AN ANALYTIC SOLUTION TO THE BUSEMANN-PETTY PROBLEM

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Abstract. We present a complete solution to the Busemann-Petty problem on sections of convex bodies using methods of Fourier analysis.

Résumé. Nous présentons une solution complète du problème de Busemann-Petty sur les sections des corps convexes en utilisant des méthodes d'analyse de Fourier.

Version francais abrégée.

Le problème de Busemann-Petty ([BP], 1956) formule la conjecture suivante: soient K et L des corps convexes symétriques par rapport à l'origine dans $\mathbb{R}^n$ tels que $vol_{n-1}(K\cap \xi^\perp) \leq vol_{n-1}(L\cap \xi^\perp)$ pour tout ξ dans la sphère unité Ω de $\mathbb{R}^n$, où $\xi^\perp = \{x \in \mathbb{R}^n : (x,\xi)=0\}$ est l' hyperplan perpendiculaire à ξ, et vol_{n-1} est le volume $(n-1)$-dimensionnel. En résulte-t-il que $vol_n(K) \leq vol_n(L)$? La réponse, négative si $n \geq 5$, et positive si $n \leq 4$, résulte du travail de nombreux auteurs (voir plus bas pour le détail). Cette note est l'annonce d'une solution unifiée au problème, obtenue dans [GKS].

Notre solution s'articule autour des trois points suivants. Tout d'abord, on utilise le lien entre les corps d'intersection et le problème de Busemann-Petty qui résulte du travail de Lutwak [Lu]:

Théorème A. *Le problème de Busemann-Petty a un réponse positive dans $\mathbb{R}^n$ si et seulement si tout corps convexe symétrique par rapport à l'origine dans $\mathbb{R}^n$ est un corps d'intersection.*

Ensuite, un lien entre les corps d'intersection et la transformée de Fourier est exhibé dans [K3,K5]:

Théorème B. *Un corps étoilé K symétrique par rapport à l'origine dans $\mathbb{R}^n$ est un corps d'intersection si et seulement si la transformée de Fourier de sa fonction radiale ρ_K est une mesure positive sur $\mathbb{R}^n$.*

Enfin, la formule suivante relie les dérivées des fonctions des sections parallèles aux transformées de Fourier des puissances de la fonction radiale:

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Théorème 1. *Soit K un corps étoilé symétrique par rapport à l'origine dans $\mathbb{R}^n$ dont le bord est C^∞, et soit $k \in \mathbb{N}$, $k \neq n-1$. Supposons que $\xi \in S^{n-1}$, et soit $A_\xi(t) = \mathrm{vol}_{n-1}\big(K \cap (\xi^\perp + t\xi)\big)$ la fonction de section parallèle de K correspondante.*

(a) *Si k est pair alors*

$$(\rho_K^{n-k-1})^\wedge(\xi) = (-1)^{k/2}\pi(n-k-1)A_\xi^{(k)}(0);$$

(b) *Si k est impair alors*

$$(\rho_K^{n-k-1})^\wedge(\xi) = c_k \int_0^\infty \frac{A_\xi(z) - A_\xi(0) - A''_\xi(0)\frac{z^2}{2} - \cdots - A_\xi^{(k-1)}(0)\frac{z^{k-1}}{(k-1)!}}{z^{k+1}}\,dz,$$

où $c_k = (-1)^{(k+1)/2}\,2(n-1-k)k!$, $A_\xi^{(k)}(0)$ est la dérivée d'ordre k de la fonction $z \to A_\xi(z)$ en zéro, et $(\rho_K^{n-k-1})^\wedge$ est la transformée de Fourier de ρ_K^{n-k-1} au sens des distributions.

Des cas particuliers $n = 3, k = 1$ et $n = 4, k = 2$ on déduit du théorème de Brunn qu'en dimension 3 et 4 les fonctions radiales des corps convexes symétriques par rapport à l'origine sont définies positives, et donc les théorèmes A et B entraînent une réponse positive au problème de Busemann-Petty pour ces dimensions. D'un autre coté, en utilisant le théorème 1 avec $n = 5, k = 3$, on construit facilement un exemple de corps convexe symétrique par rapport à l'origine dans $\mathbb{R}^5$ dont la fonction radiale n'est pas définie positive. On déduit ainsi une réponse négative au problème de Busemann-Petty pour $n \geq 5$.

Le Théorème 1 est une conséquence immédiate de la formule suivante faisant intervenir les dérivées fractionnaires $A_\xi^{(q)}$ d'ordre q de la fonction A_ξ:

Théorème 2. *Soit K un corps étoilé symétrique par rapport à l'origine dans $\mathbb{R}^n$ dont le bord est C^∞. Supposons que $\xi \in S^{n-1}$, et soit A_ξ la fonction de section parallèle de K correspondante. Si $q \in \mathbb{C}$, $Re q > 1$, $q \neq n-1$, on a:*

$$A_\xi^{(q)}(0) = \frac{\cos\frac{q\pi}{2}}{\pi(n-q-1)}(\rho_K^{n-q-1})^\wedge(\xi).$$

English version.

The 1956 Busemann-Petty problem (see [BP]) asks the following question. Suppose that K and L are origin-symmetric convex bodies in $\mathbb{R}^n$ such that

$$\mathrm{vol}_{n-1}(K \cap H) \leq \mathrm{vol}_{n-1}(L \cap H)$$

for every hyperplane H containing the origin; does it follow that

$$\mathrm{vol}_n(K) \leq \mathrm{vol}_n(L)?$$

In this note we show that the answer is negative if $n \geq 5$ and affirmative if $n \leq 4$. The problem has a long and dramatic history, and the journey to the final answer involved many people. A negative answer to the problem for $n \geq 5$ was established by Gardner [Ga1] and Papadimitrakis [Pa], after earlier negative answers of Larman and Rogers [LR] (for $n \geq 12$), Ball [Ba] ($n \geq 10$), and Giannopoulos [Gi] and Bourgain [Bo] ($n \geq 7$). Gardner [Ga2] proved that the answer to the Busemann-Petty problem is affirmative when $n = 3$. A negative answer when $n = 4$ was claimed in [Z2], but three years later, in May '97, the main argument of [Z2] was proved to be wrong in [K3, K4]. After that, in June '97, Zhang [Z3] showed that the answer is affirmative when $n = 4$, and a unified solution to the problem was given in August '97 in [GKS]. This note is an announcement of the results from [GKS].

As in the papers [Ga1, Ga2, K3, K4, Z3], a crucial role in our solution belongs to the connection between intersection bodies and the Busemann-Petty problem established by Lutwak [Lu] in 1988 and slightly modified in [Ga2] and [Z1,Z2]:

Theorem A. *The Busemann-Petty problem has an affirmative answer in $\mathbb{R}^n$ if and only if every origin-symmetric convex body in $\mathbb{R}^n$ is an intersection body.*

An origin-symmetric star body K in $\mathbb{R}^n$ is said to be an *intersection body* if there exists a finite (non-negative) Borel measure μ on the $(n-1)$-dimensional unit sphere S^{n-1} so that the radial function ρ_K of K equals the spherical Radon transform of μ. Here $\rho_K(x) = \max\{a > 0 : \ ax \in K\}, \ x \in \mathbb{R}^n$.

We also use the following connection between intersection bodies and the Fourier transform found in [K3,K5]:

Theorem B. *An origin-symmetric star body K in $\mathbb{R}^n$ is an intersection body if and only if the radial function ρ_K is a positive definite distribution on $\mathbb{R}^n$.*

Here we consider ρ_K as a tempered distribution from the class $\mathcal{S}'(\mathbb{R}^n)$. Recall that by L. Schwartz's generalization of Bochner's theorem, a distribution $f \in \mathcal{S}'(\mathbb{R}^n)$ is positive definite if and only if its Fourier transform $\hat{f}$ is a positive distribution in the sense that $\langle \hat{f}, \phi \rangle \geq 0$ for every non-negative test function $\phi \in \mathcal{S}(\mathbb{R}^n)$ (see [GV, p. 152]).

Note that in the formulation of the Busemann-Petty problem we can assume without loss of generality that the bodies K and L are infinitely smooth in the sense that the restrictions of their radial functions to the unit sphere belong to the space $C^\infty(S^{n-1})$. In fact, we can approximate the body K in the Hausdorff metric by smaller infinitely smooth bodies and L by larger infinitely smooth bodies.

For every $\xi \in S^{n-1}$ we define the *parallel section function*

$$A_\xi(t) = \mathrm{vol}_{n-1}\big(K \cap (\xi^\perp + t\xi)\big)$$

as a function of $t \in \mathbb{R}$. (The function A_ξ is sometimes called the $(n-1)$-dimensional X-ray of K orthogonal to ξ; see [Ga3, Chapter 2].) It is easily seen that if K is infinitely smooth then A_ξ is infinitely differentiable in a neighborhood of zero. Our solution to the Busemann-Petty problem is based on the following formula relating derivatives of parallel section functions to Fourier transforms of powers of radial functions:

Theorem 1. *Let K be an origin-symmetric infinitely smooth star body in $\mathbb{R}^n$, and let $k \in \mathbb{N} \cup \{0\}$, $k \neq n-1$. Suppose that $\xi \in S^{n-1}$, and let A_ξ be the corresponding parallel section function of K.*

(a) *If k is even, then*

$$(\rho_K^{n-k-1})^\wedge(\xi) = (-1)^{k/2}\pi(n-k-1)A_\xi^{(k)}(0);$$

(b) *if k is odd, then*

$$(\rho_K^{n-k-1})^\wedge(\xi) = c_k \int_0^\infty \frac{A_\xi(z) - A_\xi(0) - A_\xi''(0)\frac{z^2}{2} - \cdots - A_\xi^{(k-1)}(0)\frac{z^{k-1}}{(k-1)!}}{z^{k+1}}\, dz,$$

where $c_k = (-1)^{(k+1)/2}\, 2(n-1-k)k!$, $A_\xi^{(k)}(0)$ is the derivative of order k of the function $z \to A_\xi(z)$ at zero, and $(\rho_K^{n-k-1})^\wedge$ is the Fourier transform of ρ_K^{n-k-1} in the sense of distributions.

Note that the case $n = 3,\ k = 1$ of Theorem 1 gives the formula for the inverse spherical Radon transform of the radial function from [Ga2], and in the case $n = 4,\ k = 2$ we obtain the corresponding formula from [Z3]. Also, putting $k = 0$ in Theorem 1, we obtain [K2, Lemma 1], which essentially proves Theorem B.

Let us show how the solution to the problem follows from Theorem 1. By Brunn's theorem, for every $\xi \in S^{n-1}$ the parallel section function A_ξ of an origin-symmetric convex body has its maximum at zero (see [S, Theorem 6.1.1]). Putting $n = 4,\ k = 2$ in part (a) of Theorem 1, we conclude that the radial function of every origin-symmetric convex body in $\mathbb{R}^4$ is a positive definite distribution, and an affirmative answer to the Busemann-Petty problem in dimension 4 follows from Theorems A and B. An affirmative answer in dimension 3 follows from part (b) of Theorem 1 with $n = 3,\ k = 1$ in the same way. Finally, define an origin-symmetric convex body K in $\mathbb{R}^5$ by

$$K = \left\{ (x_1, .., x_5) \in \mathbb{R}^5 : x_5 \in [-a_\varepsilon, a_\varepsilon] \text{ and } \left(\sum_{i=1}^4 x_i^2\right)^{1/2} \leq f_\varepsilon(|x_5|) \right\},$$

where $f_\varepsilon(x) = (1 - x^2 - \varepsilon x^4)^{1/4},\ \varepsilon \in (0,1)$, and $a_\varepsilon > 0$ is such that $f_\varepsilon(a_\varepsilon) = 0$ and $1 - x^2 - \varepsilon x^4 > 0$ on $(0, a_\varepsilon)$. Applying Theorem 1 with $n = 5,\ k = 3$ to the body K we see that the radial function of K is not positive definite, and again by Theorems A and B, it follows that K is not an intersection body. Since the intersection of an intersection body with a hyperplane H containing the origin is also an intersection body in H, there are origin-symmetric convex bodies in $\mathbb{R}^n$, $n \geq 5$, that are not intersection bodies. Therefore the answer to the Busemann-Petty problem is negative in dimensions 5 or greater.

Theorem 1 is a direct consequence of the following theorem.

Theorem 2. *Let K be an origin-symmetric infinitely smooth star body in $\mathbb{R}^n$ with Minkowski functional $\|\cdot\|$. Suppose that $\xi \in S^{n-1}$, and let A_ξ be the corresponding parallel section function of K. For $q \in \mathbb{C}$ with $\operatorname{Re} q > -1$, $q \neq n-1$ we have*

$$A_\xi^{(q)}(0) = \frac{\cos\frac{q\pi}{2}}{\pi(n-q-1)}(\|x\|^{-n+q+1})^\wedge(\xi).$$

Here $A_\xi^{(q)}(0)$ is the fractional derivative of order q at zero, defined by

$$A_\xi^{(q)}(0) = \frac{1}{\Gamma(-q)}\int_0^\infty t^{-q-1}A_\xi(t)\,dt$$

if $-1 < \operatorname{Re} q < 0$, and by

$$A_\xi^{(q)}(0) = \frac{1}{\Gamma(-q)}\int_0^\infty t^{-q-1}\left(A_\xi(t) - \sum_{j=0}^{k-1}\frac{t^{2j}}{(2j)!}A_\xi^{(2j)}(0)\right)dt,$$

whenever $2k-2 < \operatorname{Re} q < 2k$, $k \in \mathbb{N}$. The function $q \mapsto A_\xi^{(q)}(0)$, $q \in \mathbb{C}$, can then be extended to an analytic function on all of $\mathbb{C}$. Note that the function A_ξ is even and that if q is an even integer the fractional derivative of order q coincides with the ordinary derivative of the same order. We refer the reader to [GS, pp. 48–56] for details.

We need the following simple fact proved in [K1].

Lemma. *For every even test function $\varphi \in \mathcal{S}$, $\xi \in S^{n-1}$, and $-1 < q < 0$ we have*

$$\int_{\mathbb{R}^n}|\langle\xi,x\rangle|^{-q-1}\varphi(x)\,dx = \frac{-1}{2\Gamma(1+q)\sin\frac{q\pi}{2}}\int_{-\infty}^\infty |t|^q\hat{\varphi}(t\xi)\,dt.$$

To prove Theorem 2, suppose first that $-1 < q < 0$. The function $A_\xi(z) = \int_{\langle x,\xi\rangle = z}\chi(\|x\|)\,dx$ is even. Applying Fubini's theorem and passing to spherical coordinates, we get

$$\begin{aligned} A_\xi^{(q)}(0) &= \frac{1}{2\Gamma(-q)}\int_{-\infty}^\infty |z|^{-q-1}A_\xi(z)\,dz \\ &= \frac{1}{2\Gamma(-q)}\int_{\mathbb{R}^n}|\langle x,\xi\rangle|^{-q-1}\chi(\|x\|)\,dx \\ &= \frac{1}{2\Gamma(-q)}\int_{S^{n-1}}|\langle\theta,\xi\rangle|^{-q-1}\int_0^\infty r^{n-q-2}\chi(r\|\theta\|)\,dr\,d\theta \\ &= \frac{1}{2(n-q-1)\Gamma(-q)}\int_{S^{n-1}}|\langle\theta,\xi\rangle|^{-q-1}\|\theta\|^{-n+q+1}\,d\theta. \end{aligned}$$

We now consider $A_\xi^{(q)}(0)$ as a function of $\xi \in \mathbb{R}^n \setminus \{0\}$. By the lemma, for every even test function $\varphi \in \mathcal{S}$ we have

$$(1)\quad \langle A_\xi^{(q)}(0),\varphi(\xi)\rangle = \frac{1}{2(n-q-1)\Gamma(-q)}\int_{S^{n-1}}\|\theta\|^{-n+q+1}\,d\theta\int_{\mathbb{R}^n}|\langle\theta,\xi\rangle|^{-q-1}\varphi(\xi)\,d\xi$$

$$= \frac{-1}{4(n-q-1)\Gamma(-q)\Gamma(q+1)\sin\frac{q\pi}{2}} \int_{S^{n-1}} \|\theta\|^{-n+q+1} \int_{-\infty}^{\infty} |t|^q \widehat{\varphi}(t\theta)\, dt\, d\theta$$

$$= \frac{\cos\frac{q\pi}{2}}{\pi(n-q-1)} \left\langle (\|x\|^{-n+q+1})^\wedge(\xi), \varphi(\xi) \right\rangle,$$

where the last equation follows from the property $\Gamma(-q)\Gamma(q+1) = -\pi/\sin(q\pi)$ of the Γ-function and the simple calculation

$$\langle (\|x\|^{-n+q+1})^\wedge(\xi), \varphi(\xi)\rangle = \int_{\mathbb{R}^n} \|x\|^{-n+q+1} \widehat{\varphi}(x)\, dx = \int_{S^{n-1}} \|\theta\|^{-n+q+1} \int_0^\infty t^q \widehat{\varphi}(t\theta)\, dt\, d\theta$$

(note that the function $\|x\|^{-n+q+1}$ is locally integrable on $\mathbb{R}^n$ because $-1 < q < 0$).

Since (1) holds for every even test function φ, Theorem 2 is proved when $-1 < q < 0$.

In order to prove the theorem for other values of q, we first observe that $(\|x\|^{-n+q+1})^\wedge$ is an analytic distribution (with respect to q) on $\{q \in \mathbb{C} : \operatorname{Re} q > -1\}$. It follows that for every even test function $\varphi \in \mathcal{S}$, the functions $q \mapsto \langle A_\xi^q(0), \varphi\rangle$ and

$$q \mapsto \left\langle \frac{\cos\frac{q\pi}{2}}{\pi(n-q-1)} (\|x\|^{-n+q+1})^\wedge(\xi), \varphi \right\rangle$$

are analytic on the connected region $\{q \in \mathbb{C}\colon \operatorname{Re} q > -1,\ q \neq n-1\}$ (for details of analytic continuation in such situations, see [GS]). These functions coincide on the interval $-1 < q < 0$, so they coincide on $\{q \in \mathbb{C}\colon \operatorname{Re} q > -1,\ q \neq n-1\}$. Since φ is an arbitrary even test function, we have proved Theorem 2.

Part (a) of Theorem 1 immediately follows from Theorem 2 and the fact that fractional derivatives coincide with ordinary derivatives. To prove part (b) of Theorem 1, divide both sides of the formula in Theorem 2 by $\cos(q\pi/2)$, and compute the limit as $q \to k$, where k is an odd integer.

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