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BOOK REVIEWS

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Schenck, H. *Computational algebraic geometry* (Cambridge University Press, 2003), xiv + 193 pp., 0521 53650 2 (paperback), £18.99, 0521 82964 9 (hardback), £50.

Mathematical work thrives on examples, in both research and teaching. Now that computers are powerful and cheap enough to be widely available, the range of feasible examples has enlarged significantly. Algorithmic methods that once seemed impractical are now used routinely. In the areas of algebraic geometry and commutative algebra, one of the most widely used packages is Macaulay (due to Grayson and Stillman). This is free, available for most platforms and fairly easy to use. The foundation stone here is the notion of Gröbner bases and the associated algorithm for computing them, both due to Buchberger. Briefly speaking, suppose we are given multivariate polynomials $f_1, f_2, \dots, f_n$ with coefficients from a field and wish to know if a polynomial f is in the ideal generated by the given polynomials. Trivially, this is so if and only if $f = c_1g_1u_1 + c_2g_2u_2 + \cdots + c_mg_mu_m$, where each c_i is a constant, each g_i is one of the given generators and each u_i is a power product in the indeterminates. This suggests introducing a notion of multivariate division with the aim of showing that f is in the ideal if and only if it leaves remainder 0 when divided by the generators. Even when this notion is made precise it is easy enough to see that it is over optimistic (if the remainder is 0, then certainly we have membership of the ideal, but the converse is not guaranteed). However, a simple analysis of how things go wrong suggests the notion of a change of basis for which the optimism is fully justified: hence Gröbner bases. Buchberger's algorithm provides a way of bringing about such a change of basis. Once we have a Gröbner basis for an ideal, a great deal of information can be derived. The idea can be generalized to modules in a fairly straightforward manner and as a reward we can, for example, compute syzygies (due to Schreyer).

The book under review covers a great deal of ground, taking in topics from commutative algebra, algebraic geometry, algebraic topology and algebraic combinatorics. Aspects of theory are presented together with illustrative examples using Macaulay 2. The book is organized into 10 chapters as well as two introductory appendices on algebra and complex analysis. Gröbner bases are discussed in an early chapter. The chapters begin with *Basics of commutative algebra* and range through *Combinatorics, topology and the Stanley–Reisner ring* to *Curves, sheaves and cohomology* and finally *Projective dimension, Cohen–Macaulay modules, upper bound theorem*. This sample of chapter headings makes it clear that this short book cannot possibly cover anything like all the details, indeed to quote the author, 'Mea maxima culpa: in this chapter we have sketched material more properly studied in a semester course'. In fact this is one of the book's strengths; it gives the reader a guided and informative tour to deep material. The author suggests further reading at the end of each chapter, discussing how it fits in with the topics and approach taken in the book.

The book is not primarily concerned with the development of algorithms in algebraic geometry but much of the presentation is informed by such considerations. For example, computing the Hilbert polynomial is discussed first in terms of free resolutions. This is then revisited in the chapter on Gröbner bases and reconsidered in this context via a well known result due to Macaulay (that the Hilbert function of an ideal is the same as that of the ideal generated by the initial terms of the ideal, under an admissible order). As another example we quote again from the author (Chapter 8): 'So what? Well, this is exactly what proves that we can compute $\mathrm{Tor}_i(M,N)$ ...'. This also indicates the informal style of the writing: friendly but not overly so. There are many exercises throughout, of both a theoretical and computational nature. They also range from the straightforward to the hard (for which the author gives helpful hints or references where the solution can be found).

To sum up, a student considering this area will find the guided-tour aspect of this book very helpful. The computing aspect is a good pointer to what can be done, but the book is not intended to act as a manual. Unless such a student is already well informed, he or she will find the suggestions for further reading essential.

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STOPPLE, J. *A primer of analytic number theory* (Cambridge University Press, 2003), 0 521 01253 8 (paperback) £22.95, 0 521 81309 3 (hardback) £65.

The author makes it immediately clear in the introduction that this book is not intended to be anything like a conventional text on analytic number theory. The starting point is the observation that courses on number theory normally appear quite early in the curriculum, and consequently avoid all results that require any input from analysis. His basic thesis is that this avoidance is unnecessary. In this book, he seeks to demonstrate that one can introduce a wide range of topics within analytic number theory assuming no more than a basic exposure to 'calculus'. It is not even assumed that the reader knows what is meant by the sum of a series: this is the subject of a 30-page digression, starting with Taylor series, in which the actual definition of convergence only appears on about the twentieth page.

The general style is user-friendly and interactive. The simpler steps of proofs are often set as exercises, so that the reader is recruited as a partner in the process of discovery. Solutions to the exercises are provided at the end. There are also a large number of numerical exercises and illustrations, based on MAPLE or MATHEMATICA. Readers are repeatedly assured that they can leave out proofs if they find them hard. The text is liberally interspersed with biographies, historical information and quotations. A good deal of space is given to informal discussion of famous conjectures and open problems. The claim that nothing is required beyond basic calculus is partly sustained by a rather free policy of importing unproved results, both from analysis and from number theory, when wanted.

Despite all this, the range of topics attempted is distinctly ambitious for a reader who is really starting from the position described. In the reviewer's opinion, any such reader is likely to find large parts of the book extremely challenging!

The early material covers topics like perfect numbers, convolutions, the Möbius function and the estimates for the partial sums of the divisor function τ and the sum-of-divisors function σ. Readers of the type postulated should get this far without much trouble. With strong enough motivation, they may also succeed with the next few topics: the Chebyshev estimates, the Bernoulli numbers and the Euler product. Here there is a typical example of the consequences of the author's self-imposed task. The cotangent series is needed for the evaluation of $\zeta(2n)$. Fourier series are not allowed, so a direct proof is given (to the reviewer's mind, this is one of the author's less satisfactory proofs; it is based on a rather casual use of the O-notation, and would seem to require some non-trivial further work for its completion).

After that, the level rises steeply. Mertens's theorems are presented in some detail, with the application to the maximal order of σ. Three chapters then develop the theory of the gamma and zeta functions. The Bernoulli numbers are used to define the extended zeta function. The functional equation is derived using the Jacobi theta function (a fairly complicated 'elementary' calculation, attributed to Polya, replaces the customary use of the Poisson summation formula). Hadamard's product is then simply assumed, and Von Mangoldt's explicit formula for $\psi(x)$ is 'derived', taking injectivity of the Mellin transform as a further assumption. A very rough and ready argument is then given to indicate why one might believe Riemann's explicit formula for $\Pi(x)$. It is then stated that the prime number theorem is a consequence of Riemann's formula, and that the proof can be found in other books; no reference is made to the possible existence of simpler proofs not based on these formulae.

The remaining three chapters are devoted to topics related to Diophantine equations: first, Pell's equation and quadratic forms with positive discriminant, with quadratic reciprocity assumed—the connection with Dirichlet L-functions is illustrated by several particular cases (the reader is informed that this is a 'miracle'!). Next, a largely descriptive introduction to elliptic curves is given, culminating in the conjecture of Birch and Swinnerton-Dyer. The final chapter, by contrast, gives a fairly detailed account of the analytic class number formula for quadratic forms with negative discriminants.

In summary, the book is a well presented and stimulating informal introduction to a wide range of topics, but it is hardly realistic to claim that most of it is accessible to students who have only done basic calculus. Among mature mathematicians wanting to extend their knowledge of the subject, it will appeal to those who are more interested in 'flavour' and background than in full proofs.

G. J. O. JAMESON

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Pisier, G. *Introduction to operator space theory* (Cambridge University Press, 2003), 0 521 81165 1 (paperback), £40.

The starting point of operator space theory is the relatively recent viewpoint that operator spaces, that is, linear spaces whose elements are operators on a Hilbert space, should not in general be studied within the category of Banach spaces. Although they are Banach spaces, Banach-space methods often do not take one very far in the analysis of such spaces. This is witnessed, for example, by the lack of overlap one sees between Banach-space and C^*-algebra literature. A 'better' category for the study of subspaces of C^*-algebras began to emerge around 1970, in the work of Arveson [**1**]. Namely, in addition to the norm on such a space X, we consider for each positive integer n the natural norm $\|\cdot\|_n$ on the space of $n\times n$ matrices $M_n(X)$ inherited from the C^*-algebra $M_n(A)$. The 'morphisms' then consist of the *completely bounded maps*, namely the linear maps T for which the norm of the map $[x_{ij}]\mapsto[T(x_{ij})]$ on $M_n(X)$, is bounded independently of n. The least upper bound of this sequence of norms, is called the *completely bounded norm* of T. The subject started to become a field in its own right with the abstract characterization of operator spaces, in terms of the just-mentioned matrix norms, done in Ruan's thesis under the direction of Effros. They had begun the creation of a variant of functional analysis appropriate to the new category, a variant that bridged (a large part of) the gap between Banach space theory and operator algebras.

As evidenced by the length of time it took in coming, the importance of the matrix norms mentioned above, and of the completely bounded maps, is not obvious from the outside. The first hint of it comes by reflecting on the basic 'tensor product' of operator algebras. Suppose that X and Y are C^*-algebras (respectively, operator spaces) contained in $B(H)$ and $B(K)$, respectively. Here H, K are Hilbert spaces. If $H\otimes K$ is the Hilbert-space tensor product of H and K, then the symbol $x\otimes y$, for $x\in X$ and $y\in Y$, corresponds canonically to an operator on $H\otimes K$. The (closure of the) span of these operators in $B(H\otimes K)$ is called the spatial (or minimal) tensor product of X and Y. It is again a C^*-algebra (respectively, operator space).

This construction is fundamental for operator algebras, and for their applications in quantum physics or non-commutative geometry. However, Banach space techniques fail miserably with this new object. For example, the 'tensor product' $S \otimes T$ of bounded maps S and T between C^*-algebras (respectively, operator spaces) is usually unbounded as a map between the spatial tensor products. On the other hand, $S \otimes T$ is *completely bounded* (and hence bounded) if S and T are completely bounded. Thus it is the maps tied to the *matrix norm structure* that behave as they should. The drawback of the operator space approach, for those unused to this subject, is that the family $\{\|\cdot\|_n\}_{n\geqslant 2}$ of norms seems to be a heavy additional burden. In fact, much of the time the matrix norms are no burden at all: often if a result can be proved for the 'first level' ($n = 1$), then the higher levels will follow in a routine way.

Perhaps the next conceptual adjustment which a newcomer to operator spaces will need to make is with the notion of duality. The dual $B(X, \mathbb{C})$ of a Banach space is again a Banach space, and, as one would wish, the dual $X^* = B(X, \mathbb{C})$ of an operator space is again an operator space. That is, there is a canonical sequence of norms on $M_n(X^*)$, for $n \geqslant 2$, with respect to which X^* is again an operator space. Indeed, with respect to the canonical identification of a matrix x in $M_n(X^*)$ with an M_n-valued map T on X, the norm of x equals the 'completely bounded norm' of T. One may then proceed to find the right operator space analogues of the basic duality theorems in functional analysis. This is an example of 'non-commutative functional analysis': maps into the scalar field $\mathbb{C}$ have been replaced by maps into the non-commutative space M_n (or $B(H)$, if one prefers to replace the sequence $\{M_n\}$ by one space). Every Banach space may be viewed as an operator space, and indeed, in a sense that one may make precise, Banach spaces are the 'commutative operator spaces'.

We have reviewed very quickly some of the 'first things' of operator space theory. Of course, the proof of the pudding is whether the ensuing theory really has substantial applications to operator algebra theory, and to 'non-commutative mathematics'. One may make the case, for example, that C^*-algebras are rather rigid objects, and that one obtains much more freedom and less rigidity by working with their linear subspaces. These subspaces often capture the important structure.

Although in the late 1980s Pisier was one of the leading figures in Banach space theory, there is evident in his work from that decade a strong drive to unify C^*-algebra and Banach space theory. By the end of that decade, Pisier began his part of the mammoth construction project to which this book testifies. Today, operator space theory is a highly regarded field of mathematics. The book under review shows why this is so, exhibiting from cover to cover the aforementioned 'proof of the pudding'. Pisier gives us a delightful and personal tour, often taking us into beautiful new places which hitherto only the builders have seen. As one might expect, there are many striking 'importations' of ideas from Banach space theory. Very commendably, the author avoids the danger of 'imports' that are not connected to the central concerns of operator algebraists. The thrust of the book is not, however, on 'Banach space imports'. Indeed, the main emphasis is on connections with important topics in C^*-algebra theory.

The lengthy introduction to the book consists of a short discussion of the early history of the subject, and some of the main ideas, followed by a detailed summary of each of the chapters of the book. The book is divided into three parts. Part I constitutes about half of the text, and concentrates on general operator space theory. In addition to the basic results in the field, there are many advanced results here, together with an excellent collection and exploration of important examples of operator spaces. Although this part is titled 'Introduction to operator spaces', the book begins early on to display an extraordinary diversity, breadth and strength. For example, the theorems are very often improvements on the existing literature; and the examples are compelling and rich, encompassing, for example, the author's non-commutative L^p theory (explored much more fully in an earlier text of his, [**6**]) and connections to classical and non-commutative probability, his fundamental 'operator Hilbert space', discrete groups and their C^*-algebras, Hankel operators, fermionic analysis, free probability and random matrix models, and much more.

Part II is devoted to deep properties of C^*-algebras and their tensor products, and related properties of operator spaces. Building on some extraordinary work of Kirchberg, and of several other of the main protagonists in operator algebras, the author exposes many of the most important results in operator-space theory. In turn, he uses these to give several amazing applications to C^*-algebra theory. The techniques used are extraordinarily impressive and inspirational. This may be seen clearly, for example, in the presentation of Junge and Pisier's solution to the long-open question of the unicity of the C^*-algebra tensor norm on $B(H) \otimes B(H)$. Some estimates of a certain constant defined in terms of sums of unitaries, needed in the course of the proof, turn out to be derivable from deep number theoretic results due to Lubotzky, Phillips and Sarnak. An alternative approach in Chapter 20 gives the precise value of the constant, using the incredible recent applications due to Haagerup and Thorbjørnsen [**4**] of free probability to C^*-algebras.

Part III of the text is the shortest, and centres on Pisier's remarkable work on similarity problems. The final chapter, for example, presents his solution to the famous Sz.-Nagy–Halmos problem: whether every polynomially bounded operator on a Hilbert space is similar to a contraction. Other chapters have their genesis in his deep work on the Kadison similarity conjecture. Contained here too are many novel results and tools for non-self-adjoint operator algebras, which the reader will not find elsewhere in the literature.

Finally, the book ends with 40 pages of worked solutions and hints to the exercises, an invaluable service in its own right.

The tone of the book is quite informal, friendly and inviting. Even to experts in the field, a large proportion of the results, and certainly of the proofs, will be new and stimulating. Also, the author, in his unique and extremely generous style, very often showcases the work or remarks of others—particularly of his brilliant former students and collaborators. Due only partly to this, there are literally thousands of wonderful results and insights in the text which the reader will not find elsewhere. The book covers an incredible amount of ground, and makes use of some of the most exciting recent work in modern analysis. Although this may in a few places be at the (necessary) expense of accessibility to a beginning graduate student, the reader is never made to feel intimidated. In most of the book, the author goes to great lengths to explain nuances, describe in other words what is going on, relate how a result or formula ought to be viewed, and so on. Part of the reason for this may be the fact that this text grew up from notes from a series of advanced courses taught by the author. A huge reference list is provided for those needing further details, or for those who wish to chase down an intriguing direction. The reference list displays the broad culture and sophistication of the author.

Operator spaces, like Banach spaces, naturally lead in many diverse directions. Thus, although there are now several books available which are centred around the subject of operator spaces (see, for example, [**2**, **3**, **5**], of which [**3**] is the closest in subject matter), Pisier's text and the others are complementary, in various ways. Also, the book under review is both the longest (479 pages) and the least expensive of the available texts. It is a magnificent book: an enormous treasure trove, and a work of love and care by one of the great analysts of our time. All students and researchers in functional analysis should have a copy. Anybody planning to work in operator space theory will need to be thoroughly immersed in it.

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