and $y = m_2x + b_2$ are two distinct lines, then $m_1m_2 = -1$ iff the lines are perpendicular.

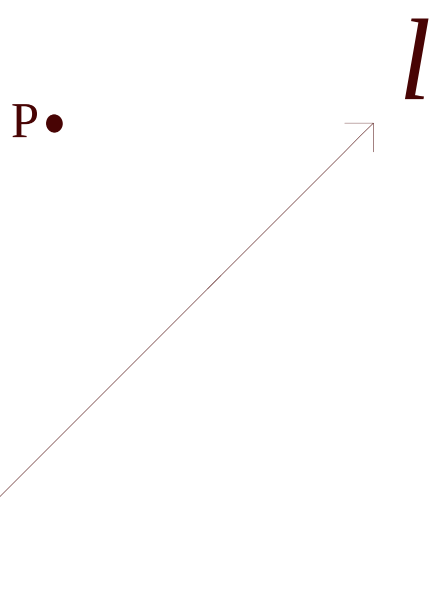
Are lines y = 2x +5 and $y = \frac{-1}{2}x + 7$ perpendicular?

12-2 Reflections and Glide Reflections

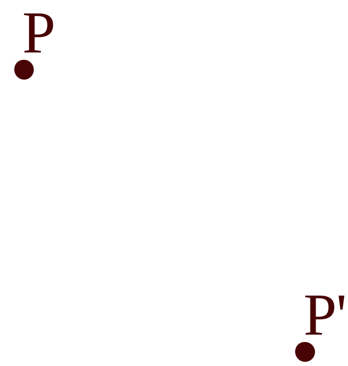
A <u>reflection</u> in a line l is a transformation of a plane that pairs each point P of the plane with a point P' in such a way that l is the perpendicular bisector of $\overline{PP'}$ as long as P is not on l. If P is on l then P = P'. A reflection is an isometry of the plane about a line l, which flips the plane using l as a hinge axis.



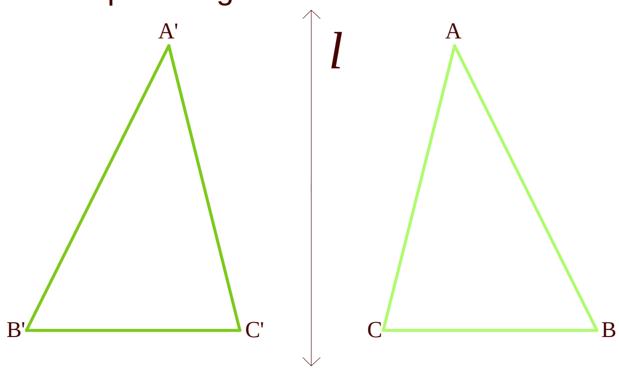
Example: Construct the image of a point P under a reflection about l. (Use rhombus technique and perpendicular bisector.)



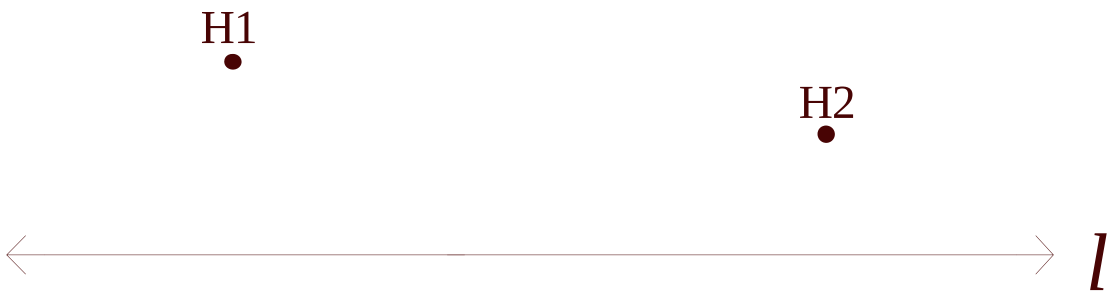
Find the line of reflection. (Construct the perpendicular bisector of $\overline{PP'}$.)



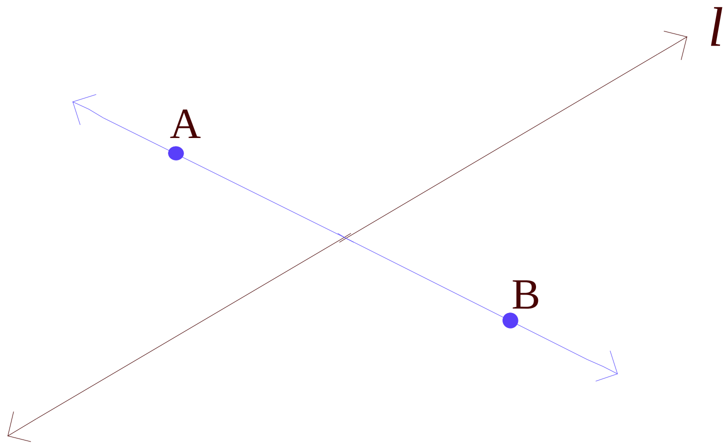
The orientation or ordering changes under a reflection, but images are congruent to their pre-images.



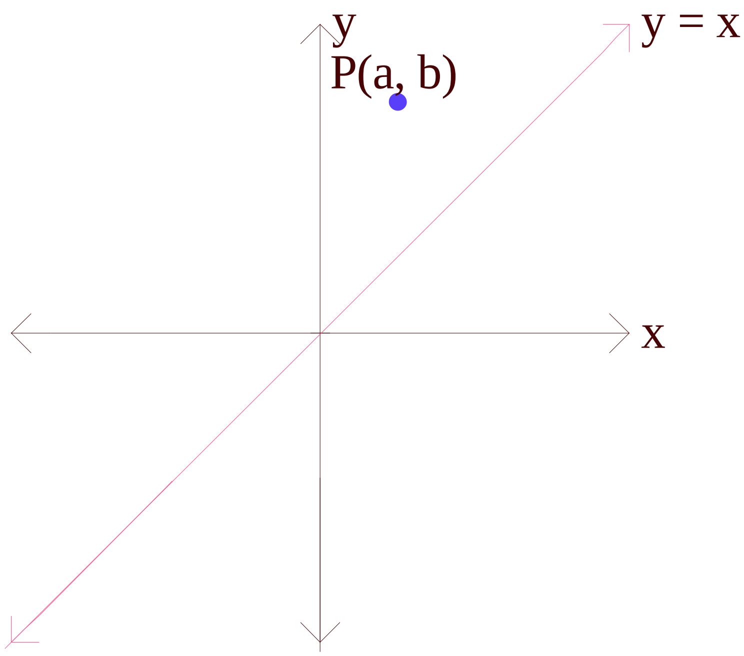
Problem: Place a telephone pole on property line l so that the total distance to the two houses H1 and H2 is smallest.



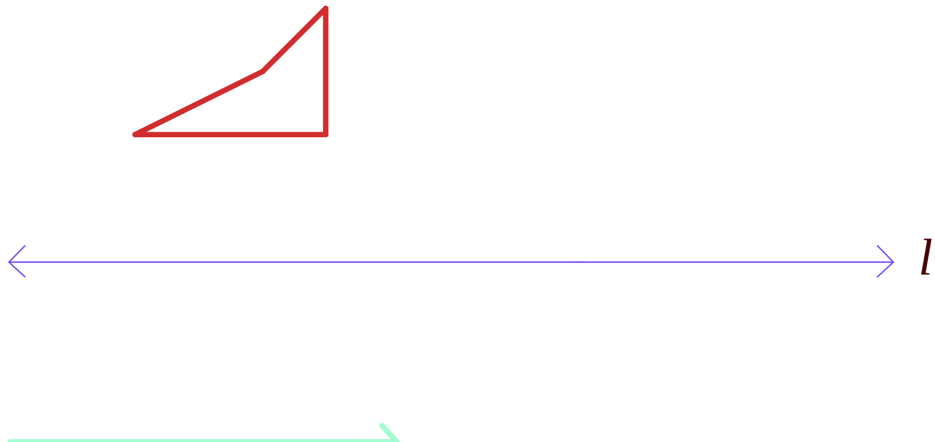
Find the image of the line reflected about line *l*.



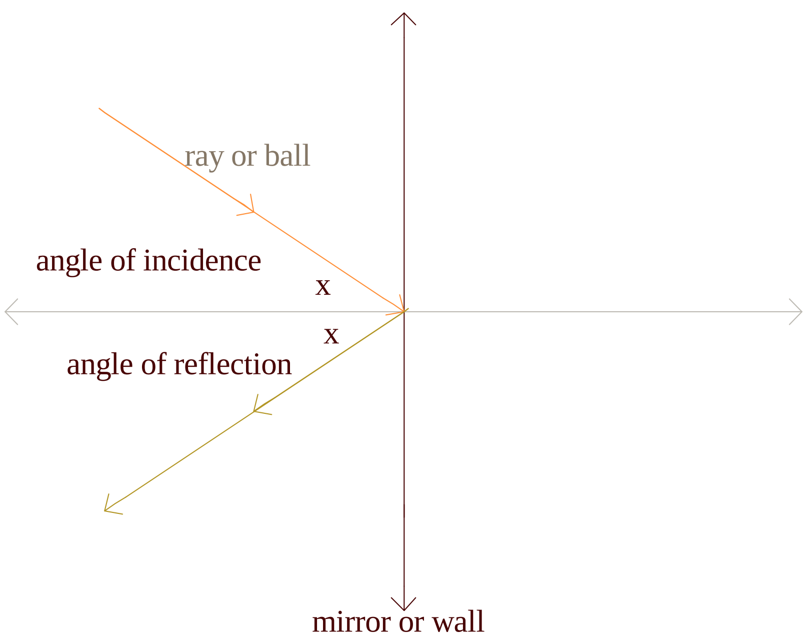
Coordinate Systems – Reflect a point P(a, b) about the line y = x.



<u>Glide Reflection</u> – a transformation consisting of a translation followed by a reflection in a line parallel to the slide arrow. If no reflecting line is given, then it is the line containing the slide arrow.



In a glide reflection, images are congruent to pre-images, and ordering or orientation is reversed.



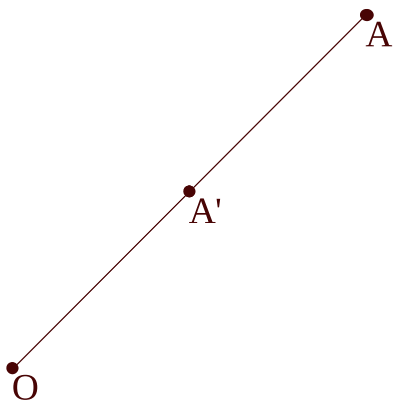
The angle of incidence is congruent to the angle of reflection.

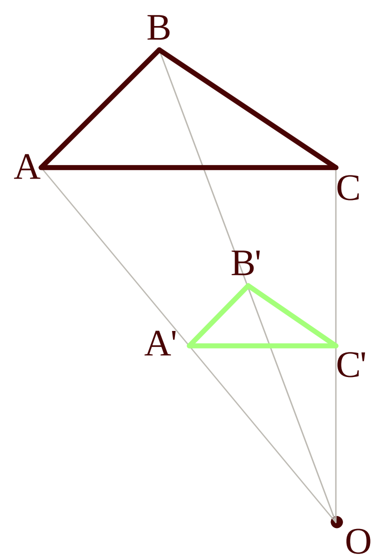
12-3 Size Transformations

Size Transformation - Choose a point O for the center of the transformation. Choose a scale factor r > 0. The size transformation assigns to each point A in the plane, the point A' where

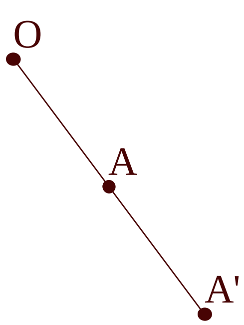
i. O, A and A' are collinear
ii. O is not between A and A'
iii. OA' = rOA

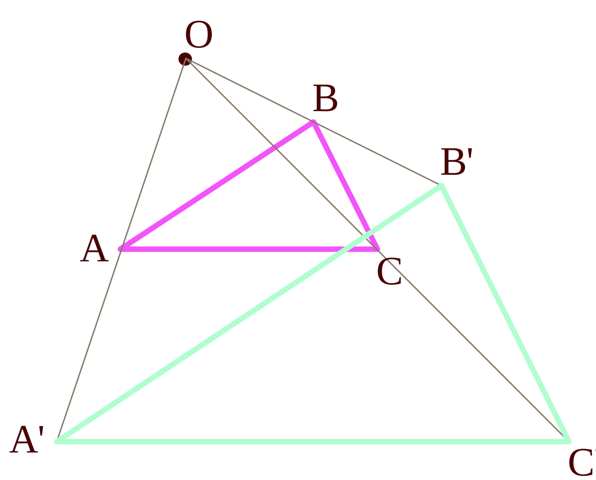
Ex: Let $r = \frac{1}{2}$ and center be O. Below are two different examples.





Ex: Let r = 2 and center be O. Below are two different examples.





Theorem 12-1 (on images of a size transformation)

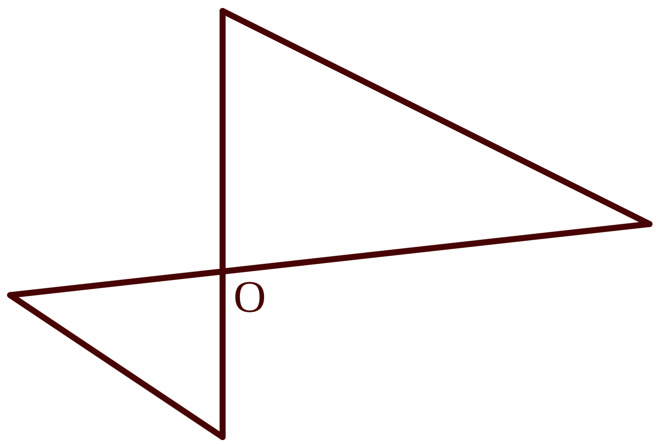
i. $\overline{AB} \parallel \overline{A'B'}$ and A'B' = r(AB)

ii. $\angle ABC \cong \angle A'B'C'$

Proof:

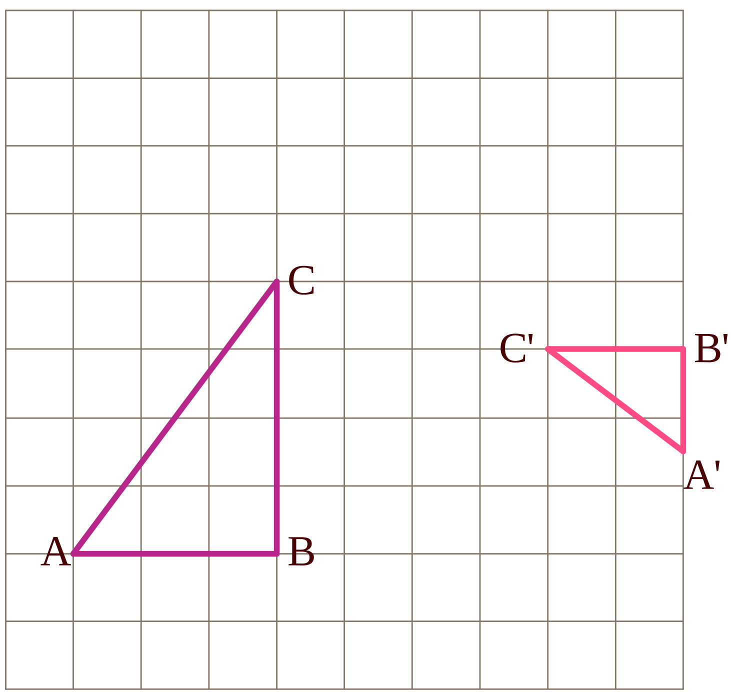
Similar Figures – Two figures are similar if it is possible to transform one onto the other by a sequence of isometries (translations, rotation, reflection) and a size transformation.

Ex:

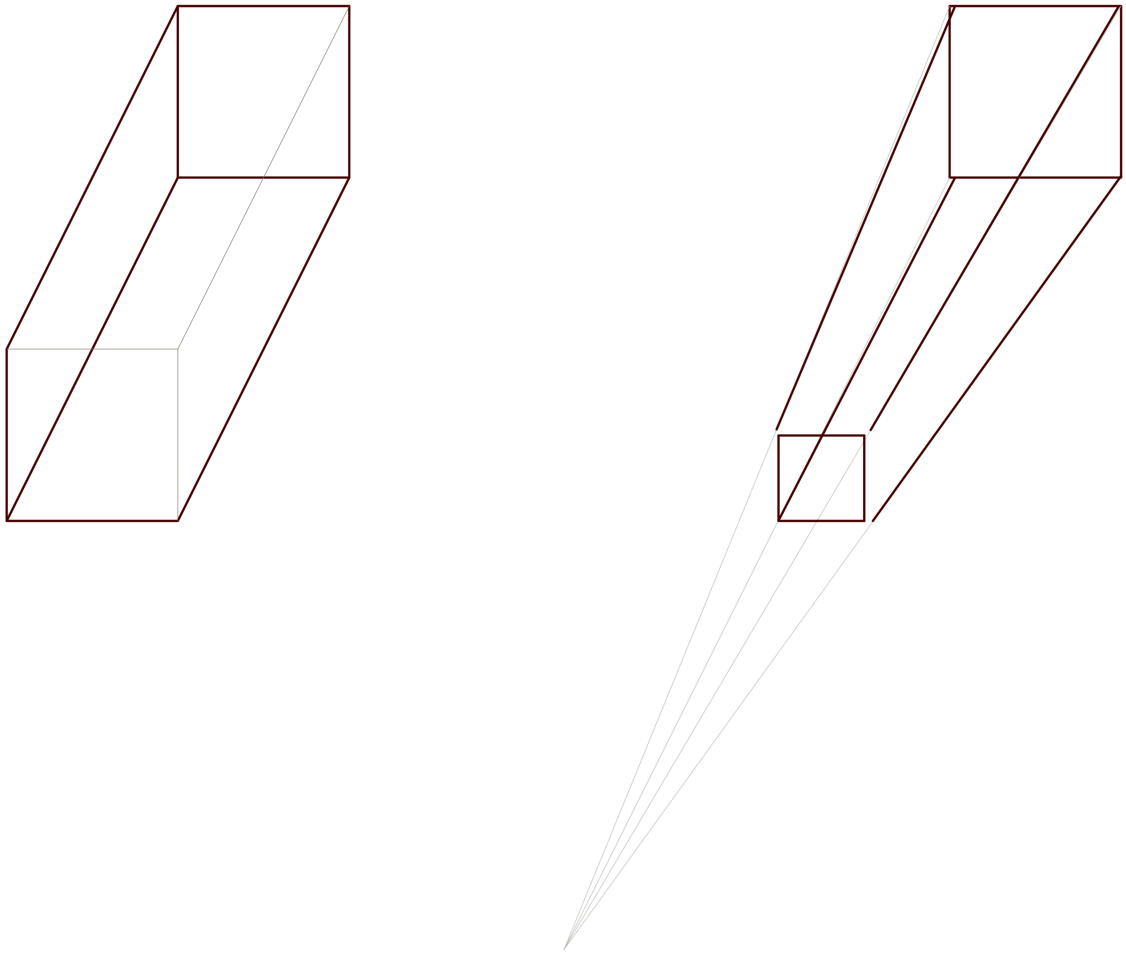


Rotate the smaller triangle 180 degrees using O as the center. Then let r = 2 and Let O be the center of the size transformation. The triangles are similar since one could be transformed onto the other by a rotation followed by a size transformation.

Describe a sequence of isometries that when followed by a size transformation changes triangle ABC to triangle A'B'C'.



Application – perspective drawings are used in architecture and art

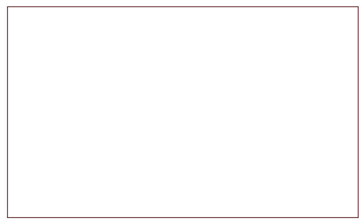


The eyes expect objects to look smaller as they get further away. The figure on the right is a perspective drawing.

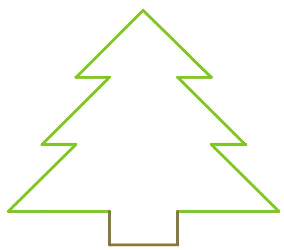
12-4 Symmetries

A. Line Symmetry – A plane region has a line l of symmetry if a reflection of the plane about l produces exactly the same figure.

Find the lines of symmetry of a rectangle.



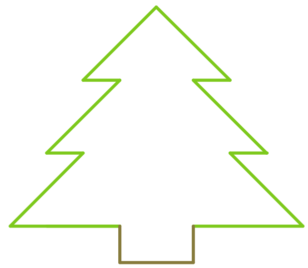
Find the line(s) of symmetry for the below picture.



B. Rotational Symmetry or Turn Symmetry – If there is a point P so that when the plane is rotated about P a certain number θ, $0^o < \theta < 360^o$, of degrees, the region is placed exactly upon itself, then the region has θ rotational symmetry (where θ is smallest possible).

Find the point P and the rotational symmetry if it exists of an equilateral triangle.

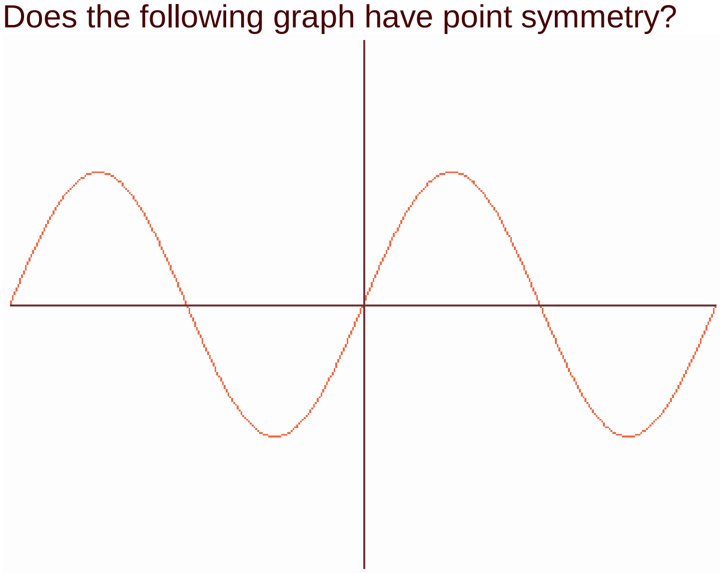
Find the point P and the rotational symmetry if it exists of the below picture.



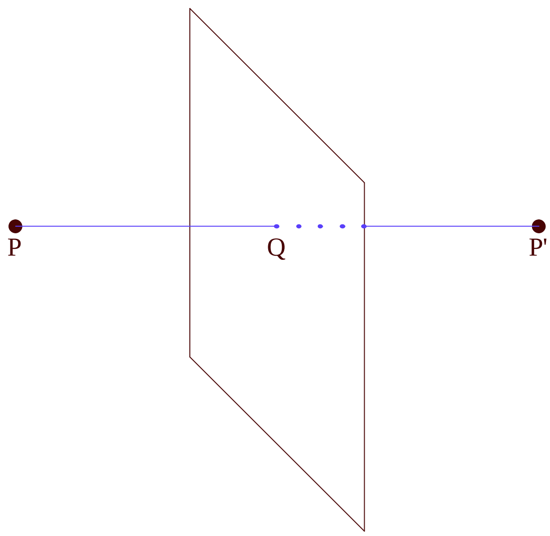
C. Point Symmetry – This is a 180-degree rotational symmetry about the turn center. Need to indicate the center.

Does a square have point symmetry?

Does the following graph have point symmetry?



D. Plane Symmetry (for solids) – Given a plane, each point is assigned to its opposite image through the plane. Plane symmetry means that the image is identical when reflected through the plane.



i. PQ = P'Q

ii. $\overline{PP'}$ is perpendicular to the plane

How many different plane symmetries does a right circular cylinder have? How many of each type?

Solids can also have point symmetry, line symmetry, and rotational symmetry (see figures on page 774).

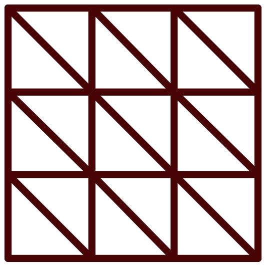
Find all the symmetries of any rectangle.

Problem: Given an arc of a circle, find its center and radius. (paper math)

12-5 Tessellations of the Plane

A tessellation is the filling of a plane or space with repetition of a figure or figures in such a manner than none overlap and there are no gaps.

Ex: tiles on a floor, mosaics, Escher's drawings



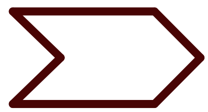
Regular Tessellation – is a tessellation made up of one type of regular polygon

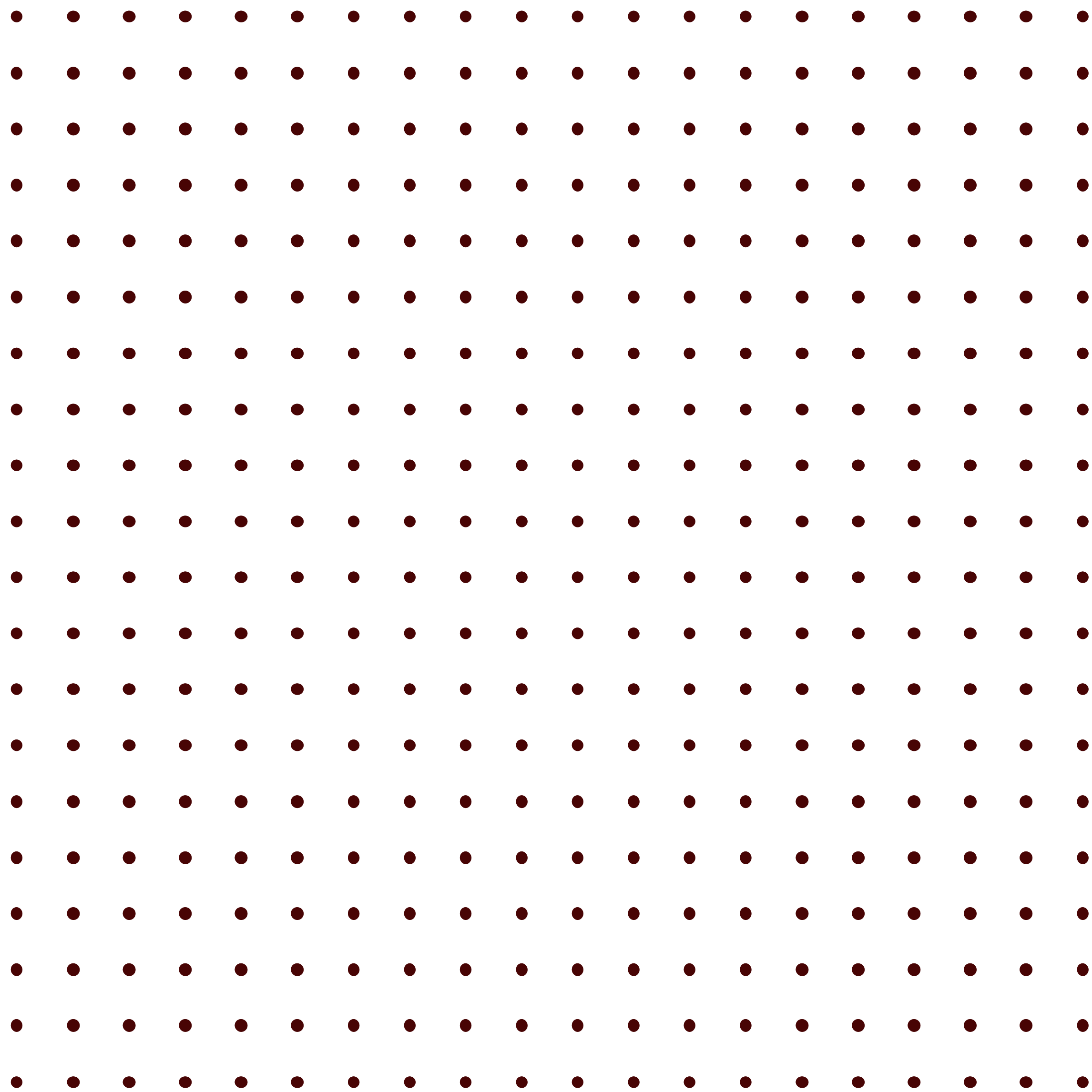
Ex: square tiles on a floor

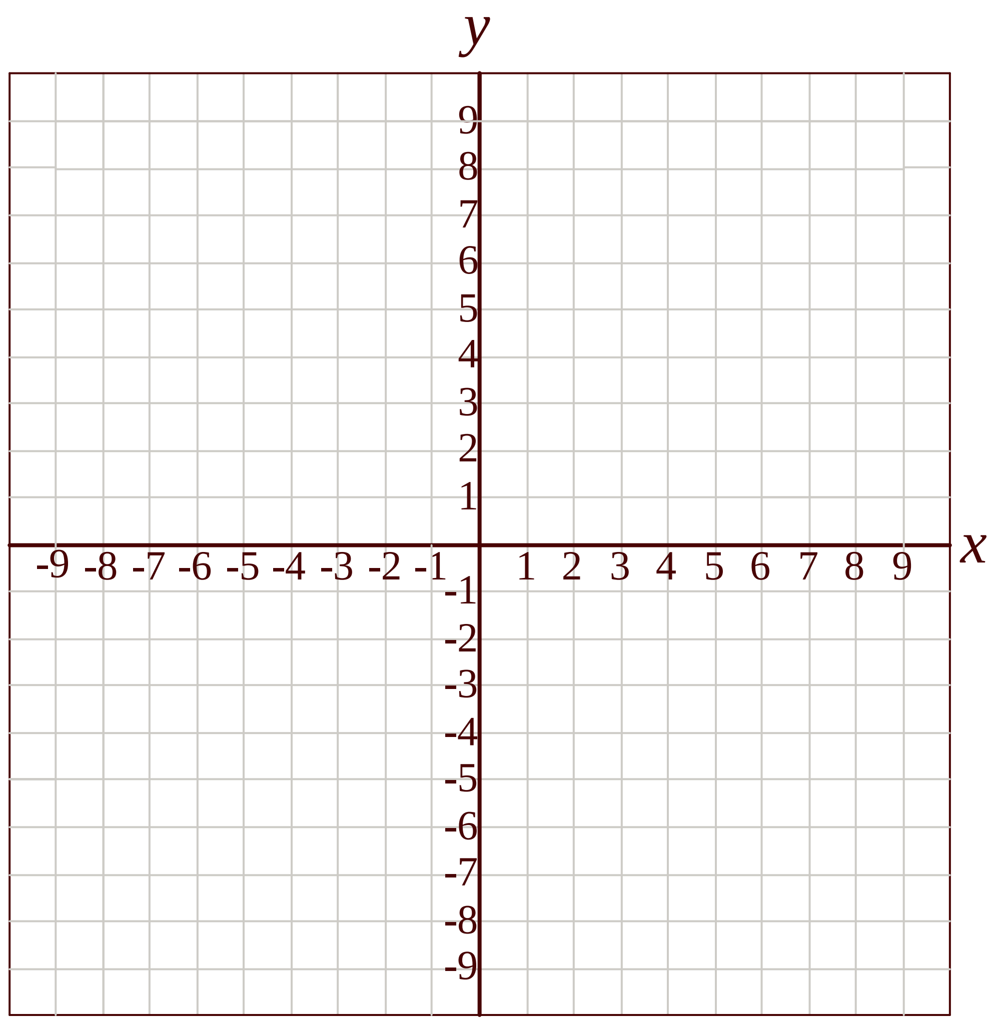
Problem: Which regular polygons tessellate the plane?

Therefore the only regular polygons that can tessellate the plane are

Other types can tessellate the plane.



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