

## Subalgebras of the Stanley–Reisner Ring\*

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**Abstract.** In [2], Billera proved that the  $\mathbf{R}$ -algebra of continuous piecewise polynomial functions ( $C^0$  splines) on a  $d$ -dimensional simplicial complex  $\Delta$  embedded in  $\mathbf{R}^d$  is a quotient of the Stanley–Reisner ring  $A_\Delta$  of  $\Delta$ . We derive a criterion to determine which elements of the Stanley–Reisner ring correspond to splines of higher-order smoothness. In [5], Lau and Stiller point out that the dimension of  $C_k^r(\Delta)$  is upper semicontinuous in the Zariski topology. Using the criterion, we give an algorithm for obtaining the defining equations of the set of vertex locations where the dimension jumps.

### 1. Introduction

Let  $\Delta$  be a  $d$ -dimensional simplicial complex, embedded in  $\mathbf{R}^d$ .  $\Delta$  is *pure* if all maximal faces of  $\Delta$  are of dimension  $d$ , and *strongly connected* if for any two  $d$ -simplices  $\sigma, \sigma'$ , with  $\sigma \cap \sigma' = \tau \neq \emptyset$ , there is a sequence of  $d$ -simplices

$$\sigma = \sigma_1, \sigma_2, \dots, \sigma_q = \sigma'$$

such that, for each  $i < q$ ,  $\sigma_i \cap \sigma_{i+1}$  has dimension  $d - 1$  and contains  $\tau$ . We require  $\Delta$  to be pure and strongly connected (in other words,  $\Delta$  is a pseudomanifold).

**Definition 1.1.** Let  $\Delta$  be a simplicial complex on the vertex set  $\{v_1, \dots, v_n\}$ . The Stanley–Reisner ring [6]  $A_\Delta = \mathbf{R}[Y_1, \dots, Y_n]/I_\Delta$ , where  $I_\Delta$  is the ideal of nonfaces of  $\Delta$ , i.e.,  $I_\Delta = \langle Y_{i_1} \cdots Y_{i_j} \mid \{v_{i_1}, \dots, v_{i_j}\} \notin \Delta \rangle$ .

Let  $C^r(\Delta)$  denote the algebra of piecewise polynomial functions on  $\Delta$  of smoothness  $r$  ( $r$ -splines). In [2], Billera proved that  $C^0(\Delta)$  is isomorphic as an  $\mathbf{R}$ -algebra to

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$A_\Delta / \sum_{i=1}^n Y_i - 1$ , where  $n$  is the number of vertices of  $\Delta$ . The key idea is to think of a variable in  $A_\Delta$  as a Courant function, centered at the corresponding vertex.

In [3], Billera and Rose noted that questions about splines are most naturally approached by embedding  $\Delta$  in the hyperplane  $\{x_{d+1} = 1\} \subseteq \mathbf{R}^{d+1}$ , and then forming the cone  $\hat{\Delta}$  of  $\Delta$  with the origin in  $\mathbf{R}^{d+1}$ . For example, the dimension of  $C_k^r(\Delta)$  (the space of splines of smoothness  $r$ , for which each polynomial is of degree at most  $k$ ) is equal to the dimension of the *graded* module  $C^r(\hat{\Delta})$  in degree  $k$ . The Stanley–Reisner ring of  $\hat{\Delta}$  has one more variable than the Stanley–Reisner ring of  $\Delta$ ; since that variable corresponds to the vertex of the cone, it does not appear in any of the monomials generating  $I_\Delta$ , hence [4, Theorem 4.2]

$$C^0(\hat{\Delta}) \simeq A_{\hat{\Delta}} / \sum_{i=1}^{n+1} Y_i - 1 \simeq A_\Delta.$$

Since  $C^{r+1}(\hat{\Delta}) \hookrightarrow C^r(\hat{\Delta})$ , Billera’s result implies that there is a descending chain of subalgebras contained in  $A_\Delta$ , each corresponding to a subalgebra of splines of increasing orders of smoothness. Viewing the variables of the (affine) Stanley–Reisner ring as Courant functions gives a geometric picture of  $C^0(\Delta)$ ; homogenizing the problem streamlines the algebra. By relating the homological approach used by Billera in [1] to a homology where  $A_\Delta$  appears, we obtain a local characterization of those elements of  $A_\Delta$  which correspond to elements of  $C^r(\hat{\Delta})$ .

## 2. Preliminaries

In order to simplify the presentation we concentrate on the planar case, i.e.,  $\Delta$  is a simplicial complex which is embedded in  $\mathbf{R}^2$ . Let  $r \geq 0$  be an integer, and let  $R = \mathbf{R}[x_1, x_2, x_3]$ .  $\Delta_i, \Delta_i^0$  denote (respectively) the sets of  $i$ -dimensional faces and  $i$ -dimensional interior faces of  $\Delta$ ; for  $\varepsilon \in \Delta_1$ , let  $l_\varepsilon \in \mathbf{R}[x_1, x_2]$  denote a nonzero linear form vanishing on  $\varepsilon$ , and  $l_\varepsilon \in R$  its homogenization.

**Definition 2.1.** The space of splines  $C^r(\hat{\Delta})_k$  is the set  $\{F : |\hat{\Delta}| \rightarrow \mathbf{R}, F|_{\hat{\sigma}_i}$  is a polynomial of degree  $k$ , for all  $\sigma_i \in \Delta_2$ , and  $F$  is continuously differentiable of order  $r\}$ .

Denote the coordinates of a vertex  $v_i$  in  $\mathbf{R}^2$  by  $(v_{i1}, v_{i2})$ . If  $\{v_i, v_j, v_k\}$  defines an element  $\sigma$  of  $\Delta_2$ , let  $x_i^\sigma$  be the unique linear function which is zero at  $v_j$  and  $v_k$ , and one at  $v_i$ :

$$x_i^\sigma = \frac{\begin{vmatrix} x_1 & v_{j1} & v_{k1} \\ x_2 & v_{j2} & v_{k2} \\ 1 & 1 & 1 \end{vmatrix}}{\begin{vmatrix} v_{i1} & v_{j1} & v_{k1} \\ v_{i2} & v_{j2} & v_{k2} \\ 1 & 1 & 1 \end{vmatrix}}.$$

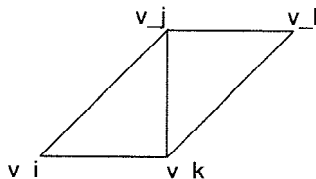
Let  $v_{ijk}$  denote the determinant which appears in the denominator of the above expression. Expanding the determinants yields  $l_\varepsilon \in \mathbf{R}[x_1, x_2]$ , where  $\varepsilon$  is the edge joining

$v_j$  and  $v_k$ . The functions  $x_i^\sigma, x_j^\sigma, x_k^\sigma$  give the unique barycentric coordinates of a point of  $\mathbf{R}^2$  in terms of the vertices  $v_i, v_j, v_k$  of  $\sigma$  (see equation 2.4 of [2]).

**Definition 2.2.** Let  $X_i^\sigma$  be the homogenization (with respect to  $x_3$ ) of  $x_i^\sigma$ . For  $\sigma \in \Delta_2$  as above, define  $A_\sigma := \mathbf{R}[X_i^\sigma, X_j^\sigma, X_k^\sigma]$ , and define  $B_\sigma$  as the linear automorphism  $R \xrightarrow{B_\sigma} A_\sigma$ ,

$$\begin{aligned} x_1 &\rightarrow v_{i1}X_i^\sigma + v_{j1}X_j^\sigma + v_{k1}X_k^\sigma, \\ x_2 &\rightarrow v_{i2}X_i^\sigma + v_{j2}X_j^\sigma + v_{k2}X_k^\sigma, \\ x_3 &\rightarrow X_i^\sigma + X_j^\sigma + X_k^\sigma. \end{aligned}$$

**Example 2.3.** Let  $\sigma$  denote the triangle on the left, and  $\sigma'$  the triangle on the right:



**Lemma 2.4.** Suppose  $\sigma, \sigma' \in \Delta_2$  share a common edge, as above. The map  $B_{\sigma'\sigma}$  given by

$$\begin{aligned} X_l^{\sigma'} &\rightarrow \frac{v_{ijk}}{v_{ljk}} X_i^\sigma, \\ X_j^{\sigma'} &\rightarrow \frac{v_{lik}}{v_{ljk}} X_i^\sigma + X_j^\sigma, \\ X_k^{\sigma'} &\rightarrow \frac{v_{lji}}{v_{ljk}} X_i^\sigma + X_k^\sigma \end{aligned}$$

is the unique homomorphism from  $A_{\sigma'}$  to  $A_\sigma$  such that  $B_{\sigma'\sigma} \circ B_{\sigma'} = B_\sigma$ . Let  $s_1, s_2$ , and  $s_3$  denote the coefficients of  $X_i^\sigma$ ;  $s_1 + s_2 + s_3 = 1$ .

In [1], Billera showed that if  $f, g$  are polynomials supported on maximal simplices of  $\hat{\Delta}$  which meet along  $\hat{\varepsilon}$  (the cone over an edge), then  $f$  and  $g$  meet  $C^r$  smoothly if and only if  $l_{\hat{\varepsilon}}^{r+1} | f - g$ .

**Lemma 2.5.** Suppose  $\sigma, \sigma' \in \Delta_2$  are as in Example 2.3, i.e., they share a common edge  $\varepsilon = \sigma \cap \sigma'$ . Let  $f, g \in R$  (think of  $f$  as supported on  $\hat{\sigma}$  and  $g$  as supported on  $\hat{\sigma}'$ ). Then  $l_{\hat{\varepsilon}}^{r+1} | f - g$  in  $R$  if and only if  $(X_i^\sigma)^{r+1} | B_\sigma(f - g)$  in  $A_\sigma$ .

*Proof.*  $B_\sigma$  is an isomorphism, and  $B_\sigma(l_{\hat{\varepsilon}}) = c \cdot X_i^\sigma, 0 \neq c \in \mathbf{R}$ . □

**Lemma 2.6.** *Fix a total order on the elements of  $\Delta_2$ . Then the following diagram is commutative and exact:*

$$\begin{array}{ccccccc}
 0 & \longrightarrow & C^r(\hat{\Delta}) & \longrightarrow & \bigoplus_{\sigma \in \Delta_2} R & \xrightarrow{\partial^r} & \bigoplus_{\substack{\varepsilon = \sigma \cap \sigma' \in \Delta_1^0 \\ \sigma < \sigma'}} R/l_{\hat{\varepsilon}}^{r+1} \\
 & & \downarrow i & & \downarrow \oplus B_{\sigma} & & \downarrow \oplus B_{\sigma} \\
 0 & \longrightarrow & \ker(\Psi^r) & \longrightarrow & \bigoplus_{\sigma \in \Delta_2} A_{\sigma} & \xrightarrow{\Psi^r} & \bigoplus_{\substack{\varepsilon = \sigma \cap \sigma' \in \Delta_1^0 \\ \sigma < \sigma'}} A_{\sigma}/B_{\sigma}(l_{\hat{\varepsilon}}^{r+1})
 \end{array}$$

*Proof.* In order to translate the smoothness condition from  $\bigoplus_{\sigma \in \Delta_2} R$  to  $\bigoplus_{\sigma \in \Delta_2} A_{\sigma}$ , we need to choose local coordinate systems in which to work (there are two choices available at each edge, corresponding to the two triangles joined by the edge). Choosing an order on  $\Delta_2$  is one way to do this, consistent with what Billera and Rose do in Section 4 of [3]. We define the maps  $\partial^r, \Psi^r$  for Example 2.3; the general definition is the natural extension of this definition. If  $\varepsilon = \sigma \cap \sigma' \in \Delta_1^0$ , and  $\sigma < \sigma'$ , then

$$\begin{aligned}
 \partial^r(f, g) &= f - g \quad \text{in } R/l_{\hat{\varepsilon}}^{r+1} \quad (f \in R_{\sigma}, g \in R_{\sigma'}), \\
 \Psi^r(F_{\sigma}, F_{\sigma'}) &= F_{\sigma} - B_{\sigma'\sigma}(F_{\sigma'}) \quad \text{in } A_{\sigma}/B_{\sigma}(l_{\hat{\varepsilon}}^{r+1}).
 \end{aligned}$$

An easy check shows the second square commutes. Since the second and third vertical maps are isomorphisms, the induced map  $i$  is then also an isomorphism by the snake lemma. □

### 3. The Map $\Phi$

We now connect  $A_{\Delta}$  to the complex defined in Lemma 2.6 (which encodes  $C^r(\hat{\Delta})$ ).

**Lemma 3.1.** *There is an inclusion of  $\mathbf{R}$ -algebras:*

$$A_{\Delta} \xrightarrow{\Phi} \bigoplus_{\sigma \in \Delta_2} A_{\sigma}.$$

*Proof.* Define  $\Phi$  via  $\Phi(1) = (1, \dots, 1)$ . On each summand  $A_{\sigma}$ , define  $\Phi$  via

$$\Phi(Y_j) = \begin{cases} 0 & \text{if } v_j \notin \sigma, \\ X_j^{\sigma} & \text{if } v_j \in \sigma. \end{cases}$$

This gives a map from  $A_{\Delta}$  to  $\bigoplus_{\sigma \in \Delta_2} A_{\sigma}$ , which may be viewed on each summand of  $\bigoplus_{\sigma \in \Delta_2} A_{\sigma}$  as the restriction map. If a monomial  $M$  corresponds to a nonface of  $\Delta$ , then  $\Phi(M)$  is zero, so we have a well-defined map. By restricting to the individual summands, it is easy to check that  $\Phi$  is a homomorphism. A polynomial  $F$  will have image zero only if it is not supported on any simplex (hence, is an element of  $I_{\Delta}$ ), so  $\Phi$  is an inclusion. □

It is easy to prove that  $\text{im } \Phi \subseteq \ker \Psi^0$ ; Billera’s theorem implies that equality holds. This, coupled with Lemma 3.1, allows us to define an  $R$ -algebra structure on  $A_\Delta$ , which is just the restriction of the  $R$ -action on  $\bigoplus_{\sigma \in \Delta_2} A_\sigma$  to the image of  $\Phi$ .

**Lemma 3.2.** *The image of  $\Phi$  is closed under the action of  $R$ .*

*Proof.* Let  $f \in R$  and  $F \in A_\Delta$ ; we need to show that, for some  $G \in A_\Delta$ ,  $B_\sigma(f) \cdot \Phi(F)|_\sigma = \Phi(G)|_\sigma$ , for all  $\sigma \in \Delta_2$ . It suffices to show that there is an  $H \in A_\Delta$  such that  $\Phi(H)|_\sigma = B_\sigma(f)$  for all  $\sigma \in \Delta_2$ .  $H = f(\sum_{i=1}^n v_{i1}Y_i, \sum_{i=1}^n v_{i2}Y_i, \sum_{i=1}^n Y_i)$  does the job.  $\square$

**Theorem 3.3.**  *$F \in A_\Delta$  corresponds to an element of  $C^r(\hat{\Delta})$  if and only if  $\Psi^r(\Phi(F)) = 0$ , i.e., if and only if  $\Psi^r(F|_{\sigma_1}, \dots, F|_{\sigma_m}) = 0$ .*

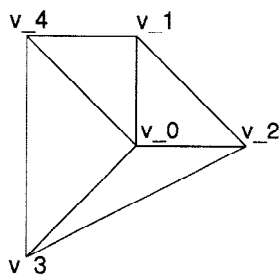
*Proof.* By Lemmas 2.4 and 2.5,  $\Psi^r(F|_{\sigma_1}, \dots, F|_{\sigma_m})$  is zero if and only if, for all  $\sigma < \sigma'$  such that  $\sigma \cap \sigma' \in \Delta_1^0$ ,  $(X_i^\sigma)^{r+1}$  divides  $F|_\sigma - B_{\sigma'\sigma}(F|_{\sigma'})$ . By Lemma 2.6,  $\ker \Psi^r \simeq C^r(\hat{\Delta})$ .  $\square$

#### 4. The Dimension of $C^r(\hat{\Delta})_k$

**Theorem 4.1.** *The dimension of  $C^r(\hat{\Delta})_k$  is stratified by a descending chain of determinantal subvarieties of  $V(I)$ , where  $I$  is the ideal of relations on the  $v_{ijk}$  (linear relations and the Plücker relations).*

*Proof.* Let  $F = \sum_{|\alpha_i|=k} a_i \cdot Y^{\alpha_i} \in A_\Delta$ .  $F|_\sigma - B_{\sigma'\sigma}(F|_{\sigma'}) \in A_\sigma$ , with coefficients which are polynomials in  $s_i$  and  $a_i$ , linear in the  $a_i$ . The smoothness condition means that coefficients of the monomials of  $F|_\sigma - B_{\sigma'\sigma}(F|_{\sigma'})$  not divisible by (the appropriate)  $(X_i^\sigma)^{r+1}$  must vanish, so the  $a_i$  are in the kernel of a matrix whose entries are polynomials in  $s_i$ . The denominators of the  $s_i$  are nonzero, so clearing them yields a matrix whose entries are polynomials in the  $v_{ijk}$ . The rank of the matrix determines the dimension of  $C^r(\hat{\Delta})_k$ , hence  $\dim C^r(\hat{\Delta})_k$  increases along a descending chain of determinantal varieties.  $\square$

**Example 4.2.** Consider the central configuration given by



For this example,  $\dim C_2^1(\hat{\Delta})$  is 7 for generic vertex positions, and 8 for nongeneric positions (see p. 123 of [3]). Identify  $Y_i$  with  $\Phi(Y_i) = X_i^\sigma$  (if  $v_i \in \sigma$ ), and let  $F$  be a quadric in  $A_\Delta$ , i.e.,

$$F = a_0 Y_0^2 + a_1 Y_0 Y_1 + \cdots + a_{12} Y_4^2.$$

Order the four maximal simplices as  $\{v_0, v_1, v_2\} < \{v_0, v_1, v_4\} < \{v_0, v_2, v_3\} < \{v_0, v_3, v_4\}$ , and apply Theorem 3.3. For example,  $F|_{\sigma_1} - B_{\sigma_2\sigma_1}(F|_{\sigma_2})$  is equal to

$$Y_0 Y_2 \cdot \left( a_2 - 2 \cdot \frac{v_{421}}{v_{401}} \cdot a_0 - \frac{v_{402}}{v_{401}} \cdot a_1 - \frac{v_{201}}{v_{401}} \cdot a_4 \right) + Y_1 Y_2 \cdot \left( a_6 - \frac{v_{421}}{v_{401}} \cdot a_1 - 2 \cdot \frac{v_{402}}{v_{401}} \cdot a_5 - \frac{v_{201}}{v_{401}} \cdot a_7 \right) + Y_2^2 \cdot \gamma.$$

$F$  is  $C^1$  across the (cone over the) face  $\sigma_1 \cap \sigma_2$  iff  $Y_2^2$  divides  $F|_{\sigma_1} - B_{\sigma_2\sigma_1}(F|_{\sigma_2})$ . The conditions from the remaining interior edges show that  $F$  is an element of  $C_2^1(\hat{\Delta})$  if and only if  $(a_0, \dots, a_{12})$  is in the kernel of the matrix  $N$ :

$$\begin{matrix} v_{124} & -v_{024} & v_{014} & 0 & -v_{012} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & v_{124} & 0 & 0 & 0 & -v_{024} & v_{014} & -v_{012} & 0 & 0 & 0 & 0 & 0 \\ v_{134} & -v_{034} & 0 & v_{014} & -v_{013} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & v_{134} & 0 & 0 & -v_{034} & 0 & 0 & 0 & v_{014} & -v_{013} \\ -v_{123} & v_{023} & -v_{013} & v_{012} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -v_{123} & 0 & 0 & 0 & v_{023} & 0 & -v_{013} & v_{012} & 0 & 0 & 0 \\ -v_{234} & 0 & v_{034} & -v_{024} & v_{023} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -v_{234} & 0 & 0 & 0 & 0 & 0 & v_{034} & -v_{024} & v_{023} & 0 \end{matrix}$$

Computing the  $i$  by  $i$  minors of  $N$ , saturating ( $v_{012}, v_{023}, v_{034}, v_{014}$  are all nonzero), and reducing mod  $I$ , we find no new relations for  $i \geq 7$ , while for  $i = 6$  we obtain  $v_{024}, v_{013}$ . So  $\dim C_2^1(\hat{\Delta}) = 8$  exactly where  $v_{013}, v_{024}$  both vanish; i.e., the vertices  $v_0, v_1, v_3$  are collinear, as are the vertices  $v_0, v_2, v_4$ .

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