Math 366 Chapter 9 Introductory Geometry

9-1 Basic Notions

The fundamental building blocks of geometry are <u>points</u>, <u>lines</u>, and <u>planes</u>. They are the undefined terms of geometry and are described intuitively to avoid a circular definition.

Point – intersection of two lines (what is a line?), has no dimension (no length, no width, no height), is a position, and has notation Point A.

Line – trace of a point (what is a point?), set of points that form a straight pattern, one dimension (has length but no width and no height), infinite in both directions, and has notation $\overleftrightarrow{AB}$, $\overleftrightarrow{BA}$ or line l.

Plane – all points in a flat pattern, has infinite length and width but no height, and has notation Plane ABC or Plane γ.

Linear Notions

<u>Collinear points</u> – points on the same line (col<u>line</u>ar),

Ex: A, B and C are collinear.

Ex: X, Y and Z are noncollinear.

Between – A point C is between points A and B if $C \neq A$, $C \neq B$, and C is on the part of the line flanked by A and B.

Line segment or segment – subset of a line between two points, including those two points, and has notation $\overline{AB}$ or $\overline{BA}$

Ray – a subset of a line that contains a point and one side of a line containing that point, and has notation $\overrightarrow{AB}$

Planar Notions

Coplanar – points that are in the same plane

Ex: Points A, B, and C are coplanar.

Ex: Points A, B, C, and D are noncoplanar since they cannot be placed in a single plane.

Skew lines – lines which do not intersect and there is no plane that contains them

Intersecting lines – two lines with exactly one point in common

Concurrent lines – lines that contain a common point

Parallel lines – two distinct coplanar lines that do not intersect, and have notation *m* || *n*.

Discussion p. 500 Now Try This 9 – 1 (***tennis ball***)

Properties (Axioms) – use to argue or prove

1. There is exactly one line through two distinct points. (***paper folding***)

2. If points A and B lie in a plane, so does $\overleftrightarrow{AB}$. (***paper folding***)

3. If two distinct planes intersect, then the intersection is a line. (***intertwine fingers***)

4. There is exactly one plane containing three noncollinear points. (***top of three students' heads***)

5. A line and a point not on the line determine a plane. (***point, pencil and paper***)

6. Two parallel lines determine a plane. (***two pencils***)

7. Two intersecting lines determine a plane. (***two pencils***)

Problem: Find the number of lines determined by 10 points, no 3 of which are collinear.

Polya's Four Step Problem-Solving Process (p. 17 – 18)

1. understanding the problem

2. devising a plan

3. carrying out the plan

4. looking back

More Notions

1. parallel planes – have no points in common

- A line is parallel to a plane if the line and plane do not intersect.
- What are all the possible relationships between a line and a plane?
- What are all the possible relationships between two distinct planes?
    - o Therefore two planes are

2. <u>half-plane</u> – A line, $\overleftrightarrow{AB}$, in a plane separates the plane into two halves, and has notation Half-Plane AB – C.

Note: The plane is the union of the three mutually disjoint sets: $\overleftrightarrow{AB}$, half-plane AB – C, and half-plane AB – D.

3. angles – An angle is the union of two rays sharing their endpoints. The rays are the sides of the angle and the common endpoint is the vertex.

Notation: $\angle$ABC or $\angle$CBA where B is the vertex. If clear from context, you can write $\angle$B or $\angle$1 or $\angle \alpha$.

Here $\angle$D is not clear from context so do not use it.

4. adjacent angles – two angles that share a common vertex and side, but the interiors do not overlap.

α and β are adjacent angles

x and *y* are not adjacent angles

5. angle measure – indicates the size of the opening. A protractor measures an angle, usually in degrees.

1^o = 60' one degree is 60 minutes
1' = 60" one minute is 60 seconds

69^o 37' 25" is 69 degrees, 37 minutes and 25 seconds.

A complete rotation about a point has measure 360^o.

6. types of angles –

<u>Straight angle</u> – measures 180^{o}

<u>Right angle</u> – measures 90^{o}

<u>Acute angle</u> – measures less than 90^{o} (and more than 0^{o})

<u>Obtuse angle</u> – measures more than 90^{o} and less than 180^{o}

7. <u>Perpendicular lines</u> – lines that intersect so that the angles formed are right angles, and has notation $m \perp n,\ \overleftrightarrow{AB} \perp \overleftrightarrow{AC},\ \overline{AB} \perp \overline{AC},\ \overrightarrow{AB} \perp \overrightarrow{AC},$ etc.

8. Line perpendicular to a plane – is a line that is perpendicular to every line in the plane through its intersection with the plane

Theorem 9 – 1: A line *r* perpendicular to two distinct lines *n* and *m* in the plane through its intersection with the plane is perpendicular to the plane.

9-2 Polygons

poly – means many
-gon – means a figure having a kind or number of angles

Draw a 'path' (bent or curved line) without lifting up your pencil (so no breaks) so it is connected. This is a curve.

Connected Connected Not Connected

Classification of Curves

1. simple – curve does not cross itself (except the starting and stopping point may be the same)

simple simple simple not simple not simple

2. closed – curve ends where it began

closed closed closed not closed not closed

3. convex curves – are simple closed curves and have no indentations (if points A and B are in the curve, then so is $\overline{AB}$)

convex convex not convex (concave)

4. <u>concave</u> – simple, closed curves which are not convex (they have an indentation)

5. <u>simple closed curve</u> – separates a plane into 3 non-intersecting pieces

<u>Polygons</u> – simple closed curve (connected) made up of segments

Polygon Polygon Not a polygon

Special Polygons, an <u>n-gon</u> is a polygon with n sides

3-gon, triangle

4-gon, quadrilateral

5-gon, pentagon

6-gon, hexagon

7-gon, heptagon

8-gon, octagon

9-gon, nonagon

10-gon, decagon

<u>diagonal</u> – segment that connects two non-consecutive vertices of a polygon

Ex: $\overline{EC}$ is a diagonal of the pentagon ABCDE

$\angle 1$ and $\angle 4$ are <u>interior angles</u>; $\angle 2$ and $\angle 3$ are <u>exterior angles</u> (determined by a side of the polygon and the extension of a contiguous side of a polygon)

<u>Congruent</u> – Two polygons are congruent if one is an exact copy of the other (can be matched exactly on top of each other).

iff = if, and only if
congruent $\cong$

1. Segments are congruent iff the segments have the same length.

$\overline{AB} \cong \overline{CD}$ iff AB = CD

$\overline{AB} \cong \overline{DE}$ iff AB = DE

2. Angles are congruent iff the angles have the same measure.

$\angle 1 \cong \angle ABC$ iff $m\angle 1 = m\angle ABC$

<u>Regular polygon</u> – is a polygon with

a. all sides congruent (equilateral)
b. all interior angles congruent (equiangular)

Ex: regular triangle (equilateral triangle, equiangular triangle)

Ex: regular quadrilateral (square)

Ex: regular pentagon

Further classifications of polygons (usually based upon angles or sides)
Also see pages 516 – 517.

Right triangle – a triangle containing one right angle

Acute triangle – a triangle in which all the angles are acute

Obtuse triangle – a triangle containing one obtuse angle

Scalene triangle – a triangle with no congruent sides

Isosceles triangle – a triangle with at least two congruent sides

Equilateral triangle – a triangle with three congruent sides

Trapezoid – a quadrilateral with at least one pair of parallel sides

Kite – a quadrilateral with at least two distinct pairs of consecutive congruent sides

Isosceles trapezoid – a trapezoid with one pair of congruent base angles (a trapezoid with two congruent nonadjacent sides)

Parallelogram – a quadrilateral in which each pair of opposite sides is parallel

Rectangle – a parallelogram with a right angle (a quadrilateral with four right angles)

Rhombus – a parallelogram with all sides congruent (a quadrilateral with all sides congruent)

Square – a rectangle with all sides congruent (a quadrilateral with four right angles and four congruent sides)

Relate the polygons by hierarchy using a Venn Diagram or a tree (see p. 520 for other examples)

Ex: Venn Diagram relating quadrilateral, parallelogram, rectangle, rhombus, square, and trapezoid (* depends upon definition)

* trapezoid – 1. a quadrilateral with at least one pair of parallel sides (p. 517) or 2. (more common definition) a quadrilateral with exactly one pair of parallel sides (p. 519)

Ex: Tree relating triangle, acute triangle, obtuse triangle, equiangular triangle, right triangle

9-3 More About Angles

Vertical angles – two angles that are formed by intersecting lines and that are opposite from each other

Complementary angles – (two angles' measures sum to 90 degrees)

Supplementary angles – (two angles' measures sum to 180 degrees)

Transversal – line that intersects a pair of lines

vertical angles

corresponding angles

alternate exterior angles

alternate interior angles

Theorem 9 – 2: Vertical angles are congruent.

Proof:

Given $\angle 2 \cong \angle 6$.

Conclude

Theorem 9 – 3: If any 2 distinct coplanar lines are cut by a transversal, then a pair of $\begin{pmatrix}\text{alternate interior}\\ \text{alternate exterior}\\ \text{corresponding}\end{pmatrix}$ angles are congruent iff the lines are parallel.

Architects use the fact that lines are parallel iff their corresponding angles are congruent. (See p. 525).

Theorem 9 – 4: The sum of the measures of the interior angles of a triangle is 180 degrees.

Theorem 9 – 5: The sum of the exterior angles (one at each vertex) of any convex polygon is 360 degrees.

Theorem 9 – 6: The sum of the interior angles of a convex polygon with *n* sides is $(n - 2)\,180^{o}$. The measure of a single interior angle of a 'regular' *n*-gon is $\frac{(n-2)180^{\circ}}{n}$.

Argue for a quadrilateral:

Teaser: You go 1 mile south, next 1 mile east, then 1 mile north, and you are back to where you started. You see a bear. What color is it and why?

9-4 Geometry in Three Dimensions

Sphere – set of all points equidistant from a given point (center)

Simple Closed Surface – distortion of a sphere; it has exactly one interior, no holes, and is hollow

A simple closed surface separates space into interior, surface, and exterior.

Donut – not a simple closed surface since it has a hole; you would have to tear the sphere to get it

Pretzel - not a simple closed surface since it has 3 holes

Solid - the set of all points on a simple closed surface with all its interior points

Polyhedron - a simple closed surface made up of polygonal regions, called faces

Special Types of Surfaces

1. prism – polyhedron in which two congruent faces (bases) are parallel and the other faces are parallelograms (see p. 535)

    a. right prism – prism where the lateral faces are all bounded by rectangles

    b. oblique prism – prism where some of the lateral faces are not bounded by rectangles

2. pyramid – determined by a polygon and a point not on the polygon; name is based upon the shape of the base

3. polyhedrons
   a. convex polyhedron – a polyhedron if a segment connecting any two points in the interior is itself in the interior

   b. concave polyhedron – a polyhedron that is not convex; it does not pass the segment test

   c. regular polyhedron – convex polyhedron whose
      i. faces are congruent regular polygonal regions
      ii. number of edges that meet at each vertex is the same for all vertices

How many regular polyheda are there?

- understand problem – Each face must be congruent to all the other faces. Each face is bounded by a regular polygon, Find the number of different regular polyhedra

- devise a plan – The sum of the measures of all the angles at a vertex of a regular polyhedron is less than 360 degrees (otherwise it would be flat at that vertex). Examine the measures of the interior angles of regular polygons to see which ones could be faces of a regular polyhedron.

- carry out plan –

<u>regular polygon</u>   <u>measure of interior angle</u>

At least three figures must fit together at a vertex to make a polyhedron. If 3 angles of a regular heptagon were together at one vertex, then the sum of the measures of these angles would be $3 \cdot \frac{900^\circ}{7} = \left(\frac{2700}{7}\right)^o$, which is greater than 360 degrees. So a heptagon cannot be used.

The measure of an interior angle increases as the number of sides increases. Any polygon with more than 6 sides has an interior angle greater than 120 degrees, and since we need at least 3 angles at a vertex, the sum of the measures of the angles would be greater than 360 degrees. So now we look the remaining four: equilateral triangle, square, regular pentagon, and regular hexagon.

Number of polygons per vertex	Resulting Polyhedra

- looking back –
  d. Consider semi-regular polyhedra (soccer ball has regular pentagons and hexagons)

  e. There are five types of regular solid polyhedra, known as the Platonic solids (p. 537 and 539). All occur in nature. (**polydrons**)

Nets (see p. 539) are flat patterns that can be used to construct the five regular polyhedra.

4. cylinder – move a segment (keeping it parallel to the original segment) to form a simple closed planar curve at its ends; the simple closed curves along with the interior formed is a cylinder

5. cone – The union of line segments connecting a point (apex, vertex) not on a simple closed non-polygonal curve to each point on this simple closed non-polygonal curve, and the interior of the curve is a cone.

9-5 Networks (Graph Theory)

Seven Bridges of Konigsberg

Is it possible to walk across all bridges exactly once on the same walk?

In the 1700's Euler was interested in this problem and represented the landmasses and bridges as a network (or graph). See p. 545-546.

A graph is <u>traversable</u> if it has a path through the graph (network) so that each edge (arc) is traversable exactly once. If the starting and stopping points of a traversable graph are the same, then the graph is a <u>Eulerian circuit</u>.

Mail Carrier Problem – Can a mail carrier find a path to go to each mailbox (are all mailboxes are on the same side of the street or are they on both sides of the street) exactly once and end back up at the post office where he/she started?

Painter Problem – paint halls

Traveling Salesman Problem – Go through each city (vertex) exactly once (and possibly get back home)

The degree of a vertex is the number of edges touching it. An even vertex means its degree is even. An odd vertex means its degree is odd.

Are the Konigsberg bridges traversable? Why or why not?