# Notes on descriptive set theory and applications to Banach spaces

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CHAPTER 1

# Notation

Let $A$ be a non empty set. By $A^{\mathbb{N}}$ we denote the infinite sequences and by $A^{<\mathbb{N}}$ the finite sequences in $A$. If $s = (s_1, s_2, \dots s_m) \in A^{<\mathbb{N}}$, $m$ is called the *length of* $s$ and denoted by $|s|$. If $s \in A^{\mathbb{N}}$ we write $|s| = \infty$. If $m, n \in \mathbb{N}$ and $s = (s_1, \dots s_n) \in A^{<\mathbb{N}}$, with $n = |s| \geq m$ or $s = (s_i) \in A^{\mathbb{N}}$ we set $s|_m = (s_1, s_2, \dots s_m)$.

Let $s = (s_1, s_2, \dots s_m) \in A^{<\mathbb{N}}$ and $t = (t_1, t_2, \dots t_n) \in A^{<\mathbb{N}}$ or $t = (t_i) \in A^{\mathbb{N}}$. We say that $t$ *is an extension of* $s$ and write $t \succ s$ if $|t| > |s|$, and $t|_{|s|} = s$. We write $t \succeq s$ if $t \succ s$ or $t = s$. The *concatenation of* $s$ *and* $t$ is the element $(s, t) = (s_1, s_2, \dots s_m, t_1, \dots t_n) \in A^{<\mathbb{N}}$ or $(s, t) = (s_1, s_2, \dots s_m, t_1, t_2, \dots) \in A^{\mathbb{N}}$, respectively.

For a set $A$ and a cardinal number $\alpha$ $[A]^{\alpha}$, $[A]^{\leq\alpha}$, $[A]^{<\alpha}$ denotes the set of all subsets of cardinality $\alpha$, at most $\alpha$ and strictly smaller than $\alpha$, respectively and $[A]$ denotes the set of all subsets of $A$.

**Convention:** We identify the elements $[\mathbb{N}]^{<\mathbb{N}}$ and $[\mathbb{N}]^{\mathbb{N}}$, with increasing sequences in $\mathbb{N}^{<\mathbb{N}}$ and $\mathbb{N}^{\mathbb{N}}$ in the obvious way.

If $\mathcal{A} \subset [A]$, $\mathcal{A}_\sigma$, $\mathcal{A}_+$, and $\mathcal{A}_\delta$ denotes the set of all countable unions of elements of $\mathcal{A}$, countable disjoint unions of elements of $\mathcal{A}$, and all countable intersections of elements of $\mathcal{A}$, respectively.

Let $(X, \mathcal{T})$ be a topological space. For $A \subset X$ $\overline{A}^{\mathcal{T}}$, or $\overline{A}$, if there is no confusion, denotes the the closed hull of $A \subset X$ and $A^{o,\mathcal{T}}$, respectively $A^\circ$, denote the open kernel of $A$. Following general convention we write $\mathcal{G}_\delta$ instead of $\mathcal{T}_\delta = \{\bigcap U_n : U_n \in \mathcal{T}$ for $n \in \mathbb{N}\}$ and $\mathcal{F}_\sigma$ for $\{F \subset X : F$ closed$\}_\sigma$.

Let $(X, d)$ be a metric space. For $\varepsilon > 0$ and for any non empty $A \subset X$ we write

$$A_\varepsilon = \{x : d(x, A) < \varepsilon\}, \text{ where } d(x, A) = \inf_{y \in A} d(x, y).$$

If $A = \{x\}$, $x \in X$, we also write $B_\varepsilon(x) = \{x\}_\varepsilon = \{y \in X : d(x, y) < \varepsilon\}$.

We denote the topology generated by $d$ by $\mathcal{T}_d$, i.e.

$$\mathcal{T}_d = \big\{U : U \subset X, \forall x \in U\, \exists \varepsilon > 0 \quad B_\varepsilon(x) \subset U\big\}.$$

We will only use *countable ordinals* which can be easily introduced as follows using *Zorn's Wellordering Lemma*: Take an uncountable set, say the real number line $\mathbb{R}$ and well order $\mathbb{R}$. We denote this ordering by $\prec$. We then define

$$0_\prec = \min \mathbb{R} \text{with respect to } \prec \; \omega_0 = \min\big\{r : \{s : s \prec r\} \text{ is finite}\big\}$$
$$\omega_1 = \min\big\{r : \{s : s \prec r\} \text{ is countable}\big\}$$

A countable ordinal is then the element of the set $[0, \omega_1) = \{\alpha : \alpha \prec \omega_1\}$. A more detailed introduction to ordianals and a proof of Zorn's Lemma using the *Axiom of Choice can be found in the appendix*

CHAPTER 2

# Polish spaces

## 1. Introduction

A topological space $(X, \mathcal{T})$ is called *completely metrizable* if the topology $\mathcal{T}$ is generated by a metric for which $X$ is complete, for which every Cauchy sequence converges. A *Polish space* is a separable completely metrizable space. Note that a Polish space has a countable basis.

EXAMPLES 1.1. The following are Polish spaces.

(i) Every countable set with its discrete topology.
(ii) $\mathbb{R}$, $\mathbb{R}^n$, $\mathbb{C}$, $\mathbb{C}^n$, $[0,1]$ with their usual topologies.
(iii) Closed subspaces of Polish spaces.
(iv) Countable products of Polish spaces. Indeed, assume that for $i \in \mathbb{N}$, $d_i$ is a metric which turns $X_i$ into a separable, complete metric space. For $\overline{x} = (x_i)$ and $\overline{y} = (y_i)$ in $\prod X_i$ define

$$d(\overline{x}, \overline{y}) = \sum_{i=1}^{\infty} 2^{-i} \max\big(1, d_i(x_i, y_i)\big).$$

Then $d(\cdot, \cdot)$ turns $\prod X_i$ into a complete metric space. If $D_i = \{\xi(i,j) : j \in \mathbb{N}\} \subset X_i$ is dense in $X_i$, then

$$D = \big\{(\xi_i)_{i\in\mathbb{N}} : \xi_i \in D_i \text{ and } \#\{i \in \mathbb{N} : \xi_i \neq \xi(i,1)\} < \infty\big\}$$

is countable and dense in $(X, d)$. Important examples of such products are:
- $\Delta = \{0,1\}^{\mathbb{N}}$ *Cantor space*
- $\mathbb{H} = [0,1]^{\mathbb{N}}$ *Hilbert cube*
- $\mathcal{N} = \mathbb{N}^{\mathbb{N}}$

(v) Countable sums of Polish spaces.

THEOREM 1.2. [Alexandrov] *Every $\mathcal{G}_\delta$ set $G$ in a completely metrizable, or Polish, space $X$ is completely metrizable, or Polish, respectively.*

PROOF. Let $d$ be a complete metric generating the topology on $X$. Assume first that $G \subset X$ is open, then

$$f : G \to X \times \mathbb{R}, \quad x \mapsto \Big(x, \frac{1}{d(x, X \setminus G)}\Big),$$

is injective, continuous, $f^{-1} : f(G) \to X$ is continuous, and $f(G)$ is closed in $X \times \mathbb{R}$ (indeed, if $f(x_n)$ converges, in $X \times R$, to $(x, r)$, then $\lim_{n\to\infty} x_n = x$ in $X$, thus

$\lim_{n\to\infty} d(x_n, X\setminus G) = d(x, X\setminus G)$ and since $\lim_{n\to\infty} 1/d(x_n, X\setminus G) = r$ it follows that $d(x, X\setminus G) > 0$ and, thus, $x \in G$).

So, we deduce that $G$ is homeomorphic to a closed subset of $X \times R$, and thus completely metrizable, respectively Polish.

In the general case, we write $G$ as $G = \bigcap G_n$, $G_n \subset X$ open and consider the map

$$F : G \to X \times \mathbb{R}^{\mathbb{N}}, \quad x \mapsto \Big(x, \frac{1}{d(x, X\setminus G_1)}, \frac{1}{d(x, X\setminus G_2)}, \dots\Big).$$

We can argue as before and deduce that $f$ embeds $G$ onto a closed subspace of $X \times \mathbb{R}^{\mathbb{N}}$. ☐

Proposition 1.3. *Let $f : A \to Z$ be a continuous map from a subset $A$ of a metrizable space $W$ into a completely metrizable space $Z$. Then $f$ can be continuously extended to a $\mathcal{G}_\delta$ set containing $A$.*

Proof. Let $d$ be a complete metric generating the topology on $Z$. For $x \in \overline{A}$ define *the oscillation of $f$ at $x$* by

$$O_f(x) = \inf\{d - \text{diam}\big(f(A \cap V)\big) : V \subset X \text{ open and } x \in V\}.$$

Since $f$ is continuous on $A$ it follows that $O_f(x) = 0$, for $x \in A$. Secondly we claim that the set $B = \{x \in W : O_f(x) = 0\}$ is a $\mathcal{G}_\delta$-set by observing that for any $\varepsilon > 0$ the set

$$B_\varepsilon = \{x \in W : O_f(x) < \varepsilon\} = \bigcup \big\{V \cap A : V \text{ open and } d - \text{diam}\big(f(A\cap V)\big) < \varepsilon\big\}$$

is open in $\overline{A}$, and that $\overline{A}$ is a $\mathcal{G}_\delta$ set in $W$.

Finally we want to extend $f$ continuously onto $B$. If $x \in B$ there is a sequence $(x_n) \subset A$ converging to $x$. Since $O_f(x) = 0$, the sequence $f(x_n)$ must be Cauchy, and thus convergent in $Z$. We put $g(x) := \lim_{n\to\infty} f(x_n)$ and we observe that $g(x)$ is well defined. Indeed, if $(\tilde{x}_n)$ is another sequence converging to $x$ we consider the sequence $(z_n)$ with $z_{2n-1} = x_n$ and $z_{2n} = \tilde{x}_n$. Moreover $g$ extends $f$ (by considering constant sequences), and is continuous (using a straightforward diagonalization argument). ☐

Theorem 1.4. [Lauvrentiev]

*Let $X, Y$ be completely metrizable spaces, let $A \subset X$ and $B \subset Y$ and let $f : A \to B$ be a homeomorphism.*

*Then $f$ can be extended to a homeomorphism between two $\mathcal{G}_\delta$-sets containing $A$ and $B$, respectively.*

Proof. By Proposition 1.3 we can choose a $\mathcal{G}_\delta$ set $A' \subset X$, containing $A$ and a $\mathcal{G}_\delta$ set $B' \subset Y$, containing $B$. And extensions $f' : A' \to Y$ of $f$ and $g' : B' \to X$ of $f^{-1}$, respectively. Put

$$H = \text{Graph}(f') = \{(x, y) \in A' \times Y : y = f'(x)\}$$
$$\text{and } K = \text{Graph}(g') = \{(x, y) \in X \times B' : x = g(x)\}$$

and let $A^* = \pi_1(H \cap K)$ and $B^* = \pi_2(H \cap K)$, where $\pi_i : X \times Y$ is the $i$th coordinate projection for $i = 1, 2$.

Since $K$ is closed in $X\times B'$, $K$ is a $\mathcal{G}_\delta$-set in $X\times Y$. Indeed, write $B'=\bigcap U_n$, $U_n\subset Y$ open ($n\in\mathbb{N}$), $K=F\cap X\times B'$ for some $F\subset X\times Y$ and $F=\bigcap V_n$, $U_n\subset X\times Y$ open for $n\in\mathbb{N}$, and thus $K=\bigcap V_n\cap(X\times B')=\bigcap V_n\cap\bigcap(X\times U_n)$. Since $A^*=\{x\in A':(x,f'(x))\in K\}=A^*=(Id,f')^{-1}(K)$, where $(Id,f'):A'\to X\times Y, x\mapsto(x,f'(x))$, is also $\mathcal{G}_\delta$. Similarly it follows that $B^*$ is a $\mathcal{G}_\delta$ set.

Note that

$$K\cap H=\{(x,y)\in A^*\times B^*:y=f'(x)\text{ and }x=g'(y)\}=\text{Graph}(f'|_{A^*})=\text{Graph}(g'|_{A^*}),$$

which implies that $f':A^*\to B^*$ is continuous and bijective, with $f'^{-1}=g$. □

THEOREM 1.5. *Let $X$ be a completely metrizable space and $Y\subset X$ a completely metrizable subspace. Then $Y$ is a $\mathcal{G}_\delta$-set in $X$.*

PROOF. let $f:Y\to Y$ be the identity map. By Theorem 1.4 it can be extended to a homeomorphism $f':A'\to Y$ where $A'\subset X$ is a $\mathcal{G}_\delta$-set in $X$. But there is no other possibility than $A'=Y$, which implies that $Y$ had to be already a $\mathcal{G}_\delta$-set in $X$. □

## 2. Transfer Theorems

DEFINITION 2.1. Let $X$ be a Polish space, $d$ a complete metric which generates the topology on $X$ and let $A$ be a non empty set.

A *Souslin scheme* on $X$ is a family $\{F_s:s\in A^{<\mathbb{N}}\}$ of subsets of $X$ so that $\overline{F}_t\subset F_s$, whenever $t$ is a (strict) extension of $s$, and $\lim_{n\to\infty}d-\text{diam}(F_{\alpha|_n})=0$ for all $\alpha\in A^{\mathbb{N}}$.

A Souslin scheme $\{F_s:s\in A^{<\mathbb{N}}\}$ is called a *Lusin scheme* if $F_s\cap F_t=\emptyset$ whenever $s$ and $t$ are incomparable.

A *Cantor scheme* is a Lusin scheme if $A=\{0,1\}$ and if $F_s\neq\emptyset$ for all $s\in A^{<\mathbb{N}}$

REMARK. The *Souslin operation* is closely related to Souslin schemes and will be introduced in subsection 4.

The following Proposition collects some easy properties.

PROPOSITION 2.2. *Let $\{F_s:s\in A^{<\mathbb{N}}\}$ be a Souslin scheme. We equip $A^{<\mathbb{N}}$ with the product of the discrete topology.*

1) *$D=\{\alpha\in A^{\mathbb{N}}:\forall n\in\mathbb{N}\quad F_{\alpha|_n}\neq\emptyset\}$ is closed. Note that for $\alpha\notin D$ there exists an $n\in\mathbb{N}$ so that $F_{\alpha|_n}=\emptyset$ and thus $\{t\in A^{\mathbb{N}}:t\succeq\alpha|_n\}\subset D^c$.*
2) *For $\alpha\in A^{\mathbb{N}}$, every sequence $(x_n)$ with $x_n\in\overline{F}_{\alpha|_n}$, $n\in\mathbb{N}$, must be a Cauchy sequence, and thus convergent to some $x_\alpha\in X$. It follows that $\bigcap\overline{F}_{\alpha|_n}=\{x_\alpha\}$.*

   *Moreover the map $D\to X$, $f:D\to X$, $\alpha\mapsto x_\alpha$ is continuous. Indeed, choose for $\varepsilon>0$ and $\alpha\in D$ $n\in\mathbb{N}$ large enough so that $d-\text{diam}(\overline{F}_{\alpha|_n})<\varepsilon$ then if $\beta\in\{\gamma\in D:\gamma\succ\alpha|_n\}$ it follows that $x_\beta\in\bigcap_{k\in\mathbb{N}}\overline{F}_{\beta|_k}\subset\overline{F}_{\alpha|_n}$ and thus $d(x_\alpha,x_\beta)<\varepsilon$.*

   *We call $f:D\to X$ the* map associated to $\{F_s:s\in A^{<\mathbb{N}}\}$.
3) *Assume that $F_\emptyset=X$ and that for all $s\in A^{<\mathbb{N}}$ and all $n\geq|s|$ it follows that $F_s=\bigcup_{t\succ s,|t|=n}F_t$. Then the associated map is onto.*

4) *If* $\{F_s : s \in A^{<\mathbb{N}}\}$ *is a Lusin scheme the associated map is injective.*

PROPOSITION 2.3. *Every dense-in-itself Polish space* $X$ *(i.e.* $X$ *does not have any isolated point) contains a homeomorph of* $\Delta$

PROOF. We will show that there is a family $\{U_s : s \in \{0,1\}^{<\mathbb{N}}$ in $X$ consisting of non empty open sets so that, $d-$ $\text{diam}(\overline{U}_s) < 2^{|s|}$, for $s \in \{0,1\}^{<\mathbb{N}}$ $\overline{U}_s \cap \overline{U}_t = \emptyset$ whenever $s, t \in A^{<\mathbb{N}}$ are incomparable and $\overline{U}_t \subset \overline{U}_s$ if $t \succ s$.

But this can be done easily by induction on the length of $s \in A^{<\mathbb{N}}$. Then, our claim follows from aforementioned remark. □

PROPOSITION 2.4. [Cantor Bendixson Theorem]
*Every separable metric space* $X$ *can be written as* $X = Y \cup Z$ *where* $Z$ *is countable,* $Y$ *is closed and has no isolated points.*

PROOF. Let $\{V_n : n \in \mathbb{N}\}$ be a basis of the topology of $X$. For each isolated point $x$ we can choose $n_x \in \mathbb{N}$ so that $\{x\} = V_{n_x}$. This implies that the set $Z$ of isolated points must be countable and that $Y = X \setminus Z = \bigcap_{x \in Z} V^c_{n_x}$ is closed. □

Propositions 2.3 and 2.4 yield the following Corollary.

COROLLARY 2.5. *Every uncountable Polish space contains a homeomorph of* $\Delta$*, and hence is of cardinality* $c$*.*

PROPOSITION 2.6. $\Delta$ *contains a subspace homeomorphic to* $\mathbb{N}^{\mathbb{N}}$*. More precisely* $\mathbb{N}^{\mathbb{N}}$ *is homeomorphic to the following* $\mathcal{G}_\delta$ *set of* $\Delta$*:*

$$\Delta_\infty = \{(\varepsilon_i) \in \{0,1\}^{\mathbb{N}} : \#\{i : \varepsilon_i = 0\} = \#\{i : \varepsilon_i = 1\} = \infty\}.$$

PROOF. (by Alejandro Chavez Dominguez) We define the following map between $\mathbb{N}_0 \times \mathbb{N}^{\mathbb{N}}$ (which is clearly homeomorphic to $\mathbb{N}^{\mathbb{N}}$) and $\Delta_\infty$

$$\begin{aligned} &\phi : \mathbb{N}_0 \times \mathbb{N}^{\mathbb{N}} \to \Delta_\infty, \\ &(k, n_1, n_2, n_3, \ldots) \mapsto (\underbrace{0, 0, \ldots 0}_{k}, \underbrace{1, 1 \ldots 1}_{n_1}, \underbrace{0, 0, \ldots 0}_{n_2} \ldots). \end{aligned}$$

It is then easy to verify that $\phi$ is a bijection which is continuous and has a continuous inverse. □

Combining Propositions 2.4 and 2.6 we deduce the following result.

COROLLARY 2.7. *Every uncountable Polish contains a subspace homeomorphic to* $\mathbb{N}^{\mathbb{N}}$*.*

The following result generalizes Corollary 2.5.

THEOREM 2.8. *Let* $X$ *be Polish and* $E \subset X \times X$ *be a closed equivalence relation (i.e.* $E$ *as subset of* $X \times X$ *is closed) with uncountably many equivalence classes. Then there is a homeomorph* $D$ *of* $\Delta$ *in* $X$ *consisting of pairwise not equivalent elements. In particular there are exactly* $c$ *equivalence classes.*

Proof. Let $\{V_n : n \in \mathbb{N}\}$ be a basis of the topology of $X$ and put

$$Z = \bigcup\{V_n : (V_n \times V_n) \cap E \text{ has countable many equivalence classes}\}.$$

Then $Z \times Z \cap E$ has only countable equivalence classes.

$Y = X \setminus Z$ is closed, thus, Polish and for all open nonempty sets $U \subset Y$, $U \times U \cap E$ has uncountable equivalence classes (otherwise choose $m$ so that $\emptyset \neq V_m \cap Y \subset U$, since $V_m = (Z \cap V_m) \cup (Y \cap U)$ , $V_m \times V_m$ has only countable equivalence classes).

W.l.o.g. $X = Y$ and $d$ is a complete metric on $X$ with $d - \text{diam}(X) = 1$.

We construct now a Cantor scheme $\{U_s : s \in \{0,1\}^{\mathbb{N}}\}$ , of non empty open sets, so that $(\overline{U_{(s,0)}} \times \overline{U_{(s,1)}}) \cap E = \emptyset$ (since $E$ contains the diagonal $\{(x,x) : x \in X\}$, this condition is stronger than only assuming that $\overline{U_{(s,0)}} \cap \overline{U_{(s,1)}} = \emptyset$) and $d - \text{diam}(U_s) \leq 2^{-|s|}$ for $s \in \{0,1\}^{\mathbb{N}}$. Secondly note that for $\alpha \neq \beta$ in $\{0,1\}^{\mathbb{N}}$ we choose $n \in \mathbb{N}_0$ so that $\alpha|_n = \beta|_n$ and $\alpha(n+1) \neq \beta(n+1)$.

It follows then from Proposition 2.2 (4) that the map associated to $\{F_s : s \in \{0,1\}^{<\mathbb{N}}\}$ is an homeomorphic embedding of $\Delta$ into $X$.

By induction we choose for each $n \in \mathbb{N}_0$ the sets $U_s$ for all $s \in \{0,1\}^n$. Put $U_\emptyset = X$. Assuming for some $n \in \mathbb{N}_0$ and $s \in \{0,1\}^n$ $U_s$ has been chosen. Then, since $U_s$ is open we can choose two non equivalent $y_0$ and $y_1$ in $U_s$. Then we choose two neighborhoods $U_{(s,0)}$ and $U_{(s,1)}$ of $y_0$ and $y_1$, respectively, so that $\overline{U_{(s,0)}}, \overline{U_{(s,1)}} \subset U_s$, $\overline{U_{(s,0)}} \times \overline{U_{(s,1)}} \cap E = \emptyset$ (this is possible since $(y_0, y_1) \notin E$ and $E$ is closed in $X \times X$) and $d - \text{diam}(U_{(s,0)}), d - \text{diam}(U_{(s,1)}) \leq 2^{-n-1}$. □

Theorem 2.9. *Every Polish space $X$ is the image of a closed subset of $\mathbb{N}^{\mathbb{N}}$ under a continuous injection.*

Proof. Let $d$ be a complete metric which generates the topology of $X$ and for which $d - \text{diam}(X) \leq 1$. We will define a Lusin scheme $\{F_s : s \in \mathbb{N}^{<\mathbb{N}}\}$ consisting of $\mathcal{F}_\sigma$ sets, such that $F_\emptyset = X$, $d - \text{diam}(F_s) \leq 2^{-|s|-1}$ and $F_s = \bigcup_{i \in \mathbb{N}} F_{(s,i)}$, for all $s \in \mathbb{N}^{<\mathbb{N}}$. The claim will then follow from Proposition 2.2 (4) if we put $D = \{\alpha \in \mathbb{N}^{\mathbb{N}} : \forall n \in \mathbb{N}\ F_{\alpha|_n} \neq \emptyset\}$.

We choose by induction for each $n \in \mathbb{N}_0$ $F_s$ for all $s \in \mathbb{N}^n$. Put $F_\emptyset = X$ and assume for $n \in \mathbb{N}_0$ and for $s \in \mathbb{N}^n$ $F_s$ has been chosen. We can write $F_s$ as $F_s = \bigcup_{n \in \mathbb{N}} C_n$ where $C_n \subset X$ is closed and moreover $d - \text{diam}(C_n) \leq 2^{-n-1}$, for $n \in \mathbb{N}$, then we put $F_{(s,n)} = C_n \setminus \bigcup_{i=1}^{n-1} C_i$. Since $F_{(s,n)}$ is open in the closed set $C_n$ it is $\mathcal{F}_\sigma$ in $X$. □

Theorem 2.10. *Every compact metric space is the continuous image of a zero dimensional space (a topological space with a basis consisting of clopen sets).*

Proof. Let $d$ be a metric generating the topology on $X$ so that $d - \text{diam}(X) \leq 1$. By induction we will choose for every $k \in \mathbb{N}_0$, $n_k \in \mathbb{N}$, and nonempty closed $F_s \subset X$ for $s \in \prod_{j=0}^k \{1,2,\ldots,n_j\}$ or $s = \emptyset$, and $k \geq 0$, so that $F_\emptyset = X$, $d - \text{diam}(F_s) \leq 2^{-k}$ and $F_s = \bigcup_{j \leq n_k} F_{(s,j)}$.

Assume that for $k \in \mathbb{N}_0$ and all $s \in \prod_{j=1}^k \{1,2,\ldots,n_j\}$ has been chosen ($F_\emptyset = X$). Since all the $F_s$, $s \in \prod_{j=0}^k \{1,2,\ldots n_k\}$ we find an $n_{k+1} \in \mathbb{N}$ and for each $s \in \prod_{j=1}^k \{1,2,\ldots n_k\}$ an open cover $(U_{s,n})_{n \leq n_k}$ of $F_s$ of sets having diameter at most

$2^{-n-1}$. After possibly taking away some of the $(U_{s,n})_{n\le n_k}$'s and possibly repeating some of them we can assume that for all $n \le n_k$ the set $F_{(s,n)} = F_s \cap \overline{U_{(s,n)}}$ is not empty. This finishes the choice of the $F_{(s,n)}$ and the induction step.

Define $Z = \prod_{k=0}^{\infty}\{1, 2, \dots n_k\}$ (which is zero dimensional and compact with respect to the product of the discrete topology) and define for $\alpha = (\alpha_i) \in Z$ $f(\alpha)$ to be the unique element $x_\alpha$ so that $\bigcap_{n\in\mathbb{N}} F_{\alpha|_n} = \{x_\alpha\}$. Then it is easy to verify that $f$ is continuous and onto. □

In in the proof of Theorem 2.10 we could have assured that the numbers $n_k$ are powers of 2 (by increasing them if necessary). In that case the constructed space $Z$ is hoemorphic to $\Delta$ and we obtain.

THEOREM 2.11. *Every compact metrizable space is the continuous image of the Cantor space* $\Delta$.

PROPOSITION 2.12. *Let* $X = A^{\mathbb{N}}$*, where* $A$ *is a non empty set with the discrete topology. Then every closed subset* $C$ *of* $X$ *is a retract to* $X$*, i.e there is a continuous map* $f : X \to C$*, so that* $f|_C$ *is the identity.*

PROOF. For $s \in A^{<\mathbb{N}}$ put $F_s = \{\alpha \in A^{\mathbb{N}} : \alpha|_{|s|} = s\}$ and note that $F_s$ is clopen in $A^{\mathbb{N}}$. For every $s \in A^{<\mathbb{N}}$, for which $C \cap F_s \ne \emptyset$, pick an element $\gamma(s) \in C \cap F_s$. If $\alpha \notin C$, it follows from the assumption that $C$ is closed, and, thus, $X \setminus C$ is open, that there is an $n \in \mathbb{N}$ so that $F_{\alpha|_n} \subset X \setminus C$. Choose $n(\alpha)$ to be the minimum of such $n$'s in $\mathbb{N}$ (since $C$ is not empty $n = 0$ is not possible). Then define

$$f : A^{\mathbb{N}} \to C, \alpha \mapsto \begin{cases} \alpha & \text{if } \alpha \in C, \\ \gamma(\alpha|_{n(\alpha)-1}) & \text{if } \alpha \notin C. \end{cases}$$

In order to verify that $f$ is continuous one needs to observe that if $(\alpha^{(k)} : k \in \mathbb{N})$ is a sequence in $X \setminus C$ which converges to some $\alpha \in C$ that then $\gamma(\alpha^{(k)}|_{n(\alpha^{(k)})-1})$ converges to $\alpha$. But this follows from the fact that since $\alpha \in C$ it follows that $\lim_{k\to\infty} n(\alpha^{(k)}) - 1 = \infty$ and thus

$$\lim_{k\to\infty} \gamma(\alpha^{(k)}|_{n(\alpha^{(k)})-1}) = \lim_{k\to\infty} \alpha^{(k)} = \alpha.$$

□

COROLLARY 2.13. *Every Polish space* $X$ *is the continuous image of* $\mathbb{N}^{\mathbb{N}}$

PROOF. By Theorem 2.9 $X$ is a continuous image of a closed subset of $\mathbb{N}^{\mathbb{N}}$, which by Proposition 2.12 is a retract of $\mathbb{N}^{\mathbb{N}}$. □

REMARK. More precisely, the proofs of Theorem 2.9 and Proposition 2.12 (or an easy direct argument) show that there is a Souslin scheme $(U_s : s \in \mathbb{N}^{<\mathbb{N}})$ of open sets so that $U_\emptyset = X$ and $U_s = \bigcup_{n\in\mathbb{N}} U_{(s,n)}$ for $s \in \mathbb{N}^{<\mathbb{N}}$. Moreover, given any null sequence $(\varepsilon_n) \subset (0, \infty)$, we can assume that $d - \operatorname{diam}(U_s) \le \varepsilon_{|s|}$ whenever $s \in \mathbb{N}^{<\mathbb{N}}$.

## 3. Spaces of compact sets

DEFINITION 3.1. Let $X$ be a topological space and $\mathcal{K}(X)$ the family of nonempty compact subsets of $X$. On $\mathcal{K}(X)$, the topology generated by the family

$$\mathcal{V}_{\mathcal{K}(X)} = \big\{\{K \in \mathcal{K}(X) : K \subset U_0, K \cap U_i \neq \emptyset\} : n \in \mathbb{N}, U_0, U_1, \ldots, U_n \subset X \text{ open}\big\},$$

is called *Vietoris topology on* $\mathcal{K}(X)$. Since $\mathcal{V}_{\mathcal{K}(X)}$ is closed under finite intersections it is a basis of it. For $n \in \mathbb{N}$ and $U_0, U_1, \ldots U_n \subset X$ open, we denote:

$$V(U_0, U_1, \ldots U_n) = \{K \in \mathcal{K}(X) : K \subset U_0\} \cap \bigcap_{i=1}^{n} \{K \in \mathcal{K}(X) : K \cap U_i \neq \emptyset\}.$$

From now on we consider $\mathcal{K}(X)$ with the Vietoris topology.

PROPOSITION 3.2. *If $D \subset X$ is dense, then the set all finite non empty subsets of $D$ is dense in $\mathcal{K}(X)$. In particular, if $X$ is separable so is $\mathcal{K}(X)$.*

PROOF. Let $U_0, U_1, \ldots U_n \subset X$ be open so that $V(U_0, U_1, \ldots U_n) \neq \emptyset$. This implies that $U_i \cap U_0 \neq \emptyset$, for $i = 1, 2 \ldots n$, and, thus, we can pick $x_i \in D \cap U_i \cap U_0$, for $i = 1, \ldots n$, and note that $F = \{x_1, \ldots, x_n\} \in V(U_0, U_1, \ldots U_n)$. Since the sets $V(U_0, U_1, \ldots U_n)$, with $n \in \mathbb{N}$, and $U_0, U_1, \ldots U_n \subset X$ open, form a basis we deduce our claim. □

PROPOSITION 3.3. *If $(X, d)$ is a metric space we define for $K, L \in \mathcal{K}(X)$*

$$\delta_H(K, L) = \max\big(\max_{x \in K} d(x, L), \max_{y \in L} d(y, K)\big).$$

*Then $\delta_H$ is a metric on $\mathcal{K}(X)$ which induces the Vietoris topology on $\mathcal{K}(X)$. We call $\delta_H$ the* Hausdorff metric on $\mathcal{K}(X)$.

REMARK. Note that for $K, L \in \mathcal{K}(X)$ and $\varepsilon > 0$,

$$\delta_H(K, L) < \varepsilon \iff K \subset L_\varepsilon \text{ and } L \subset K_\varepsilon. \tag{1}$$

PROOF OF PROPOSITION 3.3. Clearly $\delta_H$ is symmetric. If $K, L \in \mathcal{K}(X)$ and $\delta_H(K, L) = 0$, and let $x \in K$. Since $d(x, L) = 0$ and since $L$ is closed it follows that $x \in L$. Thus we showed that $K \subset L$. By symmetry $L \subset K$.

If $K, L, M \in \mathcal{K}(X)$. For $x \in K$ choose $y_x \in L$ so that $\max_{x \in K} d(x, L) = d(x, y_x)$.

$$\begin{aligned}
\max_{x \in K} d(x, L) + \max_{y \in L} d(y, M) &= \max_{x \in K} \max_{y \in L} d(x, L) + d(y, M) \\
&= \max_{x \in K} \max_{y \in L} d(x, y_x) + d(y, M) \\
&\geq \max_{x \in K} d(x, y_x) + d(y_x, M) \geq \max_{x \in K} d(x, M).
\end{aligned}$$

By the same argument we can show that

$$\max_{x \in M} d(x, L) + \max_{y \in L} d(y, K) \geq \max_{x \in M} d(x, K).$$

Assume $K \in \mathcal{K}(X)$ and $\varepsilon > 0$. We first show that there are finitely many open sets $U_0, U_1, \ldots U_n \subset X$ so that

$$K \subset V(U_0, U_1, \ldots U_n) \subset B_{(\delta_H, \varepsilon)}(K).$$

Choose a finite $\varepsilon/2$-net $(x_i)_{i=1}^n$ in $K$ and put $U_0 = K_\varepsilon$ and $U_i = B_{(\varepsilon/2,d)}(x_i)$ for $i = 1, 2, \ldots n$. Clearly $K \in V(U_0, U_1, \ldots U_n)$. Secondly, if $L \in V(U_0, U_1, \ldots U_n)$, then $L \subset U_0 = K_\varepsilon$ and for every $x \in K$ there is an $i \in \{1, 2 \ldots n\}$ with $d(x, x_i) < \varepsilon/2$, and since $K \cap U_i \neq \emptyset$ there is a $z \in L$, so that $d(x_i, z) < \varepsilon/2$. Thus, $d(x, z) < \varepsilon$, So we proved that $K \subset L_\varepsilon$, which together with $L \subset K_\varepsilon$ implies by (1) that $\delta_H(K, L) < \varepsilon$.

Conversely if open sets $U_0, U_1, \ldots U_n \subset X$ are given, and $K \in V(U_0, U_1, \ldots U_n)$. Choose $\varepsilon_0$ so that $K_{\varepsilon_0} \subset U_0$ and $\varepsilon_i > 0$, for $i = 1, 2, \ldots, n$ so that there is an $x_i \in U_i \cap K$ with $B_{\varepsilon_i}(x_i) \subset U_i$. For $\varepsilon = \min_{0 \le i \le n} \varepsilon_i$ we claim that $B_{(\delta_H,\varepsilon)}(K) \subset V(U_0, U_1, \ldots U_n)$. Indeed, if $L \in \mathcal{K}(X)$, with $\delta_H(L, K) < \varepsilon$, it follows by (1) that $L \subset K_\varepsilon$, and, thus $L \subset U_0$, and that $K \subset L_\varepsilon$, and, thus, $B_{(d,\varepsilon)}(x_i) \cap L \neq \emptyset$, which implies that $U_i \cap L \supset B_{(d,\varepsilon_i)}(x_i) \cap L \supset B_{(d,\varepsilon)}(x_i) \cap L = \emptyset$. Therefore we verified that $L \in V(U_0, U_1, \ldots U_n)$, and, thus, that $B_{(\delta_H,\varepsilon)}(K) \subset V(U_0, U_1, \ldots U_n)$. $\square$

PROPOSITION 3.4. *If $(X, d)$ is a complete metric space and $(K_n) \subset \mathcal{K}(X)$ a Cauchy sequence with respect to $\delta_H$, then $\overline{\bigcup K_n}$ is compact in $X$.*

PROOF. Since $X$ is complete it is enough to show that $K = \overline{\bigcup K_n}$ is totally bounded. Let $\varepsilon > 0$. Since $(K_n)$ is Cauchy there is an $n_0$ so that $K_n \subset [K_{n_0}]_{\varepsilon/2}$ for all $n \ge n_0$. Thus if $(x_i)_{i=1}^k$ is an $\varepsilon/2$-net of $\bigcup_{n=1}^{n_0} K_n$, then it is an $\varepsilon$-net of $\bigcup_{n=1}^\infty K_n$. $\square$

PROPOSITION 3.5. *If $(X, d)$ is a complete metric space then $(\mathcal{K}(X), \delta_H)$ is also a complete metric space and for a Cauchy sequence $(K_n)$*

$$\delta_H - \lim_{n\to\infty} K_n = \bigcap_{n\in\mathbb{N}} \overline{\bigcup_{m=n}^\infty K_m}.$$

PROOF. Let $(K_n) \subset \mathcal{K}(X)$ be a Cauchy sequence, put $K = \bigcap_{n\in\mathbb{N}} \overline{\bigcup_{m=n}^\infty K_m}$ and let $\varepsilon > 0$. Choose $n_0$ so that $\delta_H(K_n, K_m) < \varepsilon/2$ and, thus, $[K_n] \subset [K_m]_{\varepsilon/2}$ and $[K_m] \subset [K_n]_{\varepsilon/2}$, whenever $m, n \ge n_0$. This implies $K \subset \overline{\bigcup_{n=n_0}^\infty} \subset [K_n]_\varepsilon$, for $n \ge n_0$. Secondly it follows for all $n$ and all $n \ge n_0$ $K_n \subset [K_m]_{\varepsilon/2}$ and thus, for $x \in \bigcap_{m\in\mathbb{N}} [K_m]_{\varepsilon/2}$ there is a sequence $(z_m)$, $d(z_m, x) < \varepsilon/2$ and $z_m \in \bigcup_{j=m}^\infty K_j$, for $m \in \mathbb{N}$. By Proposition 3.4 $\overline{\bigcup K_n}$ is compact and, thus, $(z_n)$ has an accumulation point say $z$. it follows that $z \in K$, and $d(z, x) \le \varepsilon/2$ and thus $x \in K_\varepsilon$. Therefore we proved that $K_n \subset K_\varepsilon$. By Proposition 1 it follows $\delta_H(K, K_n) < \varepsilon$ for all $n \ge n_0$. $\square$

Propositions 3.4 and 3.5 yield the following Corollary.

COROLLARY 3.6. *If $X$ is a Polish space, so is $\mathcal{K}(X)$.*

PROPOSITION 3.7. *If $X$ is compact and metrizable, so is $\mathcal{K}(X)$.*

PROOF. It is enough to show that $(\mathcal{K}(X), \delta_H)$ is totally bounded. Let $\varepsilon > 0$ and let $D$ finite $\varepsilon$-net of $X$. Then, using (1), it is easy to verify that $\{A \subset D : A \neq \emptyset\}$ is an $\varepsilon$-net of $\mathcal{K}(X)$. $\square$

PROPOSITION 3.8. *If $X$ is compact and metrizable, then the Borel sets $\mathcal{B}_{\mathcal{K}(X)}$ are generated by*

$$\mathcal{V}' = \big\{ \{K \in \mathcal{K}(X) : K \cap U_i \neq \emptyset\} \, : \, n \in \mathbb{N} \text{ and } U_1, U_2, \ldots U_n \subset X \text{ open} \big\}.$$

PROOF. Let $\Sigma$ be the $\sigma$-algebra generated by $\mathcal{V}'$. It is enough to show that for any $U \subset X$ open the set $\{K \in \mathcal{K}(X) : K \subset U\} \in \Sigma$. But this is equivalent with $\{K \in \mathcal{K}(X) : K \cap U^c \neq \emptyset\} \in \Sigma$. But $U^c$ is an $\mathcal{G}_\delta$ set. Thus, write $U^c = \bigcap V_n$, where $V_n \subset X$ is open. and note that

$$\{K \in \mathcal{K}(X) : K \cap U^c \neq \emptyset\} = \bigcap_{n\in\mathbb{N}} \{K \in \mathcal{K}(X) : K \cap V_n \neq \emptyset\} \in \Sigma$$

Indeed, for "$\subset$" is trivial and for $K \in \bigcap_{n\in\mathbb{N}} \{K \in \mathcal{K}(X) : K \cap V_n \neq \emptyset\}$ choose $z_n \in K \cap V_n$, for $n \in \mathbb{N}$, and an accumulation point $z$ of $(z_n)$. It follows that $z \in K \cap U^c$ and, thus, $K \cap U^c \neq \emptyset$. □

## 4. The Baire Category Theorem and Applications

THEOREM 4.1. (Baire Category Theorem)
*Let $X$ be a completely metrizable space. The intersections of countably many dense open sets in $X$ is dense.*

COROLLARY 4.2. *Let $X$ be a completely metrizable space. If $F_n \subset X$ is closed for $n \in \mathbb{N}$, and $X = \bigcup F_n$ then there is an $n \in \mathbb{N}$ so that $F^\circ \neq \emptyset$.*

PROOF. Consider $U_n = X \setminus F_n$. Since $\bigcap U_n = \emptyset$, there must be an $n \in \mathbb{N}$ for which $U_n$ is not dense in $X$.

□

DEFINITION 4.3. Let $(X, \mathcal{T})$ be a topological space and $d(\cdot, \cdot)$ a metric on $X$ (not necessariy a metric which generates the topology on $X$).

We say that $(X, \mathcal{T})$ is *d-fragmentable* if for all closed sets $F \subset X$ and all $\varepsilon > 0$ there is an open set $U \subset X$ so that $F \cap U \neq \emptyset$ so that $d - \mathrm{diam}(U \cap F) < \varepsilon$.

THEOREM 4.4. *Let $(X, \mathcal{T})$ be a Polish space, and let $d$ be a metric on $X$ so that all closed d-balls, $\overline{B_{(\varepsilon,d)}(x)}^d = \{y \in X : d(x, y) \leq \varepsilon\}$, with $\varepsilon > 0$ and $x \in X$, are closed in $X$ (with respect to $\mathcal{T}$).*

*Then $X$ is d-fragmentable if and only if $(X, d)$ is separable.*

PROOF. "$\Leftarrow$" Let $F \subset X$ be closed and choose $D \subset F$ dense in $(X, d)$ and countable, and then note that since $F = \bigcup_{a\in D} F \cap \overline{B(\varepsilon, d)(a)}$ there must be an $a \in D$ with $\overline{B(\varepsilon, d)(a)}^d$ has a non empty $\mathcal{T}$-interior.
"$\Rightarrow$"(Alejandro Chavez Dominguez)
Assume that $(X, d)$ is not separable. We need to find $\varepsilon > 0$ and a $\mathcal{T}$-closed set $F$ in $X$ which has the property that $d - \mathrm{diam}(U \cap F) \geq \varepsilon$ for all $\mathcal{T}$-open $U \subset X$ with $U \cap F \neq \emptyset$.

Since $(X, d)$ is not separable we find an uncountable $A \subset X$ and an $\varepsilon > 0$ so that $d(x, z) > \varepsilon$ for all $x \neq z$ in $A$. By the Cantor Bendixson Theorem, Proposition 2.4, we can assume, after possibly taking away countably many points, that $A$ has no $\mathcal{T}$-isolated points with respect to $\mathcal{T}$.

We claim that $F = \overline{A}^{\mathcal{T}}$ has the wanted property. For $\mathcal{T}$-open set $U \subset X$ for which $F \cap U \neq \emptyset$ it follows that $A \cap U \neq \emptyset$ ($U$ is open!) since $A$ has no isolated points with respect to $\mathcal{T}$, $A \cap U$ contains two different points $x, z$ for which, by choice of $A$ it follows that $d(x, z) > \varepsilon$. $\square$

REMARK. (We are following the argument in [**Boss1**])
Let $X$ be a Polish space, and let $\mathcal{V} = \{V_n : n \in \mathbb{N}\}$ be a basis of the topology of $X$.

Assume that $d(\cdot, \cdot)$ is a metric on $X$ and that all closed $d$-balls are closed in the Polish topology of $X$.

For $F \in \mathcal{F}(X)$, the set of all non empty closed subsets of $X$ (see Section 3), and $\varepsilon > 0$ we define by transfinite induction the following closed subsets $F_\varepsilon^{(\alpha,n)}$, with $\alpha < \omega_1$ and $n \in \mathbb{N}$.

We put $F_\varepsilon^{(0,0)} = F$. Assuming that $F_\varepsilon^{(\beta,n)}$ has been defined for some $\beta < \omega_1$ and some $n \in \mathbb{N}_0$ , we let

$$F_\varepsilon^{(\beta,n+1)} = \begin{cases} F_\varepsilon^{(\beta,n)} & \text{if } d-\operatorname{diam}(V_{n+1} \cap F_\varepsilon^{(\beta,n)}) > \varepsilon \\ F_\varepsilon^{(\beta,n)} \setminus V_n & \text{if f } d-\operatorname{diam}(V_{n+1} \cap F_\varepsilon^{(\beta,n)}) \leq \varepsilon \end{cases}$$

Once $F_\varepsilon^{(\beta,n)}$ is defined for all $n \in \mathbb{N}$ we put

$$F_\varepsilon^{(\beta+1,0)} = \bigcap_{n\in\mathbb{N}} F_\varepsilon^{(\beta,n)}.$$

If $\alpha$ is a limit ordinal and $F_\varepsilon^{(\beta,0)}$ has been defined for all $\beta < \alpha$ we put

$$F_\varepsilon^{(\alpha,0)} = \bigcap_{\beta} F_\varepsilon^{(\beta,0)}.$$

it follows then

$$F \text{ is } d\text{-fragmentable} \iff \exists \alpha < \omega_1 \quad F_\varepsilon^{(\alpha,0)} = \emptyset. \tag{2}$$

In this case put

$$o_{d-\text{frag}}(F, \varepsilon) = \min\{\alpha < \omega_1 : F_\varepsilon^{(\alpha,0)} = \emptyset\} \text{ and}$$
$$o_{d-\text{frag}}(F) = \sup_{\varepsilon>0} o_{d-\text{frag}}(F, \varepsilon).$$

CHAPTER 3

# Trees

## 1. Introduction

In this subsection we introduce trees and prove some basic results.

A *tree* is a non empty partially ordered set $(T, \le)$ which has the property that for any $x \in T$ the *set of predecessors of $x$ in $T$*

$$\text{Pred}(T, x) = \{y \in T : y < x\}$$

(where we write $y < x$ if $x \le y$ and $x \ne y$) is finite and linearly ordered. The elements of a tree are called *nodes*, minimal elements of $T$ are called *initial nodes* , maximal elements are called *terminal nodes* and for $x \in T$, which is not an initial node, we call the maximal element of $\text{Pred}(T, x)$ the *immediate predecessor of $x$*. Note that every non minimal element in a tree has an immediate predecessor and it is unique.

Let $X$ be a set and note that

$$X^{<\mathbb{N}} = \bigcup_{n\in\mathbb{N}_0} X^n = \{(x_1, x_2, \dots, x_n) : n \in \mathbb{N}, x_1, x_2, \dots, x_n \in X\} \cup \{\emptyset\}$$

is with respect to the order given by extensions "*succ*" (see Section 1) is a tree. We identify $X$ with the 1-tuples in $X$, and, thus, consider $X$ to be a subset of $X^{<\mathbb{N}}$.

For a node $x$ in a tree $X$ we call $y \in T$ a *successor of $x$* if $y > x$ and *immediate successor of $x$* if it is a minimal element of the set of all successors of $x$. Note that non terminal nodes in a tree have immediate successors but they may not be unique. We denote the set of all successors of $x$ by $\text{Succ}(T, x)$. If $T = X^{<\mathbb{N}}$ for some set $X$ we write $\text{Succ}_X(s)$ instead of $\text{Succ}(X^{<\mathbb{N}}, s)$, for an $s \in X^{<\mathbb{N}}$ or simply $\text{Succ}(s)$ if there is no confusion. We also put

$$\text{Succ}_{\infty,X}(s) = \text{Succ}_\infty(s) = \big\{\alpha \in X^{\mathbb{N}} : \alpha \succ s\big\}.$$

A maximal linearly ordered subset of a tree is called a *branch*. A tree $(T, \le)$ is called *well-founded* if $T$ does not have any infinite branch. A tree $T$ is called a *pruned tree* if $T$ does not have any maximal elements. An *interval of $T$* is a set of the form $I = \{x \in T : a \le x \le b\}$ with $a \le b$ in $T$.

A nonempty subset $S$ of $X^{<\mathbb{N}}$ is called a *tree on $X$* if it is closed under taking restrictions, i.e. if $s = (x_1, x_2 \dots x_n) \in S$ and $0 \le m \le n$ then $(x_1, x_2 \dots x_m) \in S$. In particular $\emptyset \in S$ for trees on $X$. If $X$ is a topological space we call $S$ a *closed tree on $X$* if for all $n \in \mathbb{N}$ the set $X^n \cap S$ is closed with respect to the product topology.

Assume that $T$ is a tree on some cartesian product $X \times Y$ and let $\alpha = (\alpha_i) \in X^{\mathbb{N}}$ we call the set

$$T[\alpha] = \big\{t \in Y^{<\mathbb{N}} : (\alpha|_{|t|}, t) \in T\big\},$$

the *section of* $T$ *at* $\alpha$. Note that $T[\alpha]$ is a tree on $Y$.

## 2. Well founded trees and ordinal index

If $T$ is a tree on a well ordered set $(A, <)$ we define the *Kleene - Brower ordering* $<_{\mathrm{KB}}$ on $T$ as follows. For $s = (s_i)_{i=1}^m$ and $(t_i)_{i=1}^n$ we write $s <_{\mathrm{KB}} t$ if either $s$ is an extension of $t$ (note that this is opposite to the order $\prec$) , or $s_{i_0} < t_{i_0}$ for $i_0 = \min\{1 \le i \le \min(m,n) : s_i \ne t_i\}$. Clearly $<_{\mathrm{KB}}$ is a linear order on $T$.

Proposition 2.1. *A tree $T$ on a well ordered set $A$ is well founded if and only if $<_{\mathrm{KB}}$ is a well-order on $T$.*

Proof. If $T$ is not well founded then there is a sequence $(s_i)$ with $s_1 \prec s_2 \prec s_3 \ldots$. and thus $s_1 >_{\mathrm{KB}} s_2 >_{\mathrm{KB}} \ldots$. which means that $(s_i)$ has no miminum in $<_{\mathrm{KB}}$, and therefore cannot be well ordered.

Conversely, assume that $T$ is not well ordered with respect to $<_{KB}$ and let $(s_n)$, $s_n = (a(n,1), a(n,2), \ldots a(n,k_n))$ be a strictly decreasing sequence in $(T, <_{\mathrm{KB}})$. This implies that $a(1,1) \ge a(2,1) \ge a(3,1) \ldots$ in $A$, and thus, since $A$ is well ordered, the sequence $(a(n,1))_{n\in\mathbb{N}}$ has to be eventually stationary, say $a(n,1) = a_1 \in A$ for al $n \ge n_1$. Then we consider the sequence $(a(n,2) : n \ge n_1)$ and again we find an $n_2 \ge n_2$ so that $a(n,2) = a_2$ for some $a_2 \in A$. Note also that $(a_1)$ and $(a_1, a_2)$ are in $T$ and we can continue and find a sequence $(a_i) \subset A$, so that for all $k \in \mathbb{N}$ $(a_1, a_2, \ldots a_k) \in T$, which means that $T$ is not well founded. $\square$

Let $(T, \le)$ be a tree. The *derived tree of* $T$ is the subtree

$$D(T) = \big\{x \in T : x \text{ is not terminal}\big\} = \big\{x \in T : \exists y \in T \quad y > x\big\}.$$

By transfinite induction we define for $\alpha \in \mathbf{ON}$ $T^\alpha$ as follows

$$T^0 = T$$

if $\alpha$ is a successor ordinal, say $\alpha = \beta + 1$, and if $T^\beta$ has been defined, we put

$$T^\alpha = D(T^\beta)$$

and if $\alpha$ is a limit ordinal, we put,

$$T^\alpha = \bigcap_{\beta<\alpha} T^\beta.$$

If $(T, \le)$ is a tree it follows that there is an $\alpha \in \mathbf{ON}$ (actually $\alpha < \min\{\aleph : \aleph \in \mathbf{CARD}, \aleph > \mathrm{card}(X) \vee \omega)$ so that $(T^\beta)_{\beta\ge\alpha}$ is stationary. Note that in this case

$$T^\alpha = \{s \in T : \exists (s_i) \subset T \quad s = s_1 \prec s_2 \prec s_3 \ldots\}$$

and that $T^\alpha$ is pruned.

We say that a tree $T$ *has countable index* if for some $\alpha < \omega_1$ we have $T^\alpha = \emptyset$. In that case we call

$$o(T) = \min\big\{\alpha < \omega_1 : T^\alpha = \emptyset\big\}$$

the *ordinal index of* $T$. If $T$ does not have a countable index we write $o(T) = \omega_1$. Trees are by definition non empty, but we extend the definition of $o(T)$ to $\emptyset$ by letting $o(\emptyset) = 0$.

We put

$$[T] = \{\alpha \in X^{\mathbb{N}} : \forall k \in \mathbb{N} \quad \alpha|_k \in T\},$$

and call $[T]$ the *body of* $T$.

From our recent observations we deduce easily the following equivalences

PROPOSITION 2.2. (Wellfoundedness Principle for countable sets) *For a tree $T$ on $\mathbb{N}$ the following are equivalent.*

1) *$T$ is well founded,*
2) *$[T] = \emptyset$,*
3) *$T$ has countable index.*

PROPOSITION 2.3. *Let $A^{\mathbb{N}}$ be endowed with the product of the discrete topology, then $C \subset A^{\mathbb{N}}$ is closed if and only if it is the body of some tree $T$ on $A$.*

*Moreover in the case that $C \subset A^{\mathbb{N}}$ is closed it follows that $C = [T]$ with*

$$T = \big\{s \in A^{<\mathbb{N}} : \exists \alpha \in C \quad \alpha \succ s\big\}.$$

PROOF. "$\Leftarrow$" Assume

$$C = [T] = \big\{\alpha \in A^{\mathbb{N}} : \forall n \in \mathbb{N} \quad \alpha|_n \in T\big\}.$$

if $\beta \notin C$ then there is an $n \in \mathbb{N}$ so that $\beta|_n \notin T$, and, thus, $\{\gamma \in A^{\mathbb{N}} : \gamma \succ \beta|_n\} \subset C^c$, which proves that $C^c$ is open and, thus, that $C$ is closed.
"$\Rightarrow$" Assume that $C \subset A^{\mathbb{N}}$ is closed. Then define

$$T = \big\{s \in A^{<\mathbb{N}} : \exists \alpha \in C \quad \alpha \succ s\big\}.$$

Clearly $T$ is a tree and it is also easy to see that $C \subset [T]$. If $\alpha \in [T]$ and thus $\alpha|_n \in T$ for all $n \in N$, then, by definition of $T$, there is for every $n \in \mathbb{N}$ a $\beta^{(n)} \in C$ so that $\beta^{(n)} \succ \alpha|_n$. Since $C$ is closed it follows that $\alpha \in C$, which finishes the proof.

$\square$

In the following Proposition we list some easy properties of trees.

PROPOSITION 2.4. *Let $(T, \leq)$ be a tree and $\alpha < \omega_1$.*

a) *If $o(T) = \alpha$ and we define $T' = \{x_0\} \cup T$ where $x_0 \notin T$ and we put $x_0 < x$ for all $x \in T$. Then $T'^{\beta} = \{x_0\} \cup T^{\beta}$ for $\beta \leq \alpha$ and, thus, $o(T') = \alpha + 1$.*
b) *If $x \in T^{\alpha}$ then $o(\mathrm{Succ}(T, x)) \geq \alpha$.*
c) *$o(T) = \alpha + 1$ if and only if $T^{\alpha}$ is a nonempty subset of the initial nodes of $T$ and $o(\mathrm{Succ}(T, x)) = \alpha$ for any $x \in T^{\alpha}$.*
d) *If $(T_i)_{i \in I}$ is a disjoint family of trees and we put*

$$T = \bigcup_{i \in I} T_i,$$

*and do not add any relation to the existing ones, i.e.*

$$x < y \iff \exists i \in I \ x, y \in T_i \text{ and } x < y \text{ in } T_i$$

*then*

$$o(T) = \sup_{i\in I} o(T_i).$$

e) *If $\alpha < \omega_1$ is a limit ordinal then $o(T) = \alpha$ if and only the set $I$ of initial nodes is infinite, and $\sup_{x\in I} o(\mathrm{Succ}(T, x)) = \alpha$.*

PROOF. (a) follows easily by transfinite induction on $\beta \le \alpha$.
(b) We show our claim by transfinite induction for all $\alpha < \omega_1$. If $\alpha = \beta + 1$, and the claim is true for $\beta$, we proceed as follows. For $x \in T^{\beta+1}$ we choose $y \in T^\beta$ so that $y > x$, deduce form the induction hypothesis that $o(\mathrm{Succ}(T, y)) \ge \beta$, and since $\{y\}\cup\mathrm{Succ}(T, y) \subset \mathrm{Succ}(T)$ it follows from part (a) that $o(\mathrm{Succ}(T, x)) \ge \beta+1$. Now let $\alpha$ be limit ordinal and assume the claim to be true for $\beta < \alpha$. For $x \in T^\alpha = \bigcap_{\beta<\alpha} T^\beta$, it follows then from the induction hypothesis that $o(\mathrm{Succ}(T, x)) \ge \sup_{\beta<\alpha} \beta = \alpha$.
(c) Note that $o(T) = \alpha + 1$ if and only if $T^\alpha \neq \emptyset$ and $T^{\alpha+1} = \emptyset$.
(d) For any $i \in I$ it follows $o(T) \ge o(T_i)$ and thus $o(T) \ge \sup o(T_i)$. On the other hand it follows by transfinite induction on $\alpha$ that

$$T^\alpha = \bigcup_{i\in I} T_i^\alpha$$

and, thus, if $\alpha \ge \sup o(T_i)$ then $T^\alpha = \emptyset$, which implies $o(T) \le \sup o(T_i)$.
(e) Let $I$ be the set of infinite nodes of $T$ and let $\alpha$ be a limit ordinal.

If $o(T) = \alpha$ then for all $x \in I$ $o(\mathrm{Succ}(T, x)) < \alpha$. If $I$ were finite then $\alpha = \max_{x\in I} o(\mathrm{Succ}(T, x)) + 1$ which is a contradiction to the assumption that $\alpha$ is a limit ordinal. Secondly, For any $\beta < \alpha$ such that $T^\beta \neq \emptyset$ there is an $x \in I \cap T^{\beta+1}$, thus, we deduce that $\alpha \le \sup_{x\in I} o(\mathrm{Succ}(T, x)) + 1$. Trivially we have $\alpha = o(T) \ge \sup_{x\in I} o(\mathrm{Succ}(T, x) + 1$ which implies $\alpha = \sup_{x\in I} o(\mathrm{Succ}(T, x)$.

Conversely, if $\alpha = \sup_{x\in I} o(\mathrm{Succ}(T, x)) + 1$ and $I$ is infinite, it is clear that $o(T) \ge \alpha$, and it is not possible that $o(T) > \alpha$ since otherwise there is an $x \in I \cap T^\alpha$ and, thus, $o(\mathrm{Succ}(T, x)) \ge \alpha$, □

PROPOSITION 2.5. *Let $T$ be a tree on a set $X$ with countable index. For $x \in X$ define*

$$T(x) = \{s \in X^{<\mathbb{N}} : (x, s) \in T\} = \begin{cases} \{\emptyset\} \cup \{s \in X^{<\mathbb{N}} : (x, s) \in \mathrm{Succ}(T, x)\} & \text{if } x \in T \\ \emptyset & \text{if } x \notin T. \end{cases}$$

a) *The ordinal $o(T(x))$ is a successor ordinal.*
b) *If $x \in X$, and $\alpha < \omega_1$, then $T^\beta(x) = (T^\beta)(x)$*
c) *For $\alpha < \omega_1$ we have*

$$o(T) = \alpha + 1 \iff \sup_{x\in X} o(T(x)) = \alpha. \tag{3}$$

PROOF. (a) Assume that $\alpha < \omega_1$ is a limit ordinal and that $o(T) \ge \alpha$ we need to show that $T^\alpha \neq \emptyset$ and thus $o(T) > \alpha$. From $o(T) \ge \alpha$ it follows that for all $\beta < \alpha$ $T^\beta \neq \emptyset$ and thus $\emptyset \in \bigcap_{\beta<\alpha} T^\beta = T^\alpha$ which implies our claim.

(b) We prove our claim by transfinite induction for all $\beta < o(T)$, for $\beta = 0$ the claim is trivial. Assuming the claim is true for all $\gamma < \beta$ we deduce it in the case that $\beta$ is a limit ordinal

$$\begin{aligned} s \in T^\beta(x) &\iff (x, s) \in T^\beta \\ &\iff \forall \gamma < \beta \quad (x, s) \in T^\gamma \\ &\iff (x, s) \in T^\beta \iff s \in (T^\beta)(x). \end{aligned}$$

and if $\beta = \gamma + 1$

$$\begin{aligned} s \in T^\beta(x) &\iff (x, s) \in T^\beta \\ &\iff \exists z \in X \quad (x, s, z) \in T^\gamma \\ &\iff \exists z \in X \quad (s, z) \in T(x)^\gamma \iff s \in (T(x))^\beta. \end{aligned}$$

which implies our claim.

(c) Assume that $o(T) = \alpha + 1$. For any $x \in X$ it follows that $(T(x))^\alpha = \emptyset$, and thus $o(T(x)) \leq \alpha$. Indeed, otherwise, by part (b), $\emptyset \in (T(x))^\alpha = T^\alpha(x)$ and, thus, $o(T) \geq \alpha + 2$. On the other hand if there where an $x \in X$ so that $o(T(x)) > \alpha$ it would follow that $\emptyset \in (T(x))^\alpha = T^\alpha(x)$ and, thus,

$$x \in T^\alpha \Rightarrow \emptyset \in T^{\alpha+1} \Rightarrow o(T) \geq \alpha + 2,$$

which is a contradiction.

Conversely, assume that $\sup_{x \in X} o(T(x)) = \alpha$. This implies that for any $\beta < \alpha$, there is an $x \in X$ so that $\emptyset \in T^\beta(x) = (T(x))^\beta$. If $\alpha = \beta + 1$ for some $\beta < \omega_1$. it follows that for some $x \in X$ $x \in T^\beta$ and, thus, $o(T) \geq \alpha + 1$. If $\alpha$ is a limit ordinal it follows also that $o(T) \geq \alpha + 1$. On the other hand, if $o(T) \geq \alpha + 2$ we could choose an $x \in X$ so that $x \in T^\alpha$ and thus $\emptyset \in T^\alpha(x) = (T(x))^\alpha$ and, thus, $o(T(x)) \geq \alpha + 1$ which is a contradiction and finishes the proof of (c).

□

EXAMPLE 2.6. [Fine Schreier sets]
For $\alpha < \omega_1$ we define $\mathcal{F}_\alpha \subset [\mathbb{N}]^{<\mathbb{N}}$ (recall that we identified $[\mathbb{N}]^{<\mathbb{N}}$ with the subset of finite increasing sequences in $\mathbb{N}$) by transfinite induction.

$$\mathcal{F}_1 = \{\{n\} : n \in \mathbb{N}\} \cup \{\emptyset\}$$

If $\mathcal{F}_\beta$ has been defined for some $\beta < \omega_1$ put

$$\mathcal{F}_{\beta+1} = \{A \cup \{n\} : A \in \mathcal{F}_\beta \ \& \ n \in \mathbb{N}, n < A\} \cup \{\emptyset\}$$

For a limit ordinal $\alpha$ we first choose a sequence $(\alpha_n) \subset [1, \alpha)$ which increases to $\alpha$ and then put

$$\mathcal{F}_\alpha = \big\{A \in [\mathbb{N}]^{<\mathbb{N}} : \exists n \in \mathbb{N} \min A \geq n \text{ and } A \in \mathcal{F}_{\alpha_n}\big\}.$$

EXERCISE 2.7. Show that $o(\mathcal{F}_\alpha) = \alpha + 1$.

The following result will be needed later (see Subsection 6) to show that the ordinal index $o(\cdot)$ is *co-analytic rank*.

Proposition 2.8. *Let $S, T \subset \mathbb{N}^{<\mathbb{N}}$ be two subtrees. Then $o(S) \le o(T)$ if and only if there is a map $f : S \to T$ so that*

a) $f(\emptyset) = \emptyset$
b) *If $s \prec \tilde{s}$ are in $S$ then $f(s) \prec f(\tilde{s})$.*
c) *If $t \in f(S)$, and $\tilde{t} \prec t$ then there are $\tilde{s} \prec s$ in $S$ so that $f(\tilde{s}) \prec f(s)$.*

*Secondly, $o(S) < o(T)$, if and only if there is an $n \in \mathbb{N}$ so that there is a map $f : S \to T(n)$ which satisfies (a), (b) and (c) with $T$ replaced by $T(n)$. As in Proposition 2.5, we mean by $T(n)$*

$$\begin{aligned} T(n) &= \{s \in N^{<\mathbb{N}} : (n, s) \in T\} \\ &= \begin{cases} \{\emptyset\} \cup \{s \in X^{<\mathbb{N}} : (x, s) \in \mathrm{Succ}(T, n)\} & \text{if } n \in T \\ \emptyset & \text{if } n \notin T. \end{cases} \end{aligned}$$

Proof. We will prove our claim by transfinite induction that for each $\alpha$ and any two trees $S, T$ with $o(S) \le \alpha$ it follows that $o(S) \le o(T)$ if and only if there is a map $f : S \to T$ satisfying (a), (b) and (c).

Assume the equivalence to be true for all $\beta < \alpha$, and $\alpha < \omega_1$ (the case $\alpha = \omega_1$ will be handled later).

"$\Rightarrow$" Assume $o(S) \le \alpha$ and define for $n \in \mathbb{N}$ the $S(n)$ as in Proposition 2.5

$$\begin{aligned} S(n) &= \{s \in N^{<\mathbb{N}} : (n, s) \in S\} \\ &= \begin{cases} \{\emptyset\} \cup \{s \in X^{<\mathbb{N}} : (x, s) \in \mathrm{Succ}(S, n)\} & \text{if } n \in S \\ \emptyset & \text{if } n \notin S. \end{cases} \end{aligned}$$

If $T$ is a subtree of $\mathbb{N}^{<\mathbb{N}}$ with $o(T) \ge o(S)$ then by Proposition 2.5

$$\sup_{n\in\mathbb{N}} o(S(n)) \le \sup_{n\in\mathbb{N}} o(T(n)) \text{ and } \sup_{n\in\mathbb{N}} (S(n)) < \alpha.$$

For each $n \in \mathbb{N}$ either $n \notin S$ or there is a $k(n) \in \mathbb{N}$ so that $o(S(n)) \le o(T(k(n)))$. By induction hypothesis we can find for $n \in \mathbb{N} \cap S$ a map $f_n : S(n) \to T(k(n))$ satisfying (a), (b) and (c) for $S(n)$ and $T(k(n))$. For $s = (s_1, \dots s_\ell) \in S$ we put

$$f(s) = \begin{cases} \emptyset & \text{if } s = \emptyset \\ k(s_1) & \text{if } \ell = 1 \\ (k(s_1), f_{s_1}(s_2, s_3, s_\ell)) & \text{if } \ell > 1. \end{cases}$$

it is then easy to check that $f(\cdot)$ satisfies (a), (b) and (c).

"$\Leftarrow$" Assume that $o(S) \le \alpha$ and that $f : S \to T$ satisfies (a), (b) and (c). First note that (a) and (c) imply that $f$ is length preserving, i.e. that $|f(s)| = |s|$ for $s \in \mathbb{N}$ (by induction on length of $s$. For $n \in S \cap \mathbb{N}$ it follows that $k(n) = f(n) \in \mathbb{N}$, and that the map

$$\begin{aligned} &f_n : S(n) \to T(k(n)), \\ &(s_1, \dots s_\ell) \mapsto (t_1, t_2 \dots, t_\ell), \text{ so that } f(n, s_1, \dots s_\ell) = (k(n), t_1, t_2 \dots, t_\ell) \end{aligned}$$

(note that by (b) $f(n, s_1, \dots s_\ell) \succ f(n) = k(n)$ if $\ell \geq 1$) satisfies (a), (b) and (c). By Proposition 2.5 (c) it follows for all $n \in S \cap \mathbb{N}$ that $o(S((n)) < \alpha$, and since $f_n$ satisfies (a), (b) and (c) for the trees $S(n)$ and $T(k(n))$, we deduce from the induction hypothesis and again from Proposition 2.5 (c) that for some $\beta < \alpha$

$$o(S) = \beta + 1 \iff \sup_{n\in\mathbb{N}} o(S(n)) = \beta \Rightarrow \sup_{n\in\mathbb{N}} o(T(k(n))) \geq \beta \Rightarrow o(T) \geq \beta + 1,$$

which finishes the proof.

If $o(S) = \omega_1$, then $S$ is not well founded by Propostion 2.2 and we find a sequence $(m_i)_{i\in\mathbb{N}} \subset \mathbb{N}$ so that $(m_1, m_2 \dots, m_\ell) \in S$ for all $\ell$. So if there is a map $f : S \to T$ satisfying (a), (b) and (c). it follows that for some sequence $(n_i)_{i\in\mathbb{N}} \subset \mathbb{N}$and any $\ell \in \mathbb{N}$ we have $f(m_1, m_2 \dots, m_\ell) = (n_1, n_2 \dots, n_\ell)$, thus, $T$ is not well founded either.

Conversely if $\omega_1 = o(T) \geq o(S)$, and if $(n_i)_{i\in\mathbb{N}} \subset \mathbb{N}$ is a sequence such that $(n_1, n_2 \dots, n_\ell) \in T$, then the map $f : S \to T$, $s \mapsto (n_1, \dots, n_{|s|})$ (with $f(\emptyset) = \emptyset$) satsifies (a), (b), and (c).

In order to verify the second part of our claim we observe that

$$\begin{aligned}
o(S) < o(T) &\iff o(S) + 1 \leq o(T) \\
&\iff o(S) \leq \sup_{n\in\mathbb{N}} o(T(n)) \\
&\quad\text{[By Proposition 2.5 (c)]} \\
&\iff \exists n\in\mathbb{N} \quad o(S) \leq o(T(n)) \\
&\quad\text{[By Propsoition 2.5 (a) } o(S) \text{ is a successor ordinal]} \\
&\iff \exists n\in\mathbb{N}\, \exists f : S \to T(n) \quad f \text{ satisfies (a), (b) and (c)} \\
&\quad\text{[First part of Proposition].}
\end{aligned}$$

□

Remark 2.9. Let $X$ be a topological space whose topology we denote by $\mathcal{T}(X)$. We can then consider on $X^{<\mathbb{N}}$ the topology $\mathcal{T}(X^{<\mathbb{N}})$ defined by

$$\mathcal{T}(X^{<\mathbb{N}}) = \{U \subset X^{<\mathbb{N}} : \forall n\in\mathbb{N} \quad U \cap X^n \text{ is open in } X^n\}.$$

(where $X^n$ is endowed with the product topology). Then it follows that $A \subset X^{<\mathbb{N}}$ is closed with respect to $\mathcal{T}(X^{<\mathbb{N}})$ if and only if $A \cap X^n$ is closed in $X^n$ for all $n \in \mathbb{N}$. We also deduce that $X^{<\mathbb{N}}$ is separable if and only if $X$ has that property and that $S^{<\mathbb{N}}$ is dense in $X^{<\mathbb{N}}$ if and only if $S$ is dense in $X$.

Moreover, if $X$ is a metrizable with a metric $d$ then we can define for $x = (x_1, \dots x_m)$ and $y = (y_1, \dots y_n)$ in $X^{<\mathbb{N}}$

$$d(x, y) = \begin{cases} \sum_{i=1}^n d(x_i, y_i) & \text{if } n = m \\ 1 & \text{else} \end{cases}$$

Then $d(\cdot,\cdot)$ is a metric on $X^{<\mathbb{N}}$ which generates $\mathcal{T}(X^{<\mathbb{N}})$ and it is easy to see that for $(x^{(n)}) \subset X^{<\mathbb{N}}$, if $n \in \mathbb{N}$ and $x \in X^{<\mathbb{N}}$

$$d - \lim_{n\to\infty} x^{(n)} = x \iff \exists n_0 \in \mathbb{N}\ \forall n \geq n_0 \quad |x^{(n)}| = |x^{(n_0)}| = |x| \text{ and}$$
$$\lim_{n\to\infty, n\geq n_0} x^{(n)} = x \text{ in } X^{|x|}.$$

THEOREM 2.10. *If $T$ is a well-founded closed tree on a Polish space $X$ (separable and completely metrizable) then $o(T) < \omega_1$.*

For the proof of Theorem 2.10 we will need the following lemma.

LEMMA 2.11. *Assume $T$ is a well founded and closed tree on a Polish space $X$. Then*

$$\overline{D(T)} = T \iff D(T) = T.$$

PROOF. $\Leftarrow$ is trivial. Assume that $D(T) \neq T$ but $\overline{D(T)} = T$. By induction we will choose for $n \in \mathbb{N}$ $x(n,1), x(n,2), \ldots, x(n,\ell_n)$ in $X$ so that

a) $\ell_n > \ell_{n-1}$ for $n \in \mathbb{N}$ ($\ell_0 = 0$),
b) $x^{(n)} = (x(n,1), x(n,2), \ldots, x(n,\ell_n)) \in T \setminus D(T)$,
c) $d\big((x(n-1,1), x(n-1,2), .., x(n-1,\ell_{n-1})), (x(n,1), x(n,2), .., x(n,\ell_{n-1}))\big) < 2^{-n}$ if $n \geq 2$.

For $n = 1$ we simply choose an $x^{(1)} = (x(1,1), x(1,2), \ldots, x(1,\ell_1)) \in T \setminus D(T)$.

If we assume $x^{(n)} \in T \setminus D(T)$ has been chosen, we can first choose $y^{(n+1)} = (x(n+1,1), x(n+1,2), \ldots, x(n+1,\ell_n)) \in D(T)$ with $d(x^{(n)}, y^{(n+1)}) < 2^{n+1}$ (since $x^{(n)} \in \overline{D(T)}$) and then (since $T$ is well-founded) extend $y^{(n+1)}$ to a maximal element $x^{(n+1)} = (x(n+1,1), x(n+1,2), \ldots, x(n+1,\ell_{n+1}))$ in $T$ (and thus $x^{(n+1)} \notin D(T)$).

This finishes the choice of $\ell_n$ and $x^{(n)} = (x(n,1), x(n,2), \ldots x(n,\ell_n))$.

From (c) and the completeness of $X$ we deduce that for each $i \in \mathbb{N}$ the sequence $(x(n,i))_{n\in\mathbb{N}}$ converges to some $x_i$ in $X$. Since $T$ is closed it follows that for any $n \in \mathbb{N}$ $(x_1, x_2, \ldots x_n)$ is in $T$ which is a contradiction to the assumption that $T$ is well-founded. □

PROOF OF THEOREM 2.10. By Remark 2.9 $X^{<\mathbb{N}}$, is separable and metrizable and therefore a Lindelöf space, meaning that every open cover of has a countable sub cover. Since every subset of $X^{<\mathbb{N}}$ is also separable and metrizable, and, thus, Lindelöf, we deduce that every decreasing family $(A_\alpha)_{\alpha<\omega_1}$ of closed subsets of $X$ has to be eventually stationary. We apply this observation to $A_\alpha = \overline{T^\alpha}$, if $\alpha < \omega_1$.

From Lemma 2.11 we deduce that the family of closed sets $(\overline{T^\alpha})$ is decreasing and, if $\alpha < \beta < \omega_1$ with $T^\alpha \neq \emptyset$ then it follows that $\overline{T^\beta} \subsetneq \overline{T^\alpha}$.

We therefore deduce from the Lindelöf property that $T^\alpha = \emptyset$ for some $\alpha < \omega_1$. □

EXAMPLE 2.12. The following example shows that in Theorem 2.10 we need some assumption on the topology of $X$.

Take $X = [0,\omega_1]$ with its order topology (compact but not second countable) and define

$$T = \{(\alpha_1, \alpha_2, \dots \alpha_n) : \alpha_1 > \alpha_2 > \dots \alpha_n\} \subset [0, \omega_1]^{<\mathbb{N}}.$$

Since $[0, \omega_1]$ does not contain any infinite strictly decreasing sequences, we deduce that $T$ is well founded. On the other hand we easily prove by transfinite induction that for $\alpha < \omega_1$

$$T^\alpha = \{(\alpha_1, \alpha_2, \dots \alpha_n) : \alpha_1 > \alpha_2 > \dots \alpha_n \geq \alpha\}$$

which implies that $o(T) \geq \omega_1$.

A map $f : T \to T'$ between two trees $T$ and $T'$ is called a *tree isomorphism* if $f$ is one to one, onto, and an order isomorphism (if $x, y \in T$ with $x < y$ then $f(x) < f(y)$).

## 3. Addition of trees

PROPOSITION 3.1. *(Tree addition)*
*Assume that $S$ and $T$ are trees, and that $o(S) < \omega_1$. We define:*

$$S \oplus T := S \cup \{(s, t) : s \text{ maximal in } S,\, t \in T\}$$

*and add to relations in $S \oplus T$ the following additional ones.*

$$s < (s, t) < (s, t'), \text{ whenever } s \in S,\, t, t' \in T, \text{ with } t < t'.$$

*Then $S \oplus T$ is a tree with*

$$o(S \oplus T) = o(T) + o(S).$$

PROOF. It is clear that $S \oplus T$ is a tree and since $T$ is isomorphic to a subtree of $S \oplus T$ it follows that $o(S \oplus T) = \omega_1$ if $o(T) = \omega_1$.

Assume therefore that $\beta = o(T) < \omega_1$.

First note that $(S \oplus T)^\beta \subset S \oplus T^\beta = S$, which implies $(S \oplus T)^{\beta+\alpha} \subset S^\alpha = \emptyset$, and, thus, $o(S \oplus T) \leq \beta + \alpha$.

We will show by transfinite induction on $\alpha = o(S)$ that $o(S \oplus T) \geq \beta + \alpha$. If $\alpha = 1$ this follows from Proposition 2.4 (a)

Assume that the claim is true for all $\gamma < \alpha$. If $\alpha = \gamma + 1$ there is an initial node $s_0$ of $S$ so that $o(\text{Succ}(S, s_0)) = \gamma$ (recall that $\text{Succ}(S, s_0)$ denotes the successors of $s_0$ in $S$). Since $(\{s_0\} \oplus \text{Succ}(S, s_0) \oplus T = \{s_0\} \oplus (\text{Succ}(S, s_0) \oplus T)$ and since $\{s_0\} \oplus S(s_0)$ is tree isomorphic to $\{s_0\} \cup \text{Succ}(S, s_0)$, it follows from Proposition 2.4 (a)

$$o(S \oplus T) \geq o(\{s_o\} \oplus (S(s_0) \oplus T) \geq \beta + \gamma + 1 = \beta + \alpha.$$

If $\alpha = \sup_{\gamma<\alpha} \gamma$ it follows from Proposition 2.4 (e) and our induction hypothesis that (denote the initial elements of $S$ by $I$)

$$o(S \oplus T) = o\Big(\big[\bigcup_{x\in I} \{x\} \cup \text{Succ}(x, S)\big] \oplus T\Big) \leq \sup_{x\in I} o\Big(\{x\} \cup S(x) \oplus T\Big) \leq \beta + \alpha.$$

$\square$

## 4. Souslin Operation

DEFINITION 4.1. (Souslin Operation)
Let $A, X$ be sets ($A$ is usually either $\mathbb{N}$ or a finite subset of it). A family $\mathcal{G} = (G_s : s \in A^{<\mathbb{N}})$ is called *regular* if $G_t \subset G_s$ whenever $t \succ s$.

Let $\mathcal{G} = (G_s : s \in A^{<\mathbb{N}})$ be a family of subsets of $X$ We define

$$\mathcal{A}_A(\mathcal{G}) = \bigcup_{\alpha \in A^{\mathbb{N}}} \bigcap_{n \in \mathbb{N}_0} G_{\alpha|n}.$$

We view $\mathcal{A}_A(\cdot)$ as an operation (or map) on $[[X]] = \{\mathcal{F} : \mathcal{F} \subset [X]\}$,

$$\mathcal{A}_A(\cdot) : \{\mathcal{F} : \mathcal{F} \subset [X]\} \to \{\mathcal{F} : \mathcal{F} \subset [X]\}, \quad \text{with}$$
$$\mathcal{A}_A(\mathcal{F}) = \Big\{ \bigcup_{\alpha \in A^{\mathbb{N}}} \bigcap_{n \in \mathbb{N}_0} G_{\alpha|n} \,:\, (G_s : s \in A^{\mathbb{N}}) \subset \mathcal{F} \Big\}.$$

The operation $\mathcal{A}_{\mathbb{N}}(\cdot) : [X]^{N^{<\mathbb{N}}} \to [X]$ is called the *Souslin Operation.*

PROPOSITION 4.2. *Let $\mathcal{G} = (G_s : s \in A^{<\mathbb{N}})$ be a family of subsets of $X$ and put*

$$B = \bigcap_{k \in \mathbb{N}_0} \bigcup_{s \in \mathbb{N}^{<\mathbb{N}}, |s|=k} G_s \times \{\alpha \in A^{\mathbb{N}} : \alpha \succ s\}.$$

*Then $\mathcal{A}_A(\mathcal{G}) = \pi_X(B)$, where $\pi_X : X \times \mathbb{N}^{\mathbb{N}} \to X$, is the projection onto $X$.*

PROOF. Note that

$$\begin{aligned} x \in \pi_X(A) &\iff \exists \alpha \in \mathbb{N}^{\mathbb{N}} \quad (x, \alpha) \in A \\ &\iff \exists \alpha \in \mathbb{N}^{\mathbb{N}}\ \forall k \in \mathbb{N}\ \exists s \in \mathbb{N}^k, s \prec \alpha \quad x \in G_s \text{ and } \alpha \succ s \\ &\iff \exists \alpha \in \mathbb{N}^{\mathbb{N}}\ \forall k \in \mathbb{N}\ x \in G_{\alpha|k} \iff x \in \mathcal{A}_A(\mathcal{G}). \end{aligned}$$

□

PROPOSITION 4.3. *If $\mathcal{F}$ is a family of of subsets of $X$ then*

$$\mathcal{F}, \mathcal{F}_\sigma, \mathcal{F}_\delta \subset \mathcal{A}_A(\mathcal{F}) = \{\mathcal{A}_A(\mathcal{G}) : \mathcal{G} = (G_s : s \in A^{<\mathbb{N}}) \in \mathcal{F}^{\mathbb{N}^{<\mathbb{N}}}\}$$

PROOF. For $\mathcal{F}$ and $A \in \mathcal{F}$ simply choose $G_s = A$ for all $s \in A^{<\mathbb{N}}$.

Let $(A_n) \subset \mathcal{F}$. Put for $s = (s_1, \dots s_n) \in A^{<\mathbb{N}}$ put $G_s = A_{s_n}$. Then it follows that $\mathcal{A}_A((G_s)) = \bigcup_{n \in \mathbb{N}_0} A_n$.

Letting for $n \in \mathbb{N}_0$ $G_{(1,2,\dots n)} = A_n$ and $G_s = \emptyset$, whenever $s \not\in \{(1, 2, \dots, n) : n \in \mathbb{N}_0\}$ we deduce $\mathcal{A}_A((G_s)) = \bigcap_{n \in \mathbb{N}_0} A_n$. □

LEMMA 4.4. *Assume that $\mathcal{G} = (G_s : s \in A^{<\mathbb{N}})$ has the property that $A_s \cap A_t = \emptyset$, whenever $s, t \in A^{<\mathbb{N}}$ are incomparable.*

*Then*

$$\mathcal{A}_A(\mathcal{G}) = \bigcap_{k \in \mathbb{N}_0} \bigcup_{s \in \mathbb{N}^k} G_s.$$

PROOF. Note that

$$\begin{aligned}
x \in \mathcal{A}_A(\mathcal{G}) &\iff \exists \alpha \in \mathbb{N}^{\mathbb{N}} \quad x \in \bigcap_{k\in\mathbb{N}_0} G_{\alpha|_k} \\
&\iff \forall k \in \mathbb{N}_0\ \exists s \in \mathbb{N}^k \quad x \in G_s \\
&\left[\begin{array}{ll} \text{“}\Rightarrow\text{”} & \text{Choose } s_k = \alpha|_k \text{ for } k \in \mathbb{N}_0 \\ \text{“}\Leftarrow\text{”} & \text{Let } s_k \in \mathbb{N}^{\mathbb{N}} \text{ so that } x \in G_{s_k} \text{ for } k \in \mathbb{N}_0 \\ & \text{then it follows } s_1 \prec s_2 \prec s_3 \ldots, \text{ since} \\ & G_s \cap G_t = \emptyset, \text{ if } s \text{ and } t \text{ incomparable} \end{array}\right] \\
&\iff x \in \bigcap_{k\in\mathbb{N}_0} \bigcup_{s\in\mathbb{N}^k} G_s.
\end{aligned}$$

□

LEMMA 4.5. *Assume that $A$ is finite and that $\mathcal{G} = (G_s : s \in A^{<\mathbb{N}})$ is regular. Then*

$$\mathcal{A}_A(\mathcal{G}) = \bigcap_{k\in\mathbb{N}_0} \bigcup_{s\in\mathbb{N}^k} G_s.$$

PROOF. Note that by the proof of Lemma 4.4 it always follows that

$$\mathcal{A}_A(\mathcal{G}) \subset \bigcap_{k\in\mathbb{N}_0} \bigcup_{s\in\mathbb{N}^k} G_s.$$

Let $x \in \bigcap_{k\in\mathbb{N}_0} \bigcup_{s\in\mathbb{N}^k} G_s$ and consider

$$T = \big\{s \in \mathbb{N}^{<\mathbb{N}} : x \in G_s\big\}.$$

since $\mathcal{G}$ is regular it follows that $T$ is a tree and by assumption the set $S = \{s \in \mathbb{N}^{<\mathbb{N}} x \in G_s\}$ is infinite. Since $A$ is finite we can choose by induction $a_1, a_2, \ldots$ in $A$ so that $S_n = \{s \in \mathrm{Succ}(a_1, \ldots a_n) \,\&\, x \in G_s\}$ is infinite for every $n \in \mathbb{N}_0$. Thus, it follows that $x \in \bigcap_{n\in\mathbb{N}_0} G_{(a_1,a_2\ldots a_n)}$ which implies our claim. □

THEOREM 4.6. *The operation $\mathcal{A}_{\mathbb{N}}(\cdot)$ is idempotent, this means that if $\mathcal{F} \subset [X]$ then $\mathcal{A}_{\mathbb{N}}(\mathcal{A}_{\mathbb{N}}(\mathcal{F})) = \mathcal{A}_{\mathbb{N}}(\mathcal{F})$.*

PROOF. By Proposition 4.3 $\mathcal{A}_{\mathbb{N}}(\mathcal{F}) \subset \mathcal{A}_{\mathbb{N}}(\mathcal{A}_{\mathbb{N}}(\mathcal{F}))$. Conversely, assume

$$A = \bigcup_{\alpha^{\mathbb{N}}} \bigcap_{k\in\mathbb{N}_0} A_{\alpha|_k} \text{ and } A_s = \bigcup_{\beta\in\mathbb{N}^{\mathbb{N}}} \bigcap_{\ell\in\mathbb{N}_0} B(s, \beta|\ell), \text{ where } (B(s,t) : s, t \in \mathbb{N}^{<\mathbb{N}}) \subset \mathcal{F}.$$

First we note that

$$A = \bigcup_{\alpha\in\mathbb{N}^{\mathbb{N}}} \bigcap_{k\in\mathbb{N}_0} \bigcup_{\beta\in\mathbb{N}^{\mathbb{N}}} \bigcap_{\ell\in\mathbb{N}_0} B(\alpha|_k, \beta|_\ell) = \bigcup_{\alpha\in\mathbb{N}^{\mathbb{N}}} \bigcup_{(\beta^{(p)})\in(\mathbb{N}^{\mathbb{N}})^{\mathbb{N}_0}} \bigcap_{k,\ell\in\mathbb{N}_0} B(\alpha|_k, \beta^{(k)}|_\ell). \tag{4}$$

Advice to the reader: He or she might first think about how to continue on his or her own. The problem will be to choose to each pair $u \in \mathbb{N}^{<\mathbb{N}}$ a set $C(u) \in \mathcal{F}$ so that

$$\bigcup_{\gamma\in\mathbb{N}^{\mathbb{N}}} \bigcap_{d\in\mathbb{N}_0} C(\gamma|_d) = \bigcup_{(\alpha,\beta^{(p)})\in\mathbb{N}^{\mathbb{N}}\times(\mathbb{N}^{\mathbb{N}})^{\mathbb{N}_0}} \bigcap_{k,\ell\in\mathbb{N}_0} B(\alpha|_k, \beta^{(k)}|_\ell).$$

Let $(I_p)_{p=0}^\infty$ be a partition of $2\mathbb{N}+1$ into infinitely many sets of infinite cardinality. Write $I_p = \{i_j^{(p)} : j \in \mathbb{N}\}$, with $i_1^{(p)} < i_2^{(p)} < \dots$. For $u = (u_1, \dots u_n)$ and $I = \{i_1, i_2, \dots\} \subset \mathbb{N}$, with $i_1 < i_2 < \dots$, we put $u|_I = (u_{i_j} : j \in \mathbb{N} \,\&\, i_j \le n)$. If $\ell \le n$ we put $(u|_I|_\ell) = (u_{i_j} : j \le \ell, i_j \le n)$. Similarly we define $\gamma|_I$ and $\gamma|_I|_\ell$ if $\gamma \in \mathbb{N}^\mathbb{N}$.

We define an injection from $\mathbb{N}^\mathbb{N} \times (\mathbb{N}^\mathbb{N})^\mathbb{N}$ and $\mathbb{N}^\mathbb{N}$ by

$$\Phi\big(\alpha,(\beta^{(p)})_{p\in\mathbb{N}_0}\big) = \gamma, \text{ where for } i \in \mathbb{N}$$

$$\gamma_i = \begin{cases} \alpha_{i/2} & \text{if } i \in 2\mathbb{N} \\ \beta_j^{(p)} & \text{if } i = i_j^{(p)} \text{ for some (unique) } p \in \mathbb{N} \text{ and } j \in \mathbb{N}. \end{cases}$$

Secondly, let

$$\big(\kappa(\cdot), \lambda(\cdot)\big) : \mathbb{N} \to \mathbb{N}\times\mathbb{N}$$

be map so that for each $k, \ell \in \mathbb{N}$ the set $\{d : (\kappa(d), \lambda(d)) = (k, \ell)\}$ is infinite, and so that $d \ge \max(\kappa(d), \lambda(d)$, for $d \in \mathbb{N}$. Finally for $u \in \mathbb{N}^{<\mathbb{N}}$ put

$$C(u) = B(u|_{2\mathbb{N}}|_{\kappa(|u|)}, u|_{I_{\kappa(|u|)}}|_{\lambda(|u|)})).$$

Note that if $\big(\alpha, (\beta^{(p)})_{p\in\mathbb{N}_0}\big) \in \mathbb{N}^\mathbb{N} \times (\mathbb{N}^\mathbb{N})^{\mathbb{N}_0}$ and $\gamma = \Phi\big(\alpha, (\beta^{(p)})_{p\in\mathbb{N}_0}\big)$, and any $k, \ell \in \mathbb{N}$ by choosing $d$ with $(\kappa(d), \lambda(d)) = (k, \ell)$, and $d$ large enough so that the length of $\gamma|_d|_{I_k}$ is at least $\ell$, then

$$C(\gamma|_d) = B\big(\gamma|_d|_{2\mathbb{N}}|_{\kappa(d)}, \gamma|_d|_{I_{\kappa(d)}}|_{\lambda(d)}\big) = B\big(\alpha|_{\kappa(d)}, \beta^{(\kappa(d))}|_{\lambda(d)}\big) = B\big(\alpha|_k, \beta^{(k)}|_\ell\big).$$

On the other hand for any $d \in \mathbb{N}$, $C(\gamma|_d) = B\big(\alpha|_{\kappa(d)}, \beta^{(\kappa(d))}|_{\lambda(d)}\big)$, and thus

$$\bigcap_{d\in\mathbb{N}_0} C(\gamma|d) = \bigcap_{(\alpha,(\beta^{(p)})_{p\in\mathbb{N}_0})\in\mathbb{N}^\mathbb{N}\times(\mathbb{N}^\mathbb{N})^{\mathbb{N}_0}} \bigcap_{k,\ell\in\mathbb{N}_0} B_{\alpha|k,\beta^{(k)}|_\ell}.$$

Since the map $\Phi$ is a bijection we deduce our claim. □

## 5. A combinatorial Lemma for trees.

Let $T \subset A^{<\mathbb{N}}$ a tree on the set $A$.

LEMMA 5.1. *Let $T \subset A^{<\mathbb{N}}$ a tree on the set $A$ and let $(s_n) \subset T$. Then there exists an infinite subsequence $(t_n)$ of $(s_n)$ so that one of the following three cases holds:*
*Case 1: Either there is a finite sequence $(a_i)_{i=1}^\ell$ in $A$ ($\ell$ could be 0) so that $t_n = (a_1, \dots, a_\ell)$ for all $n \in \mathbb{N}$.*
*Case 2: Or there is a finite sequence $(a_i)_{i=1}^\ell$ in $A$ ($\ell$ could be 0) and an infinite sequence $(\tilde t_n)$, say $\tilde t_n = \big(\tilde a(n,1), \tilde a(n,2), \dots, \tilde a(n,\ell_n)\big)$, for $n \in \mathbb{N}$, so that for all $n \in \mathbb{N}$*

a) $t_n = (\tilde a_1, \dots, \tilde a_\ell, \tilde t_n)$,
b) $\tilde a(i,1) \ne \tilde a(j,1)$ *whenever* $i \ne j$.

*Case 3: Or there is an infinite sequence $(a(i)) \subset A$, an infinite sequence $(\tilde t_n) \subset A^{<\mathbb{N}}$, say $\tilde t_n = \big(\tilde a(n,1), a(n,2), \dots a(n,\ell_n)\big)$, for $n \in \mathbb{N}$, and an increasing sequence $m_n \subset \mathbb{N}$ so that*

(i) $t_n = \big(a(1), a(2), \dots, a(m_n), \tilde t_n\big)$
(ii) $\tilde a(n,1) \ne a(m_n+1)$, *if* $\ell_n \ne 0$.

PROOF. Write $s_n = (s(n,1), s(n,2), \ldots, s(n,k_n))$. Inductively we proceed for $\ell = 1, 2, \ldots$ as follows:

For $\ell = 1$: Either there is an infinite subsequence $(t_i)$ of $(s_i)$ for which $t_i = \emptyset$, if $i \in \mathbb{N}$. Then we are in case 1 and finished.

Or there is an infinite subsequence $N_1 \subset \mathbb{N}$ so that $a(m,1) \neq a(n,1)$ whenever $m, n \in N$. Then we choose $(a(i)) = \emptyset$ and $(t_i) = (\tilde{t}_i) = (s_{u_i})$ with $N_1 = \{u_1, u_2, \ldots\}$ and $u_1 < u_2 < \ldots$. We are therfore in Case 2 and would be done.

Otherwise there is an $a(1) \in A$ so that $N_1 = \{n \in \mathbb{N} : s(n,1) = a(1)\}$ is infinite. We then choose $N_2 \in [N_1]$ so that either $k_n = 1$ (and thus $s_n = a(1)$) for all $n \in \mathbb{N}_2$ which would imply that we are in Case 1), or $k_n \geq 2$, for all $n \in \mathbb{N}_2$, and $s(i,2) \neq s(j,2)$, for all $i \neq j$ in $N_2$, which would imply that we are in Case 2, or there is an $a(2) \in A$ so that for all $i \in N_2$ $s(i,2) = a(2)$. We continue this way until we either arrive to one of the first two cases (i.e. there is an $\ell \in \mathbb{N}$ and an infinite $N_\ell \subset N_{\ell-1}$ so that $s_n = (a(1), a(2) \ldots a(\ell))$ for all $n \in \mathbb{N}_\ell$, or $k_i \geq \ell + 1$ and $s(i, \ell+1) \neq s(j, \ell+1)$ for $i \neq j$ in $N_\ell$) or we obtain infinite sets $\mathbb{N} \supset N_1 \supset N_2 \ldots$ an infinite sequence $a(1), a(2), a(3) \ldots$ in $A$ so that for any $\ell \in \mathbb{N}$ and any $n \in \mathbb{N}_\ell$, $s(n,j) = a(j)$ for $j = 1, 2, \ldots \ell$.

In the later case we proceed as follows: choose $n_1 = \min N_1$ put

$$m_1 = \max\big\{m : a(1) = s(n_1, 1), a(2) = s(n_1, 2), \ldots a(m) = s(n_1, m)\big\}$$

(note that $m_1 \geq 1$) and write $t_1 = s_{n_1} = (a(1), a(2), \ldots, a(m_1), \tilde{t}_1)$ where $\tilde{t}_1 \in A^{<\mathbb{N}}$ is either empty or starts with an element different from $a(m_1+1)$. Then we choose $n_2 = \min N_{m_1+1}$ and put again $m_2 = \max\{m : a(1) = s(n_2, 1), a(2) = s(n_2, 2), \ldots a(m) = s(n_2, m)$ (note that $m_2 > m_1$) and write $t_2 = s_{n_2} = \big(a(1), a(2), \ldots a(m_2), \tilde{t}_2\big)$ where where $\tilde{t} \in A^{<\mathbb{N}}$ is either empty or starts with an element different from $a(m_2+1)$. We proceede in this manner and obtain a subsequence $(t_i)$ of $(s_i)$, a strictly increasing sequence $(m_i) \subset \mathbb{N}$ and a sequence$(\tilde{t}_i) \subset A^{<\mathbb{N}}$ satisfying the properties of Case 3. $\square$

CHAPTER 4

# Standard Borel spaces

## 1. Preliminaries

DEFINITION 1.1. Let $(X,d)$ be a metric space. The $\sigma$-algebra generated by the open sets of $X$ is called the *Borel $\sigma$ algebra* and denoted by $\mathcal{B}_{(X,d)}$, or, if there is no confusion, by $\mathcal{B}_X$.

A *standard Borel space* is a a measurable space $(X,\Sigma)$ which is Borel-isomorphic to a Borel subset of a Polish space $Y$, i.e. there is an $A\in\mathcal{B}_Y$ and a bimeasurable bijection $f:X\to A$.

We first recall several results on families which generate $\mathcal{B}_X$.

PROPOSITION 1.2. *Let $(X,d)$ be a metric space. Then $\mathcal{B}_X$ is the smallest family of subsets of $X$ which contains all open sets and is closed under countable intersections and unions.*

*$\mathcal{B}_X$ is also the smallest family of subsets of $X$ which contains all closed sets and is closed under countable intersections and unions.*

PROOF. Let $\mathcal{B}$ be the smallest family of subsets of $X$ which contains all open sets and is closed under countable intersections and unions. Clearly $\mathcal{B}\subset\mathcal{B}_X$.

In order to show the converse we need to verify that $\mathcal{B}\subset\mathcal{D}$, with

$$\mathcal{D}=\{A : A\in\mathcal{B} \text{ and } A^c\in\mathcal{B}\}.$$

Since every closed set $F$ is $\mathcal{G}_\delta$ (note that $F=\bigcap_{n\in\mathbb{N}}\bigcup_{x\in F}B_{1/n}(x)$) it follows that $\mathcal{T}_d\subset\mathcal{D}$. Secondly, $\mathcal{D}$ is closed under countable intersections and unions. Indeed, if $A_i\in\mathcal{D}$, for $i\in\mathbb{N}$, it follows that $A_i^c\in\mathcal{D}\subset\mathcal{B}$, for $i\in\mathbb{N}$, and since $\mathcal{B}$ is closed under countable intersections and unions we deduce that

$$\bigcap A_i\in\mathcal{B} \text{ and } \big(\bigcap A_i\big)^c=\bigcup A_i^c\in\mathcal{B}\qquad \bigcup A_i\in\mathcal{B} \text{ and } \big(\bigcup A_i\big)^c=\bigcap A_i^c\in\mathcal{B}$$

which implies that $\bigcap A_i$ and $\bigcup A_i$ are in $\mathcal{D}$.

The second part of the claim follows like the proof of the first since closed sets in $X$ are $\mathcal{F}_\sigma$. □

PROPOSITION 1.3. *Let $(X,d)$ be a metric space. Then $\mathcal{B}_X$ is the smallest family of subsets of $X$ which contains all open sets and is closed under countable intersections and countable disjoint unions.*

PROOF. Let $\mathcal{B}$ be the smallest family of subsets of $X$ which contains all open sets and is closed under countable intersections and countable disjoint unions. Since every closed set is a $\mathcal{G}_\delta$ set, the family

$$\mathcal{D} = \{A : A \in \mathcal{B} \text{ and } A^c \in \mathcal{B}\},$$

contains all open sets. It is left to show that $\mathcal{D}$ is closed under countable intersections and countable disjoint unions, which would imply that $\mathcal{B} \subset \mathcal{D}$ and, thus, that $\mathcal{B}$ is closed under taking complements.

If $A_i \in \mathcal{D}$ and thus $A_i^c \in \mathcal{D}$, for $i \in \mathbb{N}$, it follows for $i \in \mathbb{N}$ that $B_i = A_i \cap \bigcap_{j=1}^{i-1} A_j^c \in \mathcal{B}$ and $B_i^c = A_i^c \cup \bigcup_{j=1}^{i-1} A_j \in \mathcal{B}$ and thus $B_i \in \mathcal{D}$. Thus, $\bigcup_{i=1}^\infty A_i = \bigcup_{i=1}^\infty B_i \in \mathcal{B}$ and $\left(\bigcup_{i=1}^\infty A_i\right)^c = \bigcap_{i=1}^\infty A_i^c \in \mathcal{B}$, which yields that $\bigcup_{i=1}^\infty A_i \in \mathcal{D}$. Similarly we show that $\bigcap_{i=1}^\infty A_i \in \mathcal{D}$.

It follows that $\mathcal{B} \subset \mathcal{D}$. $\square$

An analogous result for closed instead of open sets is harder to show.

Proposition 1.4. [Sierpinski] *Let $(X, d)$ be a metric space. Then $\mathcal{B}_X$ is the smallest family of subsets of $X$ which contains all closed sets and is closed under countable intersections and countable disjoint unions.*

Before proving Proposition 1.4, we will need the following lemma.

Lemma 1.5. *Let $\mathcal{F}$ be the set of closed subsets of $\mathbb{R}$ then $(0,1] \in ((\mathcal{F}_+)_\delta)_+$.*

Proof. Let $\Delta$ the Cantor set in $[0,1]$ and $D$ the endpoints of the middle third intervals which one removes to obtain $\Delta$. More precisely:

$$\Delta = \bigcap_{n=0}^{\infty} \Delta_n \text{ where } \Delta_0 = [0,1]$$

$$\text{and assuming that for } n \in \mathbb{N}_0,\ \Delta_n = \bigcup_{i=1}^{2^n} [a_i^{(n)}, b_i^{(n)}],\ 0 = a_1^{(n)} < b_1^{(n)} < a_2^{(n)} < \dots b_{2^n}^{(n)} = 1$$

$$\text{we put } \Delta_{n+1} = \bigcup_{i=1}^{2^n} [a_i, b_i] \setminus \Big(a_i^{(n)} + \frac{b_i^{(n)} - a_i^{(n)}}{3}, a_i^{(n)} + 2\frac{b_i^{(n)} - a_i^{(n)}}{3}\Big).$$

$$D = \{a_i^{(n)}, b_i^{(n)} : n \in \mathbb{N}, 1 \le i \le 2^n\} \setminus \{0,1\} = \Big\{\frac{1}{3}, \frac{2}{3}, \frac{1}{9}, \frac{2}{9}, \frac{7}{9}, \frac{8}{9}, \dots\Big\}.$$

Then put

$$E = D \cup \{0\} \text{ and } P = \Delta \setminus E.$$

It follows that

$$\begin{aligned}(0,1] &= \Delta_0 \setminus \{0\} \\ &= \Big(\Delta_1 \setminus \Big\{0, \frac{1}{3}, \frac{2}{3}\Big\}\Big) \cup \Big[\frac{1}{3}, \frac{2}{3}\Big] \\ &= \Big(\Delta_2 \setminus \Big\{0, \frac{1}{3}, \frac{2}{3}, \frac{1}{9}, \frac{2}{9}, \frac{7}{9}, \frac{8}{9}\Big\}\Big) \cup \Big[\frac{1}{3}, \frac{2}{3}\Big] \cup \Big[\frac{1}{9}, \frac{2}{9}\Big] \cup \Big[\frac{7}{9}, \frac{8}{9}\Big] \\ &\vdots \\ &= P \cup \bigcup_{n=1}^{\infty} \bigcup_{i=1}^{2^{n-1}} \big[b_{2i-1}^{(n)}, a_{2i}^{(n)}\big] = \Big[\frac{1}{3}, \frac{2}{3}\Big] \cup \Big[\frac{1}{9}, \frac{2}{9}\Big] \cup \Big[\frac{7}{9}, \frac{8}{9}\Big] \dots .\end{aligned}$$

Therefore it is enough to show that $P \in (\mathcal{F}_+)_\delta$. Since $\Delta$ is zero dimensional (toology is generated by clopen sets) for each $x \in E$ the set $\Delta \setminus \{x\}$ (which is open in $\Delta$) is a countable union of clopen pairwise disjoint sets in $\Delta$. Indeed for every open set $U \subset \Delta$ there is a countable sequence $(V_n)$ of clopen set so that:

$$U = \bigcup_{n\in\mathbb{N}} V_n = \bigcup_{n\in\mathbb{N}} \Big[V_n \setminus \bigcup_{i=1}^{n-1} V_i\Big],$$

and note that the last union is a disjoint union of clopen sets.

Since clopen subsets of $\Delta$ are closed in $\mathbb{R}$ it follows that $\Delta \setminus \{x\} \in \mathcal{F}_+$. and thus since $E$ is countable

$$P = \bigcap_{x\in E} \Delta \setminus \{x\} \in (\mathcal{F}_+)_\delta.$$

□

Proof of Proposition 1.4. We need to show that an open set $U \subset X$ is in $\mathcal{B}$, the smallest family of subsets of $X$ which contains all closed sets and is closed under countable intersections and countable disjoint unions. The rest of the proof would then be similar to the proof of Proposition 1.3.

So let $U \subset X$ be open. W.l.o.g. $X \setminus U \neq \emptyset$. By Urysohn's Lemma (cf.[**Quer1**][Satz 7.]) there is a continuous map $f : X \to [0,1]$ so that $f^{-1}(\{0\}) = X \setminus U$ and thus $f^{-1}((0,1]) = U$. With Lemma 1.5 this implies that $U$ is in $\mathcal{F}_{+\delta+}$, and thus in $\mathcal{B}$, and $B_n \in \mathcal{T}'$ for $n \in \mathbb{N}$. □

## 2. Borel-Generated Topologies

The following two lemmas help reduce problems about measurability to topological problems.

Lemma 2.1. *Let $(X, \mathcal{T})$ be a metrizable space and $(B_n) \subset \mathcal{B}_{(X,\mathcal{T})}$. Then there is a metrizable topology $\mathcal{T}'$ such that $\mathcal{T} \subset \mathcal{T}' \subset \mathcal{B}_{(X,\mathcal{T})}$ and $(B_n) \subset \mathcal{T}'$ and (thus $\mathcal{B}_{(X,\mathcal{T})} = \mathcal{B}_{(X,\mathcal{T}')}$). If $(X, \mathcal{T})$ is zero dimensional, or second countable, $(X, \mathcal{T}')$ can be chosen to have the same property.*

Proof. Define

$$f : X \to X\times\Delta, \quad x \mapsto \big(x, (\chi_{B_i}(x))\big),$$

and let $\mathcal{T}' = \{f^{-1}(U) : U \subset X\times\Delta \text{ open}\}$, i.e. the coarsest topology on $X$ for which $f$ is continuous. Since $f$ is injective $(X, \mathcal{T}')$ is homeomorphic to a subset of $X \times \Delta$ and thus metrizable and zero dimensional or second countable if $X$ has this property. Since

$$B_n = f^{-1}\big(X\times\{\bar\varepsilon = (\bar\varepsilon_i) \in \Delta : \varepsilon_n = 1\}\big),$$

it follows that $B_n \in \mathcal{T}'$ for all $n \in \mathbb{N}$. □

Lemma 2.2. *Let $(X, d)$ be a metrizable space, $Y$ be a non empty Polish space, $A \subset X$ and $f : A \to Y$ Borel measurable.*

1) *There is a finer metrizable topology $\mathcal{T}'$ on $X$, which generates the same Borel $\sigma$-algebra, such that $f$ is continuous.*
2) *The map $f : A \to Y$ admits a Borel measurable extension $g : X \to Y$.*

Proof. Let $(U_n)$ be a basis for the toplogy on $Y$.
(1) Since $f$ is Borel there is for $n \in \mathbb{N}$ a Borel set $B_n \subset X$ so that $B_n \cap A = f^{-1}(U_n)$. Then apply Lemma. 2.1 to the sequence $(B_n)$.
(2) By Proposition 1.3 $f$ can be extended to $\mathcal{T}'$-continuous map $f' : A' \to Y$, where $A' \subset X$ is a $\mathcal{G}_\delta$-set with respect to $\mathcal{T}'$. Thus $A'$ is in $\mathcal{B}_{(X,\mathcal{T}_d)}$. Therefore we can define (let $y_0 \in Y$)

$$g : X \to Y, \quad x \mapsto \begin{cases} f'(x) & \text{if } x \in A' \\ y_0 & \text{if } x \notin A' \end{cases}.$$

□

Theorem 2.3. *Let $(X, \mathcal{T})$ be a Polish space and let $B \subset X$ be Borel. Then there is a finer Polish topology $\mathcal{T}_B$ on $X$ such that $B$ is clopen with respect to $\mathcal{T}_B$ and $\mathcal{B}_{(X,\mathcal{T})} = \mathcal{B}_{(X,\mathcal{T}_B)}$.*

Proof. Claim 1: If $B \subset X$ is closed there is a finer Polish topology $\mathcal{T}_B$ on $X$ such that $B$ is clopen with respect to $\mathcal{T}_B$ and $\mathcal{B}_{(X,\mathcal{T})} = \mathcal{B}_{(X,\mathcal{T}_B)}$.

Indeed, $(F, \mathcal{T} \cap F)$ as well as $(X \setminus F, \mathcal{T} \cap (X \setminus F)$ are Polish (by Theorem 1.2). By Example 1.1 (5)

$$(F, \mathcal{T} \cap F) \oplus (X \setminus F, \mathcal{T} \cap (X \setminus F)) = (X, \mathcal{T}'), \text{ with}$$
$$\mathcal{T}' = \{U \cup V : U \in \mathcal{T} \cap F \text{ and } V \in F^c, \mathcal{T} \cap F^c\}$$

is a Polish space which generates the same Borel sets as $\mathcal{T}$ and $F$is clopen in $(X, \mathcal{T}')$.
Claim 2. Let $\mathcal{T}_n$, $n \in \mathbb{N}$ be topologies on $X$ such that $\cap\mathcal{T}_n$ is separating points. Then the topology $\mathcal{T}_\infty$ generated by $\bigcup_{n\in\mathbb{N}} \mathcal{T}_n$ is Polish. Indeed, consider

$$f : X \to X^{\mathbb{N}}, \quad x \mapsto (x, x, x, \ldots).$$

Then $f$ is an embedding from $(X, \mathcal{T}_\infty)$ into $(X^{\mathbb{N}}, \prod \mathcal{T}_n)$ and $f$ has a closed image (for $(x_i) \notin f(X)$, let $m, n \in \mathbb{N}$, such that $x_m \neq x_n$, choose $U_m, U_n \in \cap\mathcal{T}_i$ disjoint, with $x_m \in U_m$ and $x_n \in U_n$, finally note that $\pi_m^{-1}(U_m) \cap \pi_n^{-1}(U_n)$ is a neighborhood of $(x_i)$ which is disjoint from range$(f)$.

In order to show the claim of our theorem we will show that the family $\mathcal{B}$ of sets $B \subset X$, for which there is a finer Polish topology $\mathcal{T}_B$ on $X$ such that $B$ is clopen with respect to $\mathcal{T}_B$ and $\mathcal{B}_{(X,\mathcal{T})} = \mathcal{B}_{(X,\mathcal{T}_B)}$, is a $\sigma$ algebra which contains the closed sets.

By Claim 1 $\mathcal{B}$ contains the closed sets. Secondly $\mathcal{B}$ is clearly closed under complementation. Finally, if $B_n \in \mathcal{B}$, for $n \in \mathbb{N}$ we apply Claim 2 to the sequence $\mathcal{T}_{B_n}$, to obtain $\mathcal{T}_\infty$. Since $B_n^c$ is clopen in $\mathcal{T}^\infty$ $(\bigcup B_n)^c = \bigcap B_n^c$ is closed in $\mathcal{T}^\infty$. Applying Claim 1 again to $\mathcal{T}_\infty$ we obtain a Polish topology $\mathcal{T}_{\bigcup B_n}$ on $X$, which is finer than $\mathcal{T}$, generates the same Borel $\sigma$-algebra and in which $\bigcup B_n$ is clopen. □

Using Claim 2 in the proof of Theorem 2.3 we obtain the following result.

Corollary 2.4. *Suppose $(X,\mathcal{T})$ is Polish and $(B_n)$ is a sequence of Borel sets in $X$. Then there is a finer Polish topology $\mathcal{T}'$ on $X$ generating the same Borel sets as $(X,\mathcal{T})$ so that all $B_n$ are clopen in $\mathcal{T}'$.*

Corollary 2.5. *Suppose $(X,\mathcal{T})$ is Polish, $Y$ a separable metric space and $f : X \to Y$ a Borel map. Then there is a finer Polish topology $\mathcal{T}'$ on $X$ generating the same Borel sets so that $f$ is contiunous.*

Proof. Let $(V_n)$ a countable basis of the topology generated by the metric on $Y$, and apply Corollary 2.4 to $(B_n)$ with $B_n = f^{-1}(V_n)$. □

Theorem 2.6. *Every uncountable Borel subset of a Polish space $X$ contains a homeomorph of the Cantor set. In particular, it is of cardinality $c$.*

We first need the following lemma.

Lemma 2.7. *If $X$ is non empty, and if $\mathcal{T}$, $\mathcal{T}'$ are two topologies , so that $\mathcal{T} \subset \mathcal{T}'$, $(X,\mathcal{T}')$ is compact and $(X,\mathcal{T})$ is metrizable, then $\mathcal{T} = \mathcal{T}'$.*

Proof. Consider the identity $I$ on $X$. $I$ is $(\mathcal{T}',\mathcal{T})$ continuous, which implies that $(X,\mathcal{T})$ is also compact. Since $I(C)$ is compact for any $C \subset X$, which is compact in $(X,\mathcal{T}')$ it follows that $I$ is also $(\mathcal{T},\mathcal{T}')$-continuous because compact subsets of $(X,\mathcal{T})$ are closed ($(X,\mathcal{T})$ is Haudorff). Thus $\mathcal{T} = \mathcal{T}'$. □

Proof of Theorem 2.6. Let $B \subset X$ be uncountable and Borel. Let $\mathcal{T}'$ a finer topology as in Theorem 2.3, so that $B$ is $\mathcal{T}'$ closed. Then $(B,\mathcal{T}')$ is closed and Polish, and, thus, contains by Corollary 2.5 a homeomorphic copy of $\Delta$, say $K$. By Lemma 2.7 $(K,\mathcal{T}\cap K) = (K,\mathcal{T}'\cap K)$, which implies our claim. □

## 3. The Effros-Borel Space

Definition 3.1. Let $X$ be a Polish space and $\mathcal{F}(X)$ the set of all non empty closed subsets of $X$. The *Effros Borel space of* $X$ is the measurable space $(\mathcal{F}(X),\mathcal{E}(X))$, where $\mathcal{E}(X)$ is generated by the family

$$\big\{\{F \in \mathcal{F}(X) : F \cap U \neq \emptyset\} : U \subset X \text{ open } \big\}.$$

Theorem 3.2. *$(\mathcal{F}(X),\mathcal{E}(X))$ is a standard Borel space.*

Proof. Every second countable metrizable space can be embedded into the Hilbert cube $[0,1]^\omega$ we can therefore view $X$ as a dense subset of a compact metrizable space $Y$. By Theorem 1.5, $X$ is a $\mathcal{G}_\delta$ set in $Y$, and we can therefore write $X$ as $X = \bigcap_{n\in\mathbb{N}} U_n$, with $U_n \subset Y$ open for $n \in \mathbb{N}$.

Define

$$\mathcal{Z} := \big\{\overline{F}^Y : F \in \mathcal{F}(X)\big\}.$$

Note that $\mathcal{Z} \subset F(Y) = K(Y)$ (compact subsets of $Y$) and that

$$K \in \mathcal{Z} \iff K \cap X \text{ is dense in } K. \tag{5}$$

By Proposition 3.8 the Effros Borel space $\mathcal{K}(Y)$ coincides with the Borel $\sigma$-algebra on $\mathcal{K}(Y)$ (with respect to the Vietoris topology which by Corollary 3.6 is a Polish space).

Therefore it is enough to show that the map

$$\Phi : \mathcal{F}(X) \to \mathcal{Z}, \quad F \mapsto \overline{F},$$

is a Borel isomorphism, and that $\mathcal{Z}$ is a $G_\delta$ set in $\mathcal{K}(X)$.

If $F_1, F_2 \in \mathcal{F}(X)$, and $\overline{F}_1 = \overline{F}_2$, then $F_1 = X \cap \overline{F}_1 = X \cap \overline{F}_2 = F_2$. In order to show that $\Phi$ is measurable, by Proposition 3.8 we only need to show that for $U \subset Y$ open $\Phi^{-1}(K \in \mathcal{Z} : K \cap U \neq \emptyset)$ is measurable. Since by (5)

$$F \in \Phi^{-1}(K \in \mathcal{Z} : K \cap U \neq \emptyset) \iff \overline{F} \cap U \neq \emptyset \iff (\overline{F} \cap X) \cap (U \cap X) \neq \emptyset,$$

and $F = \overline{F} \cap X$ is $\overline{F} \in \mathcal{Z}$ it follows that $\Phi^{-1}(K \in \mathcal{Z} : K \cap U \neq \emptyset) = \{F \in \mathcal{F} : F \cap (U \cap X) \neq \emptyset\} \in \mathcal{E}(X)$. Since the sets of the form $\{F \in \mathcal{F} : F \cap V \neq \emptyset\}$, $V \subset X$ open, generate $\mathcal{E}(X)$, and since every for every open $V \subset X$ there is an open $U \subset Y$ so that $U \cap Y = V$, it follows similarly that $\Phi^{-1}$ is measurable.

By (5) and the Baire Category Theorem it follows that ($U_n$ as above and let $V_n$ be a countable base of topology of $Y$)

$$\begin{aligned}
K \in \mathcal{Z} &\iff K \cap X \text{ is dense in } K \\
&\iff K \cap \bigcap_{n\in\mathbb{N}} U_n \text{ is dense in } K \\
&\iff \forall n \in \mathbb{N} \quad U_n \cap K \text{ is dense in } K \\
&\iff \forall m, n \in \mathbb{N} \quad (K \cap V_m \neq \emptyset \Rightarrow K \cap V_m \cap U_n \neq \emptyset \\
&\iff K \in \bigcap_{m,n} \big\{K \in \mathcal{K}(Y) : K \cap V_m = \emptyset \text{ or } K \cap V_m \cap U_n \neq \emptyset\big\}
\end{aligned}$$

Our claim follows therefore from that fact that $\{K \in \mathcal{K}(Y) : K \cap V_m \cap U_n \neq \emptyset\}$ is open in $\mathcal{K}(X)$, and that

$$\{K \in \mathcal{K}(Y) : K \cap V_m = \emptyset\} = \{K \in \mathcal{K}(Y) : K \subset V_m^c\} = \bigcap_{\ell\in\mathbb{N}} \{K \in \mathcal{K}(Y) : K \subset (V_m^c)_{1/\ell}\}.$$

$\square$

The next two propositions prove the measurability of several sets and maps, involving the Effros-Borel space

PROPOSITION 3.3. *Let $X$ and $Y$ be Polish spaces.*

1) $\{(F_1,F_2)\in\mathcal{F}(X)\times\mathcal{F}(X) : F_1\subset F_2\}$ *is Borel.*
2) *The map* $\mathcal{F}(X)\times\mathcal{F}(X)\to\mathcal{F}(X)$, $(F_1,F_2)\mapsto F_1\cup F_2$ *is Borel measurable.*
3) *The map* $\mathcal{F}(X)\times\mathcal{F}(Y)\to\mathcal{F}(Y\times X)$, $(F_1,F_2)\mapsto F_1\times F_2$ *is Borel.*
4) *The set* $\{K\in\mathcal{F}(X) : K \text{ compact}\}$ *is Borel.*
5) *If* $g:X\to Y$ *is continuous, then* $\mathcal{F}(X)\to\mathcal{F}(Y)$, $F\mapsto\overline{g(F)}$ *is measurable.*
6) *For* $F_0\subset X$ *closed the sets* $\{F\in\mathcal{F} : F\subset F_0\}$ *and* $\{F\in\mathcal{F} : F_0\subset F\}$ *are Effros Borel measurable.*

PROOF. Let $\{U_n : n\in\mathbb{N}\}$ and $\{V_n : n\in\mathbb{N}\}$ be bases of $X$ and $Y$, respectively. (1) follows from

$$\begin{aligned}\{(F_1,F_2)\in&\mathcal{F}(X)\times\mathcal{F}(X) : F_1\subset F_2\}\\ &=\bigcap_{n\in\mathbb{N}}\{(F_1,F_2)\in\mathcal{F}(X)\times\mathcal{F}(X) : F_2\cap U_n=\emptyset\Rightarrow F_1\cap U_n=\emptyset\}\\ &=\bigcap_{n\in\mathbb{N}}\big[\{(F_1,F_2)\in\mathcal{F}(X)\times\mathcal{F}(X) : F_2\cap U_n\neq\emptyset\}\\ &\qquad\qquad\cup\{(F_1,F_2)\in\mathcal{F}(X)\times\mathcal{F}(X) : F_1\cap U_n=\emptyset\}\big]\end{aligned}$$

(2) Let $U\subset X$ be open

$$\begin{aligned}&\{(F_1,F_2) : (F_1\cup F_2)\cap U\neq\emptyset\}\\ &=\big[\{F_1\in\mathcal{F}(X :)F_1\cap U\neq\emptyset\}\times\mathcal{F}(X)\big]\cup\big[\mathcal{F}(X)\times\{F_2\in\mathcal{F}(X) : F_2\cap U\neq\emptyset\}\big].\end{aligned}$$

(3) Since $\{U_m\times V_n : m,n\in\mathbb{N}\}$ is a basis for the topology on $X\times Y$ we need to show that for $m,n\in\mathbb{N}$ the set

$$\big\{(F_1,F_2)\in\mathcal{F}(X)\times\mathcal{F}(Y) : (F_1\times F_2)\cap(U_m\times V_n)\neq\emptyset\big\}$$

is in $\mathcal{B}_{\mathcal{F}(X)}\otimes\mathcal{B}_{\mathcal{F}(Y)}$. Since

$$\begin{aligned}\big\{(F_1,F_2)\in&\mathcal{F}(X)\times\mathcal{F}(Y) : (F_1\times F_2)\cap(U_m\times V_n)\neq\emptyset\big\}\\ &=\{F_1\in\mathcal{F}(X) : F_1\cap U_m\neq\emptyset\}\times\{F_2\in\mathcal{F}(Y) : F_1\cap V_n\neq\emptyset\}\end{aligned}$$

this follows easily.

(4) Since in Polish spaces the compact subsets are the closed and totally bounded sets we can choose a a countable dense set $D\subset X$ and write

$$\{F\in\mathcal{F}(X) : F \text{ compact}\}=\bigcap_{\ell\in\mathbb{N}}\bigcup_{A\in[X]^{<\mathbb{N}}}\big\{F\in\mathcal{F}(X) : F\subset\overline{\bigcup_{a\in A}B_{1/\ell}(a)}\big\}$$

and for $A\in[X]^{<\mathbb{N}}$

$$\Big\{F\in\mathcal{F}(X) : F\subset\overline{\bigcup_{a\in A}B_{1/\ell}(a)}\Big\}=\mathcal{F}(X)\setminus\Big\{F\in\mathcal{F}(X) : F\cap\Big[X\setminus\overline{\bigcup_{a\in A}B_{1/\ell}(a)}\Big]\neq\emptyset\Big\}.$$

(5) If $g : X \to Y$ is continuous, and $V \subset Y$ open

$$\begin{aligned}\big\{F \in \mathcal{F}(X) : \overline{g(F)} \cap V \neq \emptyset\big\} \\ &= \big\{F \in \mathcal{F}(X) : g^{-1}\big(\overline{g(F)} \cap V\big) \neq \emptyset\big\} \\ &= \big\{F \in \mathcal{F}(X) : g^{-1}\big(\overline{g(F)}\big) \cap g^{-1}(V) \neq \emptyset\big\} \\ &= \big\{F \in \mathcal{F}(X) : F \cap g^{-1}(V) \neq \emptyset\big\} \\ &\quad \text{[Since } g^{-1}(V) \text{ is open in } X \text{ for } V \subset Y \text{ open].}\end{aligned}$$

(6) For the first set note that

$$\{F \in \mathcal{F} : F \subset F_0\} = \mathcal{F} \setminus \{F \in \mathcal{F} : F \cap F_0^c \neq \emptyset\}.$$

Let $\{x_i :\in \mathbb{N}\}$ be dense in $F_0$, then the claim follows from the fact that

$$\{F \in \mathcal{F} : F_0 \subset F\} = \bigcap_{\varepsilon>0, \varepsilon\in\mathbb{Q}} \bigcap_{n\in\mathbb{N}} \{F \in \mathcal{F} : F \cap B_\varepsilon(x_i) \neq \emptyset\}$$

□

Proposition 3.4. *Let $X$ be a Polish space.*
*Then*

$$\big\{(x, F) \in X \times \mathcal{F}(X) :\ x \in F\big\}$$

*is measurable with respect to the product $\sigma$-algebra of $\mathcal{B}_X$ and the Effros-Borel space on $\mathcal{F}(X)$.*

Proof. Let $D \subset X$ be dense and countable. Note that

$$\big\{(x, F) \in X \times \mathcal{F}(X) :\ x \in F\big\} = \bigcap_{\varepsilon>0, \varepsilon\in\mathbb{Q}} \bigcup_{\xi\in D} B_\varepsilon(\xi) \times \{F \in \mathcal{F}(X) : B_\varepsilon(\xi) \cap F \neq \emptyset\}.$$

□

Proposition 3.5. *Put $\mathcal{F}_0(X) = \mathcal{F}(X) \cup \{\emptyset\}$ and endow it with the $\sigma$-algebra generated by*

$$\mathcal{E}_0 = \mathcal{E} \cup \{\emptyset\} = \big\{\{F \in \mathcal{F}_0(X) : F \cap U \neq \emptyset\} :\ U \subset X\big\}.$$

*Then*

$$\mathcal{F}(X) \times \mathcal{F}(X) \to \mathcal{F}_0(X), \quad (F, G) \mapsto F \cap G,$$

*is Borel measurable.*

Definition 3.6. Let $X$ be a set and $\mathcal{A} \subset [X] \setminus \{\emptyset\}$, a map $f : \mathcal{A} \to X$ with the property that $f(A) \in A$ is called a *selector*.

Theorem 3.7. (Selection Theorem of Kuratowski-Ryll-Nerdewiski)
*Let $X$ be Polish. Then there exists for each $n \in \mathbb{N}$ a Borel measurable selector $d_n : \mathcal{F}(X) \to X$, such that for each $F \in \mathcal{F}(X)$ the sequence $(d_n(F) : n \in \mathbb{N})$ is dense in $F$.*

PROOF. Let $\delta(\cdot,\cdot)$ be a complete metric on $X$ which generates the topology of $X$ and for which $\delta-\text{diam}(X)=1$. By the remark after Corollary 2.13 there is a Souslin scheme $(U_s : s\in\mathbb{N}^{<\mathbb{N}})$ consisting of open sets in $X$. For $F\in\mathcal{F}(X)$ note that the tree $T_F=\{s\in\mathbb{N}^{<\mathbb{N}} : U_s\cap F\neq\emptyset\}$ is a pruned tree (i.e. has no maximal elements). For $F\in\mathcal{F}(X)$ let $\alpha^{(F)}=(\alpha_n^{(F)} : n\in\mathbb{N})$ be the *leftmost branch of* $T_F$, i.e. by induction

$$\alpha_n^{(F)}=\min\{k\in\mathbb{N} : F\cap U_{(\alpha_1^{(F)},\dots\alpha_n^{(F)},k)}\neq\emptyset\}.$$

We claim that the map

$$g:\mathcal{F}(X)\to\mathbb{N}^{\mathbb{N}},\quad F\mapsto\alpha^{(F)},$$

is Borel. Indeed, for $s=(s_i)_{i=1}^{\ell}\in\mathbb{N}^{<\mathbb{N}}$

$$g(F)\in\{\alpha\in\mathbb{N}^{\mathbb{N}} : \alpha\succ s\}\iff U_s\cap F\neq\emptyset\ \&\ \forall n<|s|\,\forall k<s_{n+1}\quad F\cap U_{(s_1,s_2,\dots s_n,k)}=\emptyset.$$

If $f:\mathbb{N}^{\mathbb{N}}\to X$ denotes the associated map (which by Proposition 2.2 is continuous), we deduce that $f\circ g$ is measurable and since $\{f\circ g(F)\}=\bigcap_{n\in\mathbb{N}}U_{\alpha^{(F)}|_n}$ it follows that $f\circ g(F)\in F$.

Let $V_n$ be a basis of the topology of $X$, and define

$$d_n:\mathcal{F}(X)\to X,\quad F\mapsto\begin{cases} f\circ g(F\cap\overline{V_n}) & \text{if } F\cap\overline{V_n}\neq\emptyset\\ f\circ g(F) & \text{if } F\cap\overline{V_n}=\emptyset.\end{cases}$$

Since

$$\mathcal{F}(X)\to\big[\mathcal{F}(\overline{V_n})\cup\{\emptyset\}\big]\times\mathcal{F}(X),\quad F\mapsto(F\cap\overline{V_n},F)$$

is measurable (indeed $\{F\in\mathcal{F}(X) : F\cap\overline{V_n}\cap U\neq\emptyset\}=\bigcap_{\ell\in\mathbb{N}}\{F\in\mathcal{F}(X) : F\cap(\overline{V_n})_{1/\ell}\cap U\neq\emptyset\}$), since $\{F\in\mathcal{F}(X) : F\cap\overline{V_n}=\emptyset\}$ is measurable, and since $d$ is measurable it follows that $d_n$ is measurable. Secondly it is easy to see that $(d_n(F) : n\in\mathbb{N})$ is dense in $F$ for every $F\in\mathcal{F}(X)$. □

## 4. The Borel Isomorphism Theorem

DEFINITION 4.1. Assume $(X,\mathcal{A})$ and $(Y,\mathcal{B})$ are two measurable spaces. A map $f: X\to Y$ is called $(\mathcal{A},\mathcal{B})$-*bimeasurable*, or, if there is no confusion, simply *bimeasurable* if it is $(\mathcal{A},\mathcal{B})$-measurable, and if $f(A)\in\mathcal{B}$ for all $A\in\mathcal{A}$. If $f$ is furthermore bijective we call $f$ an $(\mathcal{A},\mathcal{B})$-isomorphism. If $X$ and $Y$ are topological spaces, and if $\mathcal{A}=\mathcal{B}_X$ and $\mathcal{B}=\mathcal{B}_Y$ we call an $(\mathcal{A},\mathcal{B})$-isomorphism between $X$ and $Y$ a *Borel-isomorphism*, and we call two topological spaces $X$ and $Y$ *Borel isomorphic* if there exists a Borel-isomorphism between $X$ and $Y$.

EXAMPLE 4.2. $[0,1]$ and the Cantor space $\Delta$ are Borel isomorphic.

PROOF. Let $D\subset[0,1]$ be the dyadic rationals, i.e.

$$D=\Big\{\sum_{i=1}^{k}2^{-n_i} : k \text{ and } n_1<n_2\dots<n_k \text{ in } \mathbb{N}\Big\}.$$

And $E \subset \Delta = \{0,1\}^\omega$, the set of sequences which are eventually constant. Define

$$f : \Delta \setminus E \to [0,1] \setminus D, \quad (\varepsilon_i) \mapsto \sum \sum_{i \in \mathbb{N}} \varepsilon_i 2^{-i}.$$

$f$ is a bijection, and using the fact that $\mathcal{B}_{[0,1]}$ is generated by the dyadic intervals it is easy to show that $f$ is a Borel-isomorphism between $\Delta \setminus E$ and $[0,1] \setminus D$.

Since $E$ and $D$ are both countable and infinite we can extend $f$ to a Borel isomorphism between $\Delta$ and $[0,1]$. □

Proposition 4.3. *Assume that $(X, \mathcal{A})$ is a countably generated measurable space.*

*Then there is a subset $Z$ of $\Delta$ and a bimeasurable map $g : X \to Z$ for all $x, y \in X$, $g(x) = g(y)$ if and only if $x$ and $y$ belong to the same atom of $\mathcal{A}$ ($A \in \mathcal{A} \setminus \{\emptyset\}$ is called an* atom *in $\mathcal{A}$ if $A$ has no proper non empty measurable subset).*

*In particular if the atoms of $\mathcal{A}$ are all singletons, we deduce that $(X, \mathcal{A})$ is isomorphic to a subspace of $\Delta$.*

Proof. Let $\{A_n : n \in \mathbb{N}\}$ be a countable generator of $\mathcal{A}$ and define

$$f : X \to \Delta, \quad x \mapsto (\chi_{A_n})_{n \in \mathbb{N}}).$$

$f$ is measurable, since for all $n \in \mathbb{N}$ the map $\pi_n \circ f : X \to \mathbb{R}$ is measurable, where $\pi_n : \mathbb{R}^\omega \to \mathbb{R}$ is the $n$-th coordinate projection. The atoms of $\mathcal{A}$ are exactly the non empty sets of the form $A_{\overline{\varepsilon}} = \bigcap_{n \in \mathbb{N}} A_i^{(\varepsilon_i)}$ with $\overline{\varepsilon} = (\varepsilon_i) \in \Delta$, and $A_i^{(\varepsilon_i)} = A_i$ if $\varepsilon_i = 1$ and $A_i^{(\varepsilon_i)} = A_i^c$ if $\varepsilon_i = 0$. Therefore it follows that for $x, y \in X$ we have $f(x) = f(y)$ if and only if $x$ and $y$ belong to the same atom of $\mathcal{A}$.

Let $Z = f(X)$. The family $\mathcal{B} = \{B \in \mathcal{A} : f(A) \in \mathcal{B}_\Delta \cap Z\}$ is a $\sigma$-algebra (it is closed under taking complements because $f$ as function defined on the quotient defined by the atoms of $\mathcal{A}$ is injective) and contains all $A_n$'s. Thus, $\mathcal{B} = \mathcal{A}$, which implies that $f$ is bimeasurable. □

Proposition 4.4. *Let $(X, \mathcal{A})$ be a measurable space, $Y$ a Polish space and $f : A \to Y$, measurable with $A \subset X$. Then $f$ admits a measurable extension to all of $X$.*

Proof. Let $(U_n)$ be a countable base of the topology on $Y$ and note that it is also a generator of $\mathcal{B}_Y$. Choose for $n \in \mathbb{N}$ $B_n \in \mathcal{A}$ so that $f^{-1}(U_n) = B_n \cap A$. Without loss of generality, we may assume that $\mathcal{A}$ is generated by $\{B_n : n \in \mathbb{N}\}$.

Now we can apply Proposition 4.3 in order to find $Z \subset \Delta$ and a bimeasurable map $g : X \to Z$ such that $g(x) = g(x')$ for $x, x' \in X$ if and only if $x$ and $x'$ belong to the same atom of $\mathcal{A}$. For $x, x' \in A$ this implies that $g(x) = g(x')$ if and only if $f(x) = f(x')$.

Let $B = g(A)$ and define $h : B \to Z$ by $h(z) = f(x)$ where $x \in A$ is such that $g(x) = z$ (if $g(x) = g(x') = z$ the $x$ and $x'$ belong to the same atom in $\mathcal{A}$ and $f(x) = f(x')$). Since for $n \in \mathbb{N}$

$$h^{-1}(U_n) = \{z \in B : h(z) \in U_n\} = g(B_n) \cap B,$$

it follows that $h$ is Borel measurable and, hence by Proposition 1.3 it has a Borel extension $\tilde{h} : Z \to Y$. It follows for $x \in A$ that $\tilde{h} \circ g(x) = h(g(x)) = f(x)$. Thus, $\tilde{h} \circ g$ is the wanted measurable extension of $f$. □

THEOREM 4.5. [The Borel Isomorphism Theorem]
*Any two uncountable standard Borel spaces are Borel isomorphic.*

We will need some auxiliary results first.

LEMMA 4.6. *Every standard Borel space is isomorphic to a subset of the Cantor space* $\Delta$.

PROOF. $A \subset X$ Borel measurable and $X$ Polish. Since every second countable metrizable space can be embedded into the Hilbert cube $\mathbb{H}$, $X$ embeds in $\mathbb{H}$. By Example 4.2 $[0,1]$ and $\Delta$ are Borel isomorphic, thus $\mathbb{H}$ and $\Delta^{\mathbb{N}}$ are Borel isomorphic. Since $\Delta^{\mathbb{N}}$ and $\Delta$ are homeomorphic we deduce our claim. □

PROPOSITION 4.7. *For every Borel subset $B$ of a Polish space $X$, there is a Polish space $Z$ and a continuous bijection from $Z$ to $B$.*

PROOF. Let $\mathcal{B}$ be the set of all subsets $B$ of $X$ so that there is a Polish space $Z$ and a continuous bijection from $Z$ to $B$. By Theorem 1.2 all open sets are in $\mathcal{B}$. By Proposition 1.4 we need to show that $\mathcal{B}$ is closed under countable intersections and countable disjoint unions. By Example 2 (V) It is clear that $\mathcal{B}$ is closed under taking countable disjoint unions. In order to show that it is closed under countable interesctions let $B_i \in \mathcal{B}$, $Z_i$ be Polish and $g_i : Z_i \to B_i$ be a continuous bijection for $i \in \mathbb{N}$ . Then the countable product $\prod_{i\in\mathbb{N}} Z_i$ is Polish and, thus,

$$Z = \Big\{(z_i) \in \prod_{i\in\mathbb{N}} Z_i : g_i(z_i) = g_1(z_1) \text{ for } i \in \mathbb{N}\Big\}$$

is also Polish, and the map $Z \to \bigcap B_i$, $z \mapsto g_1(z_1)$ is a continuous bijection. □

We also need a proof of the Schröder Bernstein Theorem which applies to measurable functions.

PROPOSITION 4.8. [Schröder Bernstein]
*If $X$ and $Y$ are sets and there exists injections $f : X \to Y$ and $g : Y \to X$. Then $X$ and $Y$ have the same cardinality. More precisely there is a bijection $h : X \to Y$ so that*

$$\text{(6)}\quad \text{Graph}(h) \subset \text{Graph}(f) \cup \text{Graph}(g^{-1}) = \{(x, f(x)) : x \in Y\} \cup \{(g(y), y) : y \in Y\}.$$

*Moreover assume $\mathcal{M}$ is a $\sigma$-algebra on $X$ and $\mathcal{N}$ is a $\sigma$-algebra on $Y$ and if $f$ is an $(\mathcal{M},\mathcal{N})$-isomorphism onto a measurable subset of $Y$, and vice versa $g$ is an $(\mathcal{N},\mathcal{M})$-isomorphism onto a measurable subset of $X$. Then there is an $(\mathcal{M},\mathcal{N})$-isomorphism between $X$ and $Y$.*

PROOF. We will only show the second part, involving measurability. The first part can then be deduced from the second taking for $\mathcal{M}$ and $\mathcal{N}$ the power sets.

Consider the map

$$H : \mathcal{M} \to \mathcal{N}, \quad A \mapsto X \setminus g(Y \setminus f(A)).$$

By assumption $H$ is well defined, monotone ($A \subset B$ implies $H(A) \subset H(B)$), and $H(\bigcup A_n) = \bigcup H(A_n)$ if $(A_n \subset \mathcal{M})$

By induction we define a sequence $(A_n) \subset \mathcal{M}$ as follows: $A_0 = \emptyset$ and, assuming $A_n$ has been chosen, we put $A_{n+1} = H(A_n)$. Let $E = \bigcup_{n=0}^\infty A_n$. Then $E = \bigcup_{n=1}^\infty A_n = \bigcup_{n=0}^\infty H(A_n) = H(E)$, i.e. $E = E \setminus g(Y \setminus f(E))$, thus $X \setminus E = g(Y \setminus f(E))$, which yields

$$g^{-1}(X \setminus E) = Y \setminus f(E). \tag{7}$$

Therefore we can define

$$h(x) = \begin{cases} f(x) & \text{if } x \in E, \\ g^{-1}(x) & \text{if } x \notin E. \end{cases}$$

Note that, by injectivity $X \setminus E = g(Y \setminus f(E)) \subset \text{range}(g)$, thus $h$ is well defined. The map $h$ is measurable since $E \in \mathcal{M}$, $f|_E$ is measurable, and since $g^{-1}|_{X\setminus E}$ is measurable. $h$ is injective since, by (7) $f(E) \cap g^{-1}(X \setminus E) = \emptyset$, and since $f$ and $g$ are injective. Again by (7), $h$ is surjective. Finally $h^{-1}$ is also measurable since, $f(E) \in \mathcal{N}$ and since $f^{-1}|_{f(E)}$ and $g|_{Y\setminus f(E)}$ are measurable □

Proof of Theorem 4.5. Assume that $B$ is an uncountable standard Borel space. By Lemma 4.6 we may assume that $B$ is a Borel subset of $\Delta$. On the other hand by Theorem 2.6 $B$ contains a homeomorph of $\Delta$.Thus, our claim follows from Proposition 4.8 □

Corollary 4.9. *Two standard Borel spaces of the same cardinality are Borel isomorphic.*

Proof. Let $A$ and $B$ be two standard Borel spaces of the same cardinality. Either they are countable, in which case both Borel $\sigma$-algebras are discrete. Or the claim follows from Theorem 4.5. □

Corollary 4.10. *Every Borel subset $B$ of a Polish space (thus every standard Borel space) is the continuous image of $\mathbb{N}^\mathbb{N}$ and an injective continuous image of a closed subset of $\mathbb{N}^\mathbb{N}$.*

Proof. By Proposition 4.7 $B$ is the continuous image of a Polish space $Z$ which by Theorem 2.9 is the injective and continuous image of a closed subset $D$ of $\mathbb{N}^\mathbb{N}$ or by Corollary 2.13 is the continuous image of $\mathbb{N}^\mathbb{N}$. □

## 5. Borel Point classes

Definition 5.1. A collection of pointsets, subsets of metrizable spaces, is called a *pointclass*. More, precisely a pointclass is a Functor ("function" on classes) :

$$\Gamma : \mathcal{MS} \to \text{All Sets}, \quad X \mapsto \Gamma(X)$$

with $\Gamma(X) \subset [X]$. Here $\mathcal{MS}$ denotes the class of metric spaces. For example closed sets, open sets, Borel sets, $\mathcal{G}_\delta$, $\mathcal{F}_\sigma$ sets are pointclasses.

If $\Gamma$ is a point class *the complement class of* $\Gamma$ is defined by

$$\tilde{\Gamma} : \mathcal{MS} \to \text{Sets}, \quad X \mapsto \{X \setminus A : A \in \Gamma(X)\}.$$

Let $\Gamma$ be a point class. A relation $\mathcal{R} \subset X \times X$ is clalled a $\Gamma$-*relation* if $\mathcal{R} \subset \Gamma(X \times X)$.

Definition 5.2. Let $X$ be metrizable. By transfinite induction we define for $1 \le \alpha < \omega_1$ the following pointclasses $\Sigma^{(0)}_\alpha(\cdot)$, $\Pi^{(0)}_\alpha(\cdot)$ and $\Delta^{(0)}_\alpha(\cdot)$.

$$\Sigma^{(0)}_1(X) = \{U \subset X : U \text{ open}\} \text{ and } \Pi^{(0)}_1(X) = \{F \subset X : F \text{ closed}\}.$$

If $1 < \alpha < \omega_1$ and if $\Sigma^{(0)}_\beta(X)$ and $\Pi^{(0)}_\beta(X)$ has been defined for all $\beta < \alpha$ we put

$$\Sigma^{(0)}_\alpha(X) = \Big(\bigcup_{\beta<\alpha} \Pi^{(0)}_\beta(X)\Big)_\sigma \text{ and } \Pi^{(0)}_\alpha(X) = \Big(\bigcup_{\beta<\alpha} \Sigma^{(0)}_\beta(X)\Big)_\delta.$$

We put for $\alpha < \omega_1$

$$\Delta^{(0)}_\alpha(X) = \Sigma^{(0)}_\alpha(X) \cap \Pi^{(0)}_\alpha(X).$$

The families $\Sigma^{(0)}_\alpha(X)$, $\Pi^{(0)}_\alpha(X)$ and $\Delta^{(0)}_\alpha(X)$ are called *additive*, *multiplicative* and *ambiguous* classes, respectively.

The following Proposition collects some easy facts.

Proposition 5.3. *Let $X$ be a metrizable space.*

1) *For $\alpha < \beta < \omega_1$*
$$\Sigma^{(0)}_\alpha(X) \subset \Sigma^{(0)}_\beta(X),\ \Pi^{(0)}_\alpha(X) \subset \Pi^{(0)}_\beta(X) \text{ and } \Delta^{(0)}_\alpha(X) \subset \Delta^{(0)}_\beta(X).$$
2) *Additive classes are closed under taking countable unions, and multiplicative classes are closed under taking countable intersections.*
3) *All three classes are closed under taking finite unions and intersections (proof by transfinite induction).*
4) *For $\alpha < \omega_1$*
$$\Sigma^{(0)}_\alpha(X) = \neg\Pi^{(0)}_\alpha(X) = \{X \setminus A : A \in \Pi^{(0)}_\alpha(X)\}$$
$$\Pi^{(0)}_\alpha(X) = \neg\Sigma^{(0)}_\alpha(X) = \{X \setminus A : A \in \Sigma^{(0)}_\alpha(X)\}$$
*(proof by transfinite induction).*
5) *$\Delta^{(0)}_\alpha(X)$ is an algebra.*

Proposition 5.4. *Let $X$ be a metrizable space and $\alpha < \omega_1$*

1) *$\Sigma^{(0)}_\alpha(X), \Pi^{(0)}_\alpha(X) \subset \Delta^{(0)}_{\alpha+1}(X)$.*
2) *If $\alpha > 1$ then*
$$\Sigma^{(0)}_\alpha(X) = (\Delta^{(0)}_\alpha(X))_\sigma \text{ and } \Pi^{(0)}_\alpha(X) = (\Delta^{(0)}_\alpha(X))_\delta.$$
*(if $X$ is zero dimensional that is also true for $\alpha = 1$).*
3) *If $\alpha$ is a limit ordinal and $(\alpha_n) \subset [1, \alpha)$ such that $\alpha = \sup_{n\in\mathbb{N}} \alpha_n$ then*
$$\Sigma^{(0)}_\alpha(X) = \Big(\bigcup_{n\in\mathbb{N}} \Pi^{(0)}_{\alpha_n}(X)\Big)_\sigma \text{ and } \Pi^{(0)}_\alpha(X) = \Big(\bigcup_{n\in\mathbb{N}} \Sigma^{(0)}_{\alpha_n}(X)\Big)_\delta.$$
4) $\mathcal{B}_X = \bigcup_{\alpha<\omega_1} \Sigma^{(0)}_\alpha(X) = \bigcup_{\alpha<\omega_1} \Pi^{(0)}_\alpha(X) = \bigcup_{\alpha<\omega_1} \Delta^{(0)}_\alpha(X)$.

5) *If* $\alpha > 1$*, then every element of* $\Sigma^{(0)}_\alpha(X)$ *is a disjoint union of elements in* $\bigcup_{\beta<\alpha}\Pi_\beta(X)$*, i.e.*

$$\Sigma^{(0)}_\alpha(X) = \big(\bigcup_{\beta<\alpha}\Pi_\alpha(X)\big)_+.$$

6) *If* $\alpha > 1$ *every element of* $\Sigma^{(0)}_\alpha(X)$ *is the disjoint union of elements in* $\Delta_\alpha(X)$*,*

Proof. (1) Since every closed set in $X$ is a $\mathcal{G}_\delta$ set and since every open set is an $\mathcal{F}_\sigma$ set (1) follows for $\alpha = 1$. Assume our claim is true for all $\beta < \alpha$, then

$$\Sigma^{(0)}_\alpha \cup \Pi^{(0)}_\alpha = \Big(\bigcup_{\beta<\alpha}\Pi^{(0)}_\beta(X)\Big)_\sigma \cup \Big(\bigcup_{\beta<\alpha}\Sigma^{(0)}_\beta(X)\Big)_\delta \subset \bigcup_{\beta<\alpha}\Delta^{(0)}_{\beta+1}(X) \subset \Delta^{(0)}_{\alpha+1}(X).$$

(2) Let $\alpha > 1$. From (1) it follows that

$$\Big(\Delta^{(0)}_\alpha(X)\Big)_\sigma \supset \Big(\bigcup_{\beta<\alpha}(\Delta^{(0)}_{\beta+1}(X)\Big)_\sigma \supset \Big(\bigcup_{\beta<\alpha}(\Pi^{(0)}_\beta(X))\Big)_\sigma = \Sigma^{(0)}_\alpha(X).$$

Since $\Delta^{(0)}_\alpha(X) \subset \Sigma^{(0)}_\alpha(X)$ and since $\Sigma^{(0)}_\alpha(X)$ is closed under countable unions it follows that $(\Delta^{(0)}_\alpha(X))_\sigma = \Sigma^{(0)}_\alpha(X)$.

Using similar arguments we show that $(\Delta^{(0)}_\alpha(X))_\delta = \Pi^{(0)}_\alpha(X)$.
(3) follows from the monotonicity of the families $\Sigma^{(0)}_\alpha(X)$ and $\Pi^{(0)}_\alpha(X)$ with respect to $\alpha < \omega_1$.
(4) By transfinite induction it follows that $\Sigma^{(0)}_\alpha(X) \subset \mathcal{B}_X$ for all $\alpha < \omega_1$. The family $\bigcup_{\alpha<\omega_1}\Sigma^{(0)}_\alpha(X)$ is closed under countable unions (since for very countable subset $C$ of $[1, \omega_1)$ it follows that $\sup_{\alpha\in C}\alpha < \omega$), it is closed under taking complements (since by (1) and (2) $\bigcup_{\alpha<\omega_1}\Sigma^{(0)}_\alpha(X) = \bigcup_{\alpha<\omega_1}\Delta^{(0)}_\alpha(X) = \bigcup_{\alpha<\omega_1}\Pi^{(0)}_\alpha(X)$), and it contains the open subsets of $X$.
(5) Let $A = \bigcup_{n\in\mathbb{N}} A_n$, with $A_n \in \bigcup_{\beta<\alpha}\Pi^{(0)}_\beta(X)$ for $n \in \mathbb{N}$. By Proposition 5.3 (3) and (4) it follows that $B_n = \big(\bigcup_{i\le n} A_i\big)^c \in \Sigma_{\beta_n}(X)$ for some $\beta_n < \alpha$. By (2) and Proposition 5.3 (3) and (4), and since $\alpha > 1$ we can write $B_n = \bigcup_{k\in\mathbb{N}} B_{(n,k)}$ where $(B_{(n,k)})$ is in $\bigcup_{\beta<\alpha}\Delta^{(0)}_\beta(X)$ and is pairwise disjoint. Thus we write

$$A = A_1 \cup (A_2 \cap B_1) \cup (A_3 \cap B_2) \ldots = A_1 \cup \bigcup_{n\in\mathbb{N}}(A_{n+1} \cap B_{(n,k)}),$$

which implies the claim.
(6) Follows from (2) and the fact that $\Delta^{(0)}_\alpha(X)$ is an algebra. □

Proposition 5.5. *Let* $X$ *and* $Y$ *be metrizable spaces and* $\alpha < \omega_1$*.*

1) *If* $f : X \to Y$ *is continuous then*

$$f^{-1}\big(\Sigma^{(0)}_\alpha(Y)\big) \subset \Sigma^{(0)}_\alpha(X) \text{ and } f^{-1}\big(\Delta^{(0)}_\alpha(Y)\big) \subset \Delta^{(0)}_\alpha(X).$$

2) *If* $Y \subset X$ *then*

$$\Sigma^{(0)}_\alpha(Y) = \Sigma^{(0)}_\alpha(X)|_Y = \{A\cap Y : A \in \Sigma^{(0)}_\alpha(X)\} \text{ and}$$
$$\Pi^{(0)}_\alpha(Y) = \Pi^{(0)}_\alpha(X)|_Y = \{A\cap Y : A \in \Pi^{(0)}_\alpha(X)\}$$

3) *If* $A \in \Sigma^{(0)}_\alpha(X\times Y)$ *(or in* $\Pi^{(0)}_\alpha(X\times Y)$*) then* $A_x \in \Sigma^{(0)}_\alpha(Y)$ *(respectively* $A_x \in \Pi^{(0)}_\alpha(Y)$*) for* $x\in X$.

Proof. By transfinite induction. □

Theorem 5.6. *Let* $1\le\alpha<\omega_1$ *and* $Y$ *second countable an metrizable. Then there exists a* $U\in\Sigma^{(0)}_\alpha(\mathbb{N}^\mathbb{N}\times Y)$ *(or in* $\Pi^{(0)}_\alpha(\mathbb{N}^\mathbb{N}\times Y)$*) so that for all* $A\subset Y$

$$A\in\Sigma^{(0)}_\alpha(Y) \iff \exists\, x\in\mathbb{N}^\mathbb{N} \quad A=U_x$$
$$(\text{or } A\in\Pi^{(0)}_\alpha(Y) \iff \exists\, x\in\mathbb{N}^\mathbb{N} \quad A=U_x, \text{ respectively})$$

*We call* $U$ universal for $\Sigma^{(0)}_\alpha(Y)$ *respectively* $\Pi^{(0)}_\alpha(Y)$.

Proof. We prove the claim by transfinite induction for all $\alpha<\omega_1$.

For $\alpha=1$, let $(V_n)$ a basis of the topology on $Y$ and put

$$U = \Big\{((n_k),y)\in\mathbb{N}\times Y : y\in\bigcup_{k\in\mathbb{N}} V_{n_k}\Big\}.$$

Note that $V\subset Y$ is open if and only if $V=\bigcup_{k\in\mathbb{N}}V_{n_k}$ for some sequence $(n_k)$ in $\mathbb{N}$ which means that $V=U|_{(n_k)}$. $U$ is open. Indeed, if $((n_k),y)\in U$, thus $y\in V_{n_{k_0}}$ for some $k_0\in\mathbb{N}$, which implies that:

$$\{(m_k)\in\mathbb{N}^\mathbb{N} : m_{k_0}=n_{k_0}\}\times V_{n_{k_0}}\subset U,$$

and proves our claim that $U$ is open. Since $\Delta^{(0)}_1=\neg\Sigma^{(0)}_1$ the set $N^\mathbb{N}\times Y\setminus U$ is universal for for $\Delta^{(0)}_1$.

Now suppose that the claim of our Theorem is true for all $\beta<\alpha$, $\alpha<\omega_1$.

Case 1: $\alpha=\beta+1$. Let $(N_i)_{i\in\mathbb{N}}$ be a partition of $\mathbb{N}$ into infinitely many sets of infinite cardinality. If $\overline{n}^{(i)}=(n(i,j):j\in N_j)\in\mathbb{N}^{N_i}$ for $i\in\mathbb{N}$, we mean by $\bigcup\overline{n}^{(i)}$ the sequence $(n_j)$ with $n_j=n(i,j)$ if $j\in N_i$ and for $\overline{n}\in\mathbb{N}^\mathbb{N}$ we denote by $\overline{n}|_{N_i}$ the subsequence $(n_j)_{j\in N_i}$. Let $V_i\in\Pi^{(0)}_\beta(\mathbb{N}^{N_i}\times Y)$ be a universal element for $\Pi^{(0)}_\beta(Y)$, (using that $\mathbb{N}^\mathbb{N}$ and $\mathbb{N}^{N_i}$ are homeomorphic via a bijection between $\mathbb{N}$ and $N_i$) and define

$$U=\bigcup_{i\in\mathbb{N}}\{(\overline{n},y) : (\overline{n}|_{N_i},y)\in V_i\} = \bigcup_{i\in\mathbb{N}}\pi_i^{-1}(V_i),$$

where $\pi_i:\mathbb{N}^\mathbb{N}\times Y\to\mathbb{N}^{N_i}\times Y$ is the $i$-th projection, for $i\in\mathbb{N}$. Since $\pi_i$, $i\in\mathbb{N}$, is continuous $U$ is in $\Sigma^{(0)}_\alpha(\mathbb{N}^\mathbb{N}\times Y)$. And if $A=\bigcup A_i$, with $A_i=V_i|\overline{n}^{(i)}$ for some $\overline{n}^{(i)}=(n(i,j):j\in\mathbb{N})\in\mathbb{N}^{N_i}$, and $i\in\mathbb{N}$, it follows that

$$U|_{\bigcup_{i\in\mathbb{N}}\overline{n}^{(i)}} = \Big(\bigcup_{i\in\mathbb{N}}\pi_i^{-1}(V_i)\Big)\Big|_{\bigcup_{i\in\mathbb{N}}\overline{n}^{(i)}} = \bigcup_{i\in\mathbb{N}}\pi_i^{-1}(V_i)|_{\bigcup_{i\in\mathbb{N}}\overline{n}^{(i)}} = \bigcup_{i\in\mathbb{N}}V_i|_{\overline{n}^{(i)}} = \bigcup A_i,$$

which proves that $U$ is universal for $\Sigma^{(0)}_\alpha$.

The argument to find a universal element for $\Pi^{(0)}_\alpha(Y)$ is similar.
Case 2: $\alpha$ is a Limit ordinal. Using Propostion 5.4 (3) we can choose $\alpha_n \nearrow \alpha$ so that $\Sigma^{(0)}_\alpha(Y) = \big(\bigcup_{n\in\mathbb{N}} \Pi^{(0)}_{\alpha_n}(Y)\big)_\sigma$, which means that every $A \in \Sigma^{(0)}_\alpha$ can be written as $A = \bigcup_{m,n} A_{(m,n)}$ with $A_{(m,n)} \in \Pi^{(0)}_{\alpha_m}(Y)\big)$ for $m, n \in \mathbb{N}$. Partitioning $\mathbb{N}$ into a family $(N_{(m,n)} : m, n \in \mathbb{N})$ of infinite subsets and choosing for each $m \in \mathbb{N}$ a $V_{(m,n)} \in \Pi^{(0)}_{\alpha_m}(\mathbb{N}^{N_{(m,n)}} \times Y)$ which is universal for $\Pi^{(0)}_{\alpha_m}(Y)$, we can proceed as in the proof of Case 1. □

Theorem 5.7. *Let $X$ be an uncountable Polish space and $\alpha < \omega_1$. Then $\Sigma^{(0)}_\alpha(X\times X)$ and $\Pi^{(0)}_\alpha(X\times X)$ contain an element which is universal for $\Sigma^{(0)}_\alpha(X)$ or $\Pi^{(0)}_\alpha(X)$, respectively, meaning that for $A \in \Sigma^{(0)}_\alpha(X)$, or $A \in \Pi^{(0)}_\alpha(X)$, respectively, there is a an $x \in X$ so that $U|_x = A$.*

Proof. By Corollary 2.7 $X$ contains a subspace $Y$ homeomorphic to $\mathbb{N}^\mathbb{N}$. Therefore, by Theorem 5.6 we can choose a universal element $V$ for $\Sigma_\alpha(X)$, respectively $\Pi^{(0)}_\alpha(X)$, in $\Sigma^{(0)}_\alpha(Y\times X)$ or $\Pi^{(0)}_\alpha(Y\times X)$ respectively. By Proposition 5.5 (2) there is a $U$ in $\Sigma^{(0)}_\alpha(X\times X)$, respectively $\Pi^{(0)}_\alpha(X\times X)$, so that $V = U \cap (Y\times X)$. This set $U$ is universal for $\Sigma^{(0)}_\alpha(X)$, respectively $\Pi^{(0)}_\alpha(X)$.

□

Corollary 5.8. *Let $X$ be an uncountable Polish space and $\alpha < \omega_1$ then there is an $A \in \Sigma^{(0)}_\alpha(X) \setminus \Pi^{(0)}_\alpha(X)$.*

Proof. Let $U \in \Sigma_\alpha(X\times X)$ be universal and

$$A := \{x \in X : (x, x) \in U\}.$$

Since the map $X \ni x \mapsto (x, x) \in X\times X$ is continuous it follows from Proposition 5.5 (1) that $A \in \Sigma^{(0)}_\alpha(X)$. We claim that $A \notin \Pi^{(0)}_\alpha(X)$. Indeed, if this were true it follows that $A^c \in \Sigma^{(0)}_\alpha(X)$ and, thus, that there is an $x_0 \in X$ so that $A^c \in U_{x_0}$ . Therefore

$$x_0 \in A^c \iff (x_0, x_0) \in U \iff x_0 \in A,$$

which is a contradiction. □

The proof of Corollary 5.8 can be generalized as follows.

Proposition 5.9. *Let $\Gamma$ be a point class on Polish spaces which is closed under complementation and continuous preimages. This $\Gamma(X)$ is a set of subsets of $X$ for each Polish space $X$ and and for any two Polish spaces $X$ and $Y$ and any continuous $f : X \to Y$ it follows that $f^{-1}(\Gamma(Y)) \subset \Gamma(X)$.*
*Then for no Polish space $X$ there is a universal element $U \in \Gamma(X\times X)$*

Proof. Assume $U \in \Gamma(X\times X)$ is universal for $\Gamma(X)$. Then put

$$A = \{x : (x, x) \in U\}.$$

Since $\Gamma$ is closed under continuous preimages and complements $A$ and $A^c$ are in $\Gamma$. Thus there is an $x_0 \in X$ so that $A^c = U|_{x_0}$ which as before gives raise to the following contradiction:

$$x_0 \in A \iff (x_0, x_0) \in U \iff x_0 \in A^c.$$

□

THEOREM 5.10. [Reduction Theorem for additive classes] *Let $X$ be metrizable and $1 < \alpha < \omega_1$. If $A_n \in \Sigma^{(0)}(\alpha)$ for $n \in \mathbb{N}$, then there exist pairwise disjoint sets $B_n \in \Sigma^{(0)}(\alpha)$ ($n \in \mathbb{N}$) so that, $B_n \subset A_n$, $n \in \mathbb{N}$ and $\bigcup_{n=1}^{n} B_n = \bigcup_{n=1}^{\infty} A_n$.*

PROOF. By Propososition 5.4 (2) we can write for $m \in \mathbb{N}$ $A_m$ as $A_m = \bigcup_{n=1}^{\infty} A_{(m,n)}$ with $A_{(m,n)} \in \Delta_\alpha^{(0)}(X)$ for $m, n \in \mathbb{N}$.

Reorder $(A_{(m,n)} : m, n \in \mathbb{N})$ into $(C_j : j \in \mathbb{N})$ and let $D_m = C_m \setminus \bigcup_{j<m} C_j$. Since $\Delta_\alpha^{(0)}(X)$ is an algebra we have $D_m \in \Delta_\alpha^{(0)}(X)$. Finally we let

$$B_m = \bigcup \big\{D_i : D_i \subset A_m \text{ and } D_i \not\subset A_j \text{ for } j = 1, 2, \dots m-1\big\}.$$

Since for all $m, n \in \mathbb{N}$ it follows that $D_m \cap A_n \neq \emptyset \Rightarrow D_m \subset A_n$ our claim follows, □

THEOREM 5.11. [Separation Theorem for multiplicative classes] *Let $X$ be metrizable and $1 < \alpha < \omega_1$. For every sequence $(A_n) \subset \Pi_\alpha^{(0)}(X)$ for which $\bigcap A_n = \emptyset$ there is a sequence $(B_n) \subset \Delta_\alpha^{(0)}(X)$ so that $B_n \supset A_n$, for $n \in \mathbb{N}$, and $\bigcap_{n\in\mathbb{N}} B_n = \emptyset$. In particular, if $A, B \in \Pi_\alpha^{(0)}(X)$ are disjoint, there is a $C \in \Delta_\alpha^{(0)}(X)$ so that $A \subset C$ and $B \cap C = \emptyset$.*

PROOF. By Theorem 5.10 there exists a pairwise disjoint sequence $(D_n) \subset \Sigma_\alpha^{(0)}(X)$ so that $D_n \subset A_n^c$, $n \in \mathbb{N}$, and $\bigcup_{n\in\mathbb{N}} D_n = \bigcup_{n\in\mathbb{N}} A_n^c = X$. Thus, letting $B_n = D_n^c$, for $n \in \mathbb{N}$, yields our claim. □

CHAPTER 5

# Analytic sets

## 1. Projective Sets

DEFINITION 1.1. Let $X, Y$ be sets and $B \subset X \times Y$.

The *projection of $B$ to $X$* is defined by

$$\exists^Y B = \pi_X(B) = \{x \in X : (x, y) \in B \text{ for some } y \in Y\}$$

and the *coprojection of $B$ to $X$* is defined by

$$\forall^Y B = \{x \in X : (x, y) \in B \text{ for all } y \in Y\}.$$

Clearly

$$\forall^Y B = \big(\exists^Y B^c\big)^c \text{ and } \exists^Y B = \big(\forall^Y B^c\big)^c.$$

For any pointclass $\Gamma$ and any Polish space $Y$

$$\exists^Y \Gamma = \big\{\exists^Y B : B \in \Gamma(X \times Y), \quad X \text{ is Polish space}\big\} \text{ and}$$
$$\forall^Y \Gamma = \big\{\forall^Y B : B \in \Gamma(X \times Y), \quad X \text{ is Polish space}\big\}.$$

Let $X$ be a Polish space. $A \subset X$ is called *analytic* if it is the projection of a Borel subset of $X \times X$. $C \subset X$is called *co-analytic* if $X \setminus C$ is analytic.

We denote by $\Sigma_1^{(1)}$ the pointclass of analytic sets and by $\Pi_1^{(1)}$ the pointclass of co-analytic sets and we put $\Delta_1^{(1)} = \Sigma_1^{(1)} \cap \Pi_1^{(1)}$. Clearly $\Delta_1^{(1)}$ contains the Borel sets. Later (Subsection 4) we will show that, conversely, every element of $\Delta_1^{(1)}$ is Borel.

PROPOSITION 1.2. *Let $X$ be a Polish space. For $A \subset X$ the following statements are equivalent.*

1) *$A$ is analytic*
2) *There is a Polish space $Y$ and a Borel set $B \subset Y \times X$ whose projection is $A$.*
3) *There is a continuous map $f : \mathbb{N}^{\mathbb{N}} \to X$ whose range is $A$.*
4) *There is a closed subset $C$ of $\mathbb{N}^{\mathbb{N}} \times X$ whose projection is $A$.*
5) *For every uncountable Polish space $Y$ there is $\mathcal{G}_\delta$-set $B$ in $X \times Y$ whose projection is $A$.*

REMARK. Note that the equivalence between (2) and (4) implies that the following conditions are equivalent to $A \subset X$ being analytic:

6) There is a Borel set $B \subset X \times \mathbb{N}^{\mathbb{N}}$ whose projection is $A$.
7) There is a $\mathcal{G}_\delta$ set $B \subset X \times \mathbb{N}^{\mathbb{N}}$ whose projection is $A$.
8) There is a Polish space $Y$ and a closed set $C \subset Y \times X$ whose projection is $A$.

Remark. Note if $Y$ is a compact space and $C \subset Y \times X$ is closed then its projection onto $X$ is also closed. Indeed if $(x_n) \subset \pi_X(C)$ converges to some $x \in X$. We can choose $y_n \in Y$ so that $(y_n, x_n) \in C$. Since $Y$ is compact there is a subsequence $(y_{n_k})$ converging to some $y \in Y$, and thus $(y, x) \in C$, which implies that $x \in \pi_X(C)$.

Proof. (1)$\Rightarrow$(2) is clear.
(2)$\Rightarrow$(3) Let $Y$ be a Polish space and $B \subset X \times Y$ whose projection is $A$. Then by Corollary 4.10 $B$ is the image of some continuous function $g : \mathbb{N}^{\mathbb{N}} \to X \times Y$, we can therefore choose $f = \pi_X \circ g$.
(3)$\Rightarrow$(4) let $f : \mathbb{N}^{\mathbb{N}} \to X$ continuous with range $A$. Let

$$C = \text{Graph}(f) = \{(x, f(x)) : x \in \mathbb{N}^{\mathbb{N}}\}$$

then $C$ is closed and $\pi_X(C) = A$.
(4)$\Rightarrow$(5) If $Y$ is an uncountable Polish space it contains a homeomorphic copy of $\mathbb{N}^{\mathbb{N}}$ by Corollary 2.7 which by Theorem 1.5 must be a $\mathcal{G}_\delta$ set in $Y$.
(5)$\Rightarrow$ (1) trivial. □

Proposition 1.3. *For a Polish space the point classes.* $\Sigma_1^{(1)}(X)$ *and* $\Pi_1^{(1)}(X)$ *are closed under countable intersections and unions, and under taking inverse images of Borel functions.*

Proof. Let $A_i \subset X$ be analytic for $i \in \mathbb{N}$. By (6) $\iff$ (1) in Proposition 1.2 we can choose $B_i \subset X \times \mathbb{N}^{\mathbb{N}}$ Borel so that $\pi_X(B_i) = A_i$, for $i \in \mathbb{N}$. Choose $B = \bigcup B_i$, then $B$ is Borel and $\pi_X(B) = \bigcup A_i$.

Let $(N_i)_{i\in\mathbb{N}}$ be a partition of $\mathbb{N}$ into infinitely many sets of infinite cardinality. and let $\tilde{B}_i \subset X \times \mathbb{N}^{N_i}$ Borel so that $\pi_X(B_i) = A_i$, for $i \in \mathbb{N}$. Choose

$$D = \big\{(x, \overline{n}) \in X \times \mathbb{N}^{\mathbb{N}} : \forall i \in \mathbb{N}\ (x, \overline{n}|_{N_i}) \in \tilde{B}_i\big\}$$

(recall if $(\overline{n} = (n_j)_{j\in\mathbb{N}}$ then $\overline{n}|_N = (n_j)_{j\in\mathbb{N}}$ for $N \subset \mathbb{N}$). It follows $\pi_X(D) = \bigcap_{i\in\mathbb{N}} A_i$.

If $f : Z \to X$ is Borel and $A \subset X$ analytic, say $A = \pi_1(B)$ $B \subset X \times X$ Borel. Then

$$f^{-1}(A) = \pi_Z\big(\{(z, x) \in Z \times X : (f(z), x) \in B\}\big)$$

$$\Big(z \in f^{-1}(A) \iff f(z) \in A \iff f(z) \in \pi_Z(B) \iff \exists (y, x) \in \pi_Z(B)\ f(z) = y\Big)$$

Since

$$\big\{(z, x) \in Z \times X : (f(z), x) \in B\big\} = (f, Id_X)^{-1}(B)$$

it follows that $\big\{(z, x) \in Z \times X : (f(z), x) \in B\big\}$ is Borel in $Z \times X$ and, thus, $f^{-1}(A)$ is analytic.

The properties for $\Pi_1^{(1)}$ follow from the fact that $\Pi_1^{(1)} = \neg\Sigma_1^{(1)}$. □

Proposition 1.4. $\Sigma_1^{(1)}$ *is closed under taking images under Borel maps.*

*Secondly,* $\Sigma_1^{(1)}$ *is closed under taking* $\exists^Y$ *and* $\Pi_1^{(1)}$ *is closed under taking* $\forall^Y$. *I.e.*

$$\begin{aligned}\exists^Y\Sigma_1^{(1)}(X \times Y) &= \big\{\exists^Y A : A \in \Sigma_1^{(1)}(X \times Y)\big\}\\ &= \big\{\{x \in X : (x, y) \in A \textit{ for some } y \in Y\} : A \in \Sigma_1^{(1)}(X \times Y)\big\} \subset \Sigma_1^{(1)}(X),\end{aligned}$$

*and*

$$\forall^Y\Pi_1^{(1)}(X\times Y) = \big\{\forall^Y B : B\in\Pi_1^{(1)}(X\times Y)\big\}$$
$$= \big\{\{x\in X : (x,y) \in A \textit{ for all } y\in Y\} : A\in \Pi_1^{(1)}(X\times Y)\big\} \subset \Pi_1^{(1)}.$$

Proof. Let $X$ and $Y$ be Polish spaces, $A \subset X$ be analytic and $g : A \to Y$ Borel. By Lemma 2.2 we can extend $g$ to a measurable map on all of $X$ which we still denote by $g$. By Corollary 2.5 there is a finer Polish Topology on $\mathcal{T}'$ on $X$ which generates the same Borel sets (and thus also the same analytic sets) so that $g$ is continuous. Since $A$ is also analytic in $(X,\mathcal{T}')$ there is by Proposition 1.2 a continuous (with respect to $\mathcal{T}'$ on $X$) map $f : \mathbb{N}^{\mathbb{N}} \to X$ whose range is $A$, and, thus, since $g(A)$ is the range of $g\circ f$ it follows, again using the equivalences in Proposition 1.2, that $g(A) = g(f(\mathbb{N}^{\mathbb{N}})$ is analytic in $Y$.

For the second claim, let $A \subset X\times Y$ analytic, thus, $A = \pi_{X\times Y}(B)$ for some Borel set $B \subset X\times Y\mathbb{N}^{\mathbb{N}}$. Then

$$\exists^Y A = \{x : \exists y\in Y \ (x,y) \in A\} \qquad = \{x : \exists y\in Y\, \exists \overline{n}\in \mathbb{N}^{\mathbb{N}}(x,y,\overline{n}) \in B\} = \pi_X(B)$$

which by Proposition 1.2, implies that $\exists^Y A$ is analyitc in $X$. A similar argument works for the pointclass $\Pi^{(1)}$. □

Theorem 1.5. *For every Polish space $X$, there is an analytic set $U \subset \mathbb{N}^{\mathbb{N}}\times X$ such that $A \subset X$ is analytic if and only if there is an $\overline{n} \in \mathbb{N}^{\mathbb{N}}$ so that $A = U_{\overline{n}}$. Thus $U$ is universal for $\Sigma_1^{(1)}(X)$.*

Proof. By Theorem 5.6 there is a closed (i.e. $\Pi_1^{(0)}$) set $C \subset \mathbb{N}^{\mathbb{N}}\times(X\times\mathbb{N}^{\mathbb{N}})$ which is universal for the closed sets in $X\times\mathbb{N}^{\mathbb{N}}$. Choose

$$U = \big\{(\overline{n},x) \in \mathbb{N}^{\mathbb{N}}\times X \ : \ \exists \overline{m}\in\mathbb{N}^{\mathbb{N}} \quad (\overline{n},x,\overline{m}) \in C\big\}.$$

Let $A \subset X$ analytic, by Proposition 1.2 there is a closed set $D \subset X\times\mathbb{N}^{\mathbb{N}}$ so that $A = \pi_X(D)$. Since $C$ is universal in for the closed sets in $X\times\mathbb{N}^{\mathbb{N}}$ we can write it as $D = C_{\overline{n}_0}$ for some $\overline{n}_0 \in \mathbb{N}^{\mathbb{N}}$ and thus

$$\begin{aligned} A &= \pi_X(D) \\ &= \pi_X(C_{\overline{n}_0}) \\ &= \pi_X\{(x,\overline{m})X\times\mathbb{N}^{\mathbb{N}} \ : \ (\overline{n}_0,x,\overline{m}) \in C\} \\ &= \{x : \exists \overline{m}\in\mathbb{N}^{\mathbb{N}} \quad (\overline{n}_0,x,\overline{m}) \in C\} \\ &= \{x : (\overline{n}_0,x,\overline{m}) \in U\} = U_{\overline{n}_0}. \end{aligned}$$

which proves our claim that $U$ is universal for $\Sigma_1^{(1)}$. □

Theorem 1.6. *Let $X$ be an uncountable Polish space. Then*

1) *There is an analytic set $U \subset X\times X$ which is universal for all analytic sets in $X$.*
2) *There is a subset of $X$ which is analytic but not Borel.*

PROOF. (1) By Corollary 2.7 $X$ contains a subset, say $Y$, homomorphic to $\mathbb{N}^{\mathbb{N}}$ which by Theorem 1.5 is a $\mathcal{G}_\delta$ set in $X$. By Theorem 1.5 we find a analytic set $U \subset Y\times$ which is universal for all the analytic sets in $X$. Since analytic sets are closed under taking continuous images it follows that $U$ is also analytic in t $X \times Y$.
(2) Let $U \subset X \times X$ as in part (1) and define

$$A = \{x : (x, x) \in U\}.$$

Then $A$, being a continuous image of an analytic set is also analytic. If $A$ where Borel its complement $A^c$ would also be Borel and, thus, analytic. Therfore there is an $x_0 \in X$ with $A = U|_{x_0}$, and we deduce the following contradiction:

$$x_0 \in A \iff (x_0, x_0) \in U \iff x_0 \in U|_{x_0} = A^c.$$

□

The Borel Isomorphism Theorem 4.5 and Theorem 1.6 yield the following corollary.

COROLLARY 1.7. *Every uncountable standard Borel space contains an analytic set that is not Borel.*

DEFINITION 1.8. Let $X$ be a Polish space. By induction we define $\Sigma^{(1)}_n(X)$, $\Pi^{(1)}_n(X)$ and $\Delta_n(X)$, for all $n \in \mathbb{N}$. For $n = 1$ these point classes were already defined.

Assuming $\Sigma^{(1)}_n(X)$, $\Pi^{(1)}_n(X)$ and $\Delta_n(X)$ has been defined we put

$$\begin{aligned} \Sigma^{(1)}_{n+1}(X) &= \exists^X \Pi^{(1)}_n(X \times X) = \big\{\pi_X(B) : B \in \Pi^{(1)}_n(X \times X)\big\} \\ \Pi^{(1)}_{n+1}(X) &= \neg\Sigma^{(1)}_{n+1}(X) = \big\{X \setminus A : A \in \Sigma^{(1)}_{n+1}(X)\big\} \\ \Delta^{(1)}_{n+1} &= \Sigma^{(1)}_{n+1}(X) \cap \Pi^{(1)}_{n+1}(X) \end{aligned}$$

The point sets $\Sigma^{(1)}_n(X)$, $\Pi^{(1)}_n(X)$ and $\Delta_n(X)$, are called *projective sets.*

PROPOSITION 1.9. *Let* $n \in \mathbb{N}$.

1) *The pointclasses* $\Sigma^{(1)}_n(X)$ *and* $\Pi^{(1)}_n(X)$ *are closed under countable unions and interesections, under Borel preimages, and under restrictions to* $\mathcal{G}_\delta$ *sets (i.e. if* $Z \subset X$ *is* $\mathcal{G}_\delta$ *then* $\Sigma^{(1)}_n(Z) \subset \Sigma^{(1)}_n(X)$ *and* $\Pi^{(1)}_n(Z) \subset \Pi^{(1)}_n(X)$*).*
2) *The following are equivalent for* $A \subset X$ *and* $n \in \mathbb{N}$ *(replace* $\Pi^{(1)}_0$ *by the Borel sets)*
   (i) $A \in \Sigma^{(1)}_n(X)$
   (ii) *There exists a Polish space* $Y$ *and* $B \in \Pi^{(1)}_{n-1}(X \times Y)$ *such that* $A = \pi_X(B)$.
   (iii) *There exists a* $C \in \Pi^{(1)}_{n-1}(X \times Y)$ *such that* $A = \pi_X(C)$.
   (iv) *For all uncountable Polish spaces* $Y$ *there is a* $B \in \Pi^{(1)}_{n-1}(X \times Y)$ *such that* $A = \pi_X(B)$.
3) $\Delta^{(1)}_n$ *is a* $\sigma$*-algebra.*
4) $\Sigma^{(1)}_n$ *is closed under* $\exists^Y$ *and* $\Pi^{(1)}_n$ *is closed under* $\forall^Y$*,* $Y$ *being a Polish space.*

PROOF. W.l.o.g. we assume that $X$ is uncountable.

We will show (1), (2), (3) and (4) simlutanuously by induction for every $n \in \mathbb{N}$. Note that for $n = 1$ Proposition 1.3 implies (1) Proposition 1.2 implies (2), (3) follows from (1) and the fact that $\Delta_1^{(1)}$ is closed under taking complements, and Proposition 1.4 implies (4).

Assume we showed our claim for $n-1$, $n > 1$. We first show that if $Z \subset X$ is $\mathcal{G}_\delta$ then $\Sigma_n^{(1)}(Z) \subset \Sigma_n^{(1)}(X)$ and $\Pi_n^{(1)}(Z) \subset \Pi_n^{(1)}(X)$. From (1) and(2) for $n-1$ we deduce that

$$\Sigma_n^{(1)}(Z) = \exists^X \Pi_{n-1}^{(1)}(Z \times X) \subset \exists^X \Pi_{n-1}^{(1)}(X \times X) = \Sigma_n^{(1)}(X),$$

and, thus, since $Z \in \mathcal{G}_\delta$,

$$\Pi_n^{(1)}(Z) = \{Z \setminus A : A \in \Sigma_n^{(1)}(Z)\} \subset \{X \setminus A : A \in \Sigma_n^{(1)}(X)\} = \Pi_n^{(1)}(X).$$

Secondly, we verify the equivalences in (2).

(i)$\Rightarrow$(ii) clear.

(ii)$\Rightarrow$(iii) Assume that $Y$ is Polish and that $B \in \Pi_{n-1}^{(1)}(X \times Y)$ and $A = \pi_X(B)$ By Corollary 2.13, $Y$ is image of $\mathbb{N}^\mathbb{N}$ under some continuous map $g : \mathbb{N}^\mathbb{N} \to Y$. Therefore, the set

$$C := (Id, g)^{-1}(B) = \{(x, \overline{n}) : (x, g(\overline{n})) \in B\},$$

is by our induction hypothesis in $\Pi_{n-1}^{(1)}(X \times \mathbb{N}^\mathbb{N})$ and since $g$ is surjective

$$\exists^{\mathbb{N}^\mathbb{N}} C = \big\{x : \exists \overline{n} \in \mathbb{N}^\mathbb{N} \quad (x, \overline{n}) \in C\big\} = \big\{x : \exists y \in Y \quad (x, y) \in B\big\} = \exists^Y B = A.$$

(iii)$\Rightarrow$(iv) Let $Y$ be an uncountable Polish space. By Corollary 2.7 $Y$ contains a subset $Z$ which is homeomorphic to $\mathbb{N}^\mathbb{N}$, and by Theorem 1.5 $Z$ is a $\mathcal{G}_\delta$-set in $Y$. Assuming (iii) there is a $C \in \Pi_{n-1}^{(1)}(X \times Z)$ so that $A = \pi_X(C)$. Since we have already shown that $\Pi_{n-1}^{(1)}(X \times Z) \subset \Pi_{n-1}^{(1)}(X \times X)$ it follows that $C \in \Pi_{n-1}^{(1)}(X \times X)$ and, thus, we deduce (iv).

(iv)$\Rightarrow$(i) clear.

Thirdly we verify that $\Sigma_n^{(1)}(X)$ and $\Pi_n^{(1)}(X)$ are closed under countable unions and interesections. Let $A_i \in \Sigma_n^{(1)}(X)$. By the already verified equivalences in (2) we can choose $B_i \in \Pi_{n-1}^{(1)}(X \times \mathbb{N}^\mathbb{N})$ Borel so that $\pi_X(B_i) = A_i$, for $i \in \mathbb{N}$. Choose $B = \bigcup B_i$, then, by our induction hypothesis, $B \in \Sigma_{n-1}^{(1)}(X \times \mathbb{N}^\mathbb{N})$ and $\pi_X(B) = \bigcup A_i$.

Let $(N_i)_{i \in \mathbb{N}}$ be a partition of $\mathbb{N}$ into infinitely many sets of infinite cardinality. and let $\tilde{B}_i \in \Pi_{n-1}^{(1)}(X \times \mathbb{N}^{N_i})$ Borel so that $\pi_X(B_i) = A_i$, for $i \in \mathbb{N}$. Choose

$$D = \big\{(x, \overline{n}) \in X \times \mathbb{N}^\mathbb{N} : \forall i \in \mathbb{N} \ (x, \overline{n}|_{N_i}) \in \tilde{B}_i\big\}$$

(recall if $(\overline{n} = (n_j)_{j \in \mathbb{N}}$ then $\overline{n}|_N = (n_j)_{j \in \mathbb{N}}$ for $N \subset \mathbb{N}$). Since

$$D = \bigcap_{i \in \mathbb{N}} \pi_i^{-1} \tilde{B}_i, \text{ with } \pi_i : \mathbb{N}^\mathbb{N} \to \mathbb{N}^{N_i}, \quad \overline{n} \mapsto \overline{n}|_{N_i}.$$

It follows from our induction hypothesis that $D \in \Pi_{n-1}^{(1)}(X \times \mathbb{N}^\mathbb{N})$ and $\pi_X(D) = \bigcap_{i \in \mathbb{N}} A_i$.

The fact that $\Pi_n^{(1)}$ is closed under taking countable unions and interesections follows easily.

Finally we show that $\Sigma_n^{(1)}$ and $\Pi_n^{(1)}$ are closed under taking inverse images of Borel maps. If $f : Z \to X$ is Borel and $A \subset X$ is in $\Sigma_n^{(1)}(X)$, say $A = \pi_1(B)$, with $B \in \Pi_{n-1}^{(1)}(X \times X)$. Then

$$f^{-1}(A) = \pi_Z\big(\{(z, x) \in Z \times X : (f(z), x) \in B\}\big)$$

$$\Big(z \in f^{-1}(A) \iff f(z) \in A \iff f(z) \in \pi_Z(B) \iff \exists (y, x) \in \pi_Z(B)\ f(z) = y\Big)$$

Since

$$\{(z, x) \in Z \times X : (f(z), x) \in B\} = (f, Id_X)^{-1}(B) \in \Pi_{n-1}^{(1)}(Z \times X),$$

by the induction hypothesis, it follows from above equivalences that

$$f^{-1}(A) = \pi_Z\big((f, Id_X)^{-1}(B)\big)$$

is in $\Sigma_n^{(1)}(Z)$.

The fact that $\Pi_n^{(1)}$ is closed under taking preimages with respect to Borel maps follows easily.

(3) follows from (1) and the fact that $\Delta^{(1)}(X)$ is closed under taking complements.

(4) Follows easily from the equivalences in (2). □

THEOREM 1.10. *Let $X$ be Polish and $n \in \mathbb{N}$. Then*

$$\Sigma_n^{(1)}(X) \cup \Pi_n^{(1)}(X) \subset \Delta_{n+1}^{(1)}(X).$$

PROOF. We prove the claim by induction on $n \in \mathbb{N}$. As $\mathcal{B}_X \subset \Delta_1^{(1)}(X) \subset \Pi_1^{(1)}(X)$ it follows that $\Sigma_1^{(1)}(X) \subset \Sigma_2^{(1)}(X)$. Let $A \in X$ be analytic, then $C = A \times X$ is analytic in $X \times \times X$ (by Proposition 1.9 analytic sets are closed under contininuos preimages) and thus $A = \forall^X C$ is in $\Pi_2^{(1)}(X)$. It follows that $A \in \Pi_2^{(1)}(X) \cap \Sigma_2^{(1)}(X) = \Delta_2^{(1)}$.

If $A \in \Pi_1^{(1)}(X) \subset \Pi_2^{(1)}(X)$ then, similarly $A = \exists^X (X \times A) \in \Sigma_2^{(1)}(X)$, and thus, $A \in \Delta_2^{(1)}(X)$

The induction steps works similarly. □

## 2. Analytic Sets and the Souslin operation

THEOREM 2.1. *Let $X$ be a Polish space, and let $d$ be a metric generating the topology on $X$. For $A \subset X$ the following statements are equivalent.*

1) *$A$ is analytic.*
2) *There exists a Souslin scheme $\mathcal{F} = (F_s : s \in \mathbb{N}^{<\mathbb{N}})$ of closed subsets of $X$ such that*
$$A = \mathcal{A}(\mathcal{F}) = \bigcup_{\alpha \in \mathbb{N}^{\mathbb{N}}} \bigcap_{n \in \mathbb{N}} F_{\alpha|_n}.$$
3) *There exists a family $\mathcal{F} = (F_s : s \in \mathbb{N}^{\mathbb{N}})$ of closed subsets of $X$ such that $A = \mathcal{A}(\mathcal{F})$.*

PROOF. (1)⇒ (2) By Proposition 1.2 there is a continuous map $f:\mathbb{N}^{\mathbb{N}}\to X$ whose range is $A$. For $s=(s_1,s_2,\dots s_\ell)\in\mathbb{N}^{<\mathbb{N}}$, choose

$$F_s=\overline{f(\{\alpha\in\mathbb{N}^{\mathbb{N}}:\alpha\succ s\})}.$$

If $\alpha\in\mathbb{N}^{\mathbb{N}}$ and $\varepsilon>0$, it follows from the continuity of $f$ that there is an $\ell$ so that $d(\alpha,\beta)<\varepsilon/2$ whenever $\alpha|_\ell=\beta|_\ell$, which implies that $d-\mathrm{diam}(F_{\alpha|_\ell})\le\varepsilon$. Thus, $\mathcal{F}=\{F_s:s\in\mathbb{N}^{\mathbb{N}}\}$ is a Souslin scheme.

We need to show that $A=\mathcal{A}(\mathcal{F})$. If $x\in A=f(\mathbb{N}^{\mathbb{N}})$, say $x=f(\alpha)$, for some $\alpha\in\mathbb{N}^{\mathbb{N}}$. Thus $x\in F_{\alpha|_\ell}$ for all $\ell\in\mathbb{N}$, and thus $x\in\mathcal{A}(\mathcal{F})$, conversely if $x\in\mathcal{A}(\mathcal{F})$, there is an $\alpha\in\mathbb{N}^{\mathbb{N}}$ so that $x\in\bigcap_{\ell\in\mathbb{N}}\overline{f(\{\beta\in\mathbb{N}^{\mathbb{N}}:\beta|\succ\alpha|_\ell\})}$. Thus there exist a sequence $\beta_\ell\in\mathbb{N}^{\mathbb{N}}$ so that $\alpha|_\ell=\beta_\ell|_\ell$ so that $d(f(\beta_\ell),x)\le 2^{-\ell}$. this implies that

$$x=\lim_{\ell\to\infty}f(\beta_\ell)=f(\alpha),$$

which implies that $x\in f(\mathbb{N}^{\mathbb{N}})=A$.
(2)⇒(3) obvious.
(3)⇒(1) Assume that $\mathcal{F}=\{F_s:s\in\mathbb{N}^{<\mathbb{N}}\}$ is given and $A=\mathcal{A}(\mathcal{F})$,.i.e

$$x\in A\iff\exists\alpha\in\mathbb{N}^{\mathbb{N}}\forall\ell\in\mathbb{N}\quad x\in F_{\alpha|_\ell}.$$

Define

$$C=\big\{(x,\alpha)\in X\times\mathbb{N}^{\mathbb{N}}:\forall n\in\mathbb{N}\ x\in F_{\alpha|_n}\big\}=\bigcap_{n\in\mathbb{N}}\bigcup_{s\in\mathbb{N}^n}F_s\times\{\alpha\in\mathbb{N}^n:\alpha\succ s\}.$$

It is easy to see that $C$ is closed and that $A=\exists^{\mathbb{N}^{\mathbb{N}}}C$, which implies that $A$ is analytic. □

THEOREM 2.2. *For $A\subset\mathbb{N}^{\mathbb{N}}$ the following are equivalent.*

1) *$A$ is co-analytic.*
2) *There is a tree $T$ on $\mathbb{N}\times\mathbb{N}$ such that*

$$\alpha\in A\iff T[\alpha]\text{ is well founded}\quad\iff T[\alpha]\text{ is well ordered with respect to }\le_{\mathrm{KB}}$$

*(second equivalence was shown in Proposition 2.1).*

PROOF. (1)⇒(2) Assume that $A\subset\mathbb{N}^{\mathbb{N}}$ is co-analytic. By Proposition 1.2 there is a closed set $C$ in $\mathbb{N}^{\mathbb{N}}\times\mathbb{N}^{\mathbb{N}}$ such that $A^c=\pi_1(C)$. By Proposition 2.3 $C$ is the body $[T]$ of a tree $T$ on $\mathbb{N}\times\mathbb{N}$. Now note that

$$\begin{aligned}\alpha\in A^c&\iff\exists\beta\in\mathbb{N}^{\mathbb{N}}\quad(\alpha,\beta)\in C\\&\iff\exists\beta\in\mathbb{N}^{\mathbb{N}}\quad(\alpha,\beta)\in[T]\\&\iff\exists\beta\in\mathbb{N}^{\mathbb{N}}\,\forall k\in\mathbb{N}\quad(\alpha|_k,\beta|_k)\in T\iff T[\alpha]\text{ not well founded.}\end{aligned}$$

(2)⇐(1) Let $A \subset \mathbb{N}^{\mathbb{N}}$ satisfy (2) and let $T$ be a tree on $\mathbb{N}\times\mathbb{N}$ such that above equivalence holds. But this means that $A^c = \pi_1([T])$. Indeed,

$$\begin{aligned} \alpha \in \pi_1([T]) &\iff \exists \beta \in \mathbb{N}^{\mathbb{N}} \quad (\alpha,\beta) \in [T] \\ &\iff \exists \beta \in \mathbb{N}^{\mathbb{N}}\, \forall k \in \mathbb{N} \quad (\alpha|_k, \beta|_k) \in [T] \\ &\iff \exists \beta \in \mathbb{N}^{\mathbb{N}}\, \forall k \in \mathbb{N} \quad \beta_k \in T[\alpha] \\ &\iff T[\alpha] \text{ is not well founded } \iff \alpha \in A^c. \end{aligned}$$

By Proposition 2.2 $[T]$ is closed, thus $A^c$ is analytic, which means that $A$ is co-analytic. □

## 3. $\mathbf{\Sigma^{(1)}}$ and $\mathbf{\Pi^{(1)}}$ complete sets

Definition 3.1. Let $X$ be Polish and $A \subset X$.

We say $A$ is $\Sigma_1^{(1)}$-*complete* or $\Pi_1^{(1)}$-*complete* if $A$ is analytic, respectively co-analytic and for every Polish space $Y$ and every analytic, respectively co-analytic, set $B \subset Y$, there is a Borel map $f : Y \to X$, such that $f^{-1}(A) = B$.

Proposition 3.2. *Let $X$ be Polish.*

1) *A completely analytic or co-analytic set is not Borel.*
2) *If $f : X \to Z$ , $Z$ Polish, is Borel measurable, $C \subset Z$ analytic or co-analytic, and $f^{-1}(C)$ is completely (co-)analytic, then $C$ is completely (co-)analytic.*
3) *In order to show that a set $A$ in $\Sigma_1^{(1)}(X)$ or in $\Pi_1^{(1)}(X)$ is $\Sigma_1^{(1)}$-complete, or $\Pi_1^{(1)}$-complete, respectively, it is enough show that for any set $B$ in $\Sigma_1^{(1)}(\mathbb{N}^{\mathbb{N}})$ or in $\Pi_1^{(1)}(\mathbb{N}^{\mathbb{N}})$ there is a Borel function $f : \mathbb{N}^{\mathbb{N}} \to X$ so that $f^{-1}(A) = B$.*

Proof. (1) By Corollary 1.7 there exists an analytic set $B$ which are not Borel in an uncountable Polish space $Y$. If $A \subset X$ is completely analytic it follws that for some Borel measurable $f : Y \to X$ we have $f^{-1}(A) = B$. Thus, $B$ would be measurable if $A$ had this property. The argumentation for co-analytic sets is the same.

(2) If $Y$ is Polish space and $B \subset Y$ is (co-)analytic, then there exists a Borel measurable $g : Y \to X$, so that $g^{-1}(f^{-1}(C)) = (f \circ g)^{-1} = B$.

(3) For general $Y$, we first can assume w.l.o.g. that $Y$ is uncountable (otherwise, for any $C \subset Y$ there is easily a Borel function $g : Y \to X$ so that $g^{-1}(A) = C$). If $Y$ is uncountable we find a Borel isomorphism $\phi$ between $\mathbb{N}^{\mathbb{N}}$ and $Y$ by Theorem 4.5. □

Definition 3.3. Let $X$ and $Y$ be Polish spaces, and $A \subset X$ and $B \subset Y$. We say that $A$ *is Borel reducible to* $B$ if there is a Borel map $f : X \to Y$, so that $f^{-1}(B) = A$. Thus, if $B$ is Borel then also $A$ is. Note also that if $A$ is Borel reducible to $B$ and $A$ is $\Sigma_1^{(1)}$-complete or $\Pi_1^{(1)}$-complete, and $B \in \Sigma_1^{(1)}$ or $B \in \Pi_1^{(1)}$, respectively, then $B$ is $\Sigma_1^{(1)}$-complete or $\Pi_1^{(1)}$-complete,

Example 3.4. Let Tr be the set of all trees on $\mathbb{N}$, since every tree on $\mathbb{N}$ is a subset of $\mathbb{N}^{\mathbb{N}}$ Tr is a subset of $\left[\mathbb{N}^{<\mathbb{N}}\right]$ which we can identify with the set $\{0,1\}^{\mathbb{N}^{<\mathbb{N}}}$ (via mapping each $A \subset \mathbb{N}^{<\mathbb{N}}$ onto its characteristic function).

The set $\{0,1\}^{\mathbb{N}^{<\mathbb{N}}}$ together with its product topology is a Polish space, and since

$$T \in \mathrm{Tr} \iff \forall s,t\in\mathbb{N}^{<\mathbb{N}}, s \prec t \quad (t \notin T) \vee (s \in T \,\&\, t \in T),$$

it follows that Tr is a $\mathcal{G}_\delta$-set in $\left[\mathbb{N}^{<\mathbb{N}}\right]$ and thus, a standard Borel space.

Let WF be the set of well founded elements of Tr. We claim that WF is $\Pi_1^{(1)}$-complete.

We first claim that the set

$$E = \left\{(T,\beta) \in \left[\mathbb{N}^{<\mathbb{N}}\right] \times \mathbb{N}^{\mathbb{N}} : T \in \mathrm{Tr} \,\&\, \exists n\in\mathbb{N}\; T \cap \left\{t \in \mathbb{N}^{<\mathbb{N}} : t \succ \beta|_n\right\} = \emptyset\right\},$$

is Borel. Indeed, note that

$$E = \bigcup_{s\in\mathbb{N}^{<\mathbb{N}}} \{T \in \mathrm{Tr} : s \notin T\} \times \{\beta \in \mathbb{N}^{\mathbb{N}} : \beta \succeq s\}.$$

Since $\mathrm{WF} = \forall^{\mathbb{N}^{\mathbb{N}}}(E)$ it follows that WF is co-analytic.

Now let $C$ be any co-analytic set in $\mathbb{N}^{\mathbb{N}}$. By Theorem 2.2 there is a tree on $\mathbb{N}\times\mathbb{N}$ so that

$$\alpha \in C \iff T[\alpha] \text{ is well founded},$$

and define

$$f : \mathbb{N}^{\mathbb{N}} \to \mathrm{Tr}, \quad \alpha \mapsto T[\alpha].$$

The map is continuous, indeed for any $s \in \mathbb{N}^{<\mathbb{N}}$ (recall that we identified $[\mathbb{N}^{<\mathbb{N}}]$ with $\{0,1\}^{\mathbb{N}^{<\mathbb{N}}}$ endowed with its product topology)

$$f(\alpha)(s) = \chi_{T[\alpha]}(s) = 1 \iff s \in T[\alpha] \iff (\alpha|_{|s|}, s) \in T,$$

which easily implies that for fixed $s$ the map $\alpha \mapsto f(\alpha)(s)$ is continuous and, thus, that $f$ is continuous.

We showed that for any co-analytic set $C$ in $\mathbb{N}^{\mathbb{N}}$ there is a Borel map $f$ so that $f^{-1}(\mathrm{WF}) = C$, which implies by the Remark after Definition 3.1 that WF is $\Pi_1^{(1)}$-complete.

PROPOSITION 3.5. *For $\alpha < \omega_1$ the set*

$$\mathrm{Tr}_\alpha = \{T \in \mathrm{Tr} : o(T) \leq \alpha\}$$

*is Borel measurable in $\{0,1\}^{\mathbb{N}^{<\mathbb{N}}}$.*

PROOF. We prove our claim by transfinite induction for all $\alpha < \omega_1$. For $\alpha = 0$ we have $\mathrm{Tr}_0 = \{\emptyset\}$ and our claim is trivial.

Assume the claim is proved for all $\beta < \alpha$. If $\alpha$ is a limit ordinal we deduce from Proposition 2.5 (a) (which says that $o(T)$ is allways a successor ordinal) that $\mathrm{Tr}_\alpha = \bigcup_{\beta<\alpha} \mathrm{Tr}_\beta$, which implies our claim for $\alpha$. If $\alpha = \beta + 1$ we deduce from Proposition 2.5 (c) that

$$\mathrm{Tr}_\alpha = \{T \in \mathrm{Tr} : \sup_{n\in\mathbb{N}} o(T(n)) \leq \beta\} = \bigcap_{n\in\mathbb{N}} \{T \in \mathrm{Tr} : T(n) \in \mathrm{Tr}_\beta\}$$

where $T(n) = \{s \in \mathbb{N}^{<\mathbb{N}} : (n,s) \in T\}$. Consider now for $n \in \mathbb{N}$ the map

$$F_n : \mathrm{Tr} \to \mathrm{Tr}, \quad T \mapsto T(n).$$

Since for a fixed $s\in\mathbb{N}^{<\mathbb{N}}$ we have

$$F_n^{-1}\big(\{T\in\mathrm{Tr}:s\in T\}\big)=\{T\in\mathrm{Tr}:(n,s)\in T\},$$

it follows that $F_n$ is Borel measurable, and, thus, that $\mathrm{Tr}_\alpha$ is measurable. □

EXAMPLE 3.6. We consider the strict linear orderings LO on $\mathbb{N}$ as a subset of $[\mathbb{N}\times\mathbb{N}]\equiv\{0,1\}^{\mathbb{N}\times\mathbb{N}}$ and claim that they are closed in $[\mathbb{N}\times\mathbb{N}]$. Indeed,

$$\begin{aligned}\mathrm{LO}=\bigcap_{(m,n)\in\mathbb{N}\times\mathbb{N},m\neq n}&\Big[\{\mathcal{R}\subset[\mathbb{N}\times\mathbb{N}]:(m,n)\in\mathcal{R}\text{ or }(n,m)\in\mathcal{R}\}\setminus\\ &\qquad\{\mathcal{R}\subset[\mathbb{N}\times\mathbb{N}]:(m,n)\in\mathcal{R}\text{ and }(n,m)\in\mathcal{R}\}\Big]\\ &\cap\bigcap_{k,m,n\in\mathbb{N}}\{\mathcal{R}\subset[\mathbb{N}\times\mathbb{N}]:(k,m)\notin\mathcal{R}\text{ or }(m,n)\notin\mathcal{R}\text{ or }(k,n)\in\mathcal{R}\}\\ &\cap\bigcap_{n\in\mathbb{N}}\{\mathcal{R}\subset[\mathbb{N}\times\mathbb{N}]:(n,n)\notin\mathcal{R}\}.\end{aligned}$$

We will show that the set of all well orderings is co-analytic, similarly as in previous example. We define:

$$\begin{aligned}\mathcal{G}&=\big\{(\mathcal{R},\alpha)\in\mathrm{LO}\times\mathbb{N}^{\mathbb{N}}:\exists n\in\mathbb{N}\quad(\alpha_{n+1},\alpha_n)\notin\mathcal{R}\big\}\\ &=\bigcup_{s\in\mathbb{N}^{<\mathbb{N}}}\big\{\mathcal{R}\in\mathrm{LO}:(s_{|s|},s_{|s|-1})\notin\mathcal{R}\big\}\times\big\{\beta\in\mathbb{N}^{\mathbb{N}}:\beta\succ s\big\}.\end{aligned}$$

and observe that $\mathrm{WO}=\forall^{\mathbb{N}^{\mathbb{N}}}\mathcal{G}$.

Using Example 3.4 it is left to show that WF is Borel reducible to WO and that there is a Borel measurable map $F:\mathrm{Tr}\to\mathrm{LO}$, so that $F^{-1}(\mathrm{WO})=\mathrm{WF}$.

We identify LO with the linear orderings of $\mathbb{N}^{<\mathbb{N}}$ (via a bijection between $\mathbb{N}$ and $\mathbb{N}^{<\mathbb{N}}$), and fix a well ordering $<_o$ on $\mathbb{N}^{<\mathbb{N}}$. We define $F:\mathrm{Tr}\to\mathrm{LO}$ with $F(T)\in\mathrm{LO}$ is defined as a follows

$$(s,t)\in F(T)\iff\begin{cases}\text{if } s,t\notin T\text{ and } s<_o t\\ \text{if } s\in T\text{ and } t\notin T\\ \text{if } s,t\in T\text{ and } s<_{\mathrm{KB}(T)} t\end{cases},$$

where $<_{KB(T)}$ is the Kleene - Brower ordering associated to the tree $T$. which by Proposition 2.1 on $T$ is well ordered if and only if $T$ is well founded. Thus, $F^{-1}(\mathrm{WO})=\mathrm{WF}$ (since $F(T)$ on $T$ coincides $<_{KB}$).

It is left to show that $F$ is continuous. Fix $s \neq t$ in $\mathbb{N}^{\mathbb{N}}$. We need to show that the set $\{T \in \mathrm{Tr}(s,t) \in F(T)\}$ is open. But since

$$\{T \in \mathrm{Tr}(s,t) \in F(T)\} = \begin{cases} \{T \in \mathrm{Tr}:\ s,t \notin T\} & \text{if } s <_0 t \\ \emptyset & \text{if } s >_0 t \end{cases}$$
$$\cup \begin{cases} \{T \in \mathrm{Tr} s,t \notin T\} & \text{if } s <_{KB} t \\ \emptyset & \text{if } s >_{KB} t \end{cases}$$
$$\cup \{T \in Tr : s \in T \text{ and } t \notin T\}$$

this follows easily.

The following examples are variant of Example 3.6.

Example 3.7. Let $\mathrm{LO}^*$ be the set of all (strict) linear orderings on a subset of $\mathbb{N}$ and let $\mathrm{LO}^*_1$ be the set of all (strict) linear orderings on a subset of $\mathbb{N}$ whose least element is 1. Thus

$$\mathrm{LO}^* = \{\mathcal{R} \in [\mathbb{N}\times\mathbb{N}] : \mathcal{R} \text{ is a linear ordering on } \mathrm{dom}(\mathcal{R})\}$$
$$\mathrm{LO}^*_1 = \left\{\mathcal{R} \in [\mathbb{N}\times\mathbb{N}] : \begin{matrix} 1 \in \mathrm{dom}(\mathcal{R}) \text{ and } \mathcal{R} \text{ is a linear ordering on } \mathrm{dom}(\mathcal{R}), \\ \text{with 1 being the smallest element} \end{matrix}\right\},$$

where

$$\mathrm{dom}(\mathcal{R}) = \{m \in \mathbb{N} : \exists n \in \mathbb{N} \quad (m,n) \text{ or } (n,m) \in \mathbb{N}\}$$

As in the previous example it is easy to see that $\mathrm{LO}^*$ and $\mathrm{LO}^*_1$ are closed in $[\mathbb{N}\times\mathbb{N}]$ and thus Polish. We put

$$\mathcal{G}^* = \big\{(\mathcal{R},\alpha) \in \mathrm{LO}^*\times\mathbb{N}^{\mathbb{N}} : \exists n \in \mathbb{N}\ (\alpha_n, \alpha_{n+1}) \notin \mathcal{R}\big\} \text{ and}$$
$$\mathcal{G}^* = \big\{(\mathcal{R},\alpha) \in \mathrm{LO}^*_1\times\mathbb{N}^{\mathbb{N}} : \exists n \in \mathbb{N}\ (\alpha_n, \alpha_{n+1}) \notin \mathcal{R}\big\}$$

and observe that $\mathrm{WO}^* = \forall^{\mathbb{N}^{\mathbb{N}}}\mathcal{G}^*$ $\mathrm{WO}^*_1 = \forall^{\mathbb{N}^{\mathbb{N}}}\mathcal{G}^*_!$ consists of all the well orderings on subsets of $\mathbb{N}$, and is therefore co-analytic.

In order to show that $\mathrm{WO}^*$ and $\mathrm{WO}^*_1$ are completely co-analytic we first note that with the same prove as in Example 3.7 we could have shown that

$$\mathrm{WO}_1 = \{\mathcal{R} \in [\mathbb{N}\times\mathbb{N}] : \mathcal{R} \text{ is well order on } \mathbb{N} \text{ with } \min(\mathcal{R}) = 1\}$$

is completely analytic in

$$\mathrm{LO}_1 = \{\mathcal{R} \in \mathrm{LO} : \forall n \in \mathbb{N}(n,1) \notin \mathcal{R}\}$$

and then note that the embedding of $\mathrm{LO}_1$ into $\mathrm{LO}^*_1$ is Borel and thus that $\mathrm{WO}_1$ Borel reduces to $\mathrm{WO}^*_1$. Since LO embeds into $\mathrm{LO}^*$ it also follows that WO Borel reduces to $\mathrm{WO}^*$

## 4. The first Separation Theorem for Analytic sets

THEOREM 4.1. *Let $A$ and $B$ two disjoint analytic sets in a Polish space $X$. Then there is a Borel set $C \subset X$ so that*

$$A \subset C \text{ and } B \cap C = \emptyset.$$

*(we say $C$* separates *$A$* from *$B$).*

Since for $A \in \Delta_1^{(1)}(X)$ $A$ as well as $A^c$ are analytic we deduce the following Corollary.

COROLLARY 4.2. *(Theorem of Souslin) For a Polish space $X$ the system $\Delta_1^{(1)}(X)$ coincides with the Borel sets on $X$.*

In order to prove Theorem 4.1 we first need the following Lemma.

LEMMA 4.3. *Suppose that $E = \bigcup_{n\in\mathbb{N}} E_n \subset X$ cannot be separated from $F = \bigcup_{n\in\mathbb{N}} F_n \subset X$ by a Borel set. Then there are $m, n \in \mathbb{N}$ so that. $E_m$ and $F_n$ cannot be separated by a Borel set.*

PROOF. Assume that for $m, n \in \mathbb{N}$ there is a Borel set $C_{(m,n))}$ which separates $E_m$ from $F_n$,. We claim that

$$C = \bigcup_{m\in\mathbb{N}} \bigcap_{n\in\mathbb{N}} C_{(m,n)},$$

separates $E$ from $F$. Indeed, Since $E_m =\subset \bigcup_{n\in\mathbb{N}} C_{(m,n)}$ it follows that $E \subset C$. Secondly, $F = \bigcup_{n\in\mathbb{N}} F_n \subset \bigcup_{n\in\mathbb{N}} C^c_{(m,n)}$ , for all $m \in \mathbb{N}$, and, thus,

$$F \subset \bigcap_{m\in\mathbb{N}} \bigcup_{n\in\mathbb{N}} C^c_{(m,n)} = C^c.$$

□

PROOF OF THEOREM 4.1. Let $A$ and $B$ two disjoint analytic sets. By Proposition 1.2 there are continuous maps $f, g : \mathbb{N}^\mathbb{N} \to X$, with $f(\mathbb{N}^\mathbb{N}) = A$ and $g(\mathbb{N}^\mathbb{N}) = B$. Now assume that $A$ and $B$ cannot be separated by a Borel set.
Claim. There are $\alpha, \beta \in \mathbb{N}^\mathbb{N}$ so that for all $n \in \mathbb{N}$ $f(\mathrm{Succ}_\infty(\alpha|_n))$ and $g(\mathrm{Succ}_\infty(\beta|_n))$ cannot be separated by a Borel set.

In order to show the claim we choose by induction for each $k$ $\alpha_k$ and $\beta_k$ in $\mathbb{N}$, so that $f(\mathrm{Succ}_\infty(\alpha|_1, \alpha_2 \ldots, \alpha_k))$ and $g(\mathrm{Succ}_\infty(\beta_1, \beta_2 \ldots, \beta|_k))$ cannot be separated by a Borel set. We do so by applying at the $k$-th step, $k \in \mathbb{N}_0$ Lemma 4.3 to the sets

$$E_m = f\big(\mathrm{Succ}_\infty(\alpha_1, \alpha_2, \ldots, \alpha_k, m)\big) \text{ and}$$
$$F_n = g\big(\mathrm{Succ}_\infty(\beta_1, \beta_2, \ldots, \beta|_k, n)\big).$$

Let $a = f(\alpha)$ and $b = g(\beta)$. Since $a \neq b$ there are disjoint neighborhoods $U$ and $V$ of $a$ and $b$, respectively, Since $f$ and $g$ are continuous and since the sets $\{\mathrm{Succ}_\infty(\alpha|_n) : n \in \mathbb{N}\}$ and $\{Succ_\infty(\beta|_n) : n \in \mathbb{N}\}$ are neighborhood bases for $\alpha$ and $\beta$, respectively, we can find an $n$ so that $f(\mathrm{Succ}_\infty(\alpha|_n)) \subset U$ and $g(\mathrm{Succ}_\infty(\beta|_n)) \subset V$, which is a contradiction to the conclusion of our claim above. □

The following result states a "countable infinite" version of Theorem 4.1

COROLLARY 4.4. *Assume* $(A_n)$ *is a sequence of pairwise disjoint analytic subsets of a Polish space* $X$. *Then there exists a sequence of pairwise disjoint Borel sets* $(B_n)$ *with* $A_n \subset B - n$ *for* $n \in \mathbb{N}$.

PROOF. By Proposition 1.3 $A'_m = \bigcup n \neq m A_n$ is analytic for any $m \in \mathbb{N}$, and by Theorem 4.1 we can therefore choose a Borel set $C_m \subset X$ so that $A_m \subset C_m$ and $C_m \cap A'_m = \emptyset$. Finally choose for $m \in \mathbb{N}$

$$B_m = C_m \cap \bigcap_{n \neq m} X \setminus C_n.$$

□

## 5. One-to-One Borel Functions

DEFINITION 5.1. If $X$ and $Y$ are Polish spaces $A \subset X$ analytic, we call a map $f : A \to Y$ *Borel measurable,* if it is $(A \cap \mathcal{B}_X, \mathcal{B}_Y)$-measurable, i.e. if for any Borel set $B \subset Y$ there is a Borel set $C \subset Y$ so that $C \cap A = f^{-1}(B)$.

REMARK. Let $A$ be an analytic subset of a Polish space $X$. thus $A = \pi_1(C)$ where $C \subset X \times X$ is Borel and $\pi_1 : X \times X \to A$, $(x_1, x_2) \mapsto x_1$. If $f : A \to Y$ is Borel measurable then for any Borel set $B \subset Y$ there is a Borel set $B' \subset X$ so that $f^{-1}(B) = B' \cap A$ and thus $\pi_1|_C^{-1}(f^{-1}(B)) = \pi|_C^{-1}(B' \cap A) = B' \cap C$. Thus $f \circ \pi_1|_C : C \to Y$ is measurable and, thus, by Proposition 1.3 $f(A) = f \circ \pi_1(C)$ is analytic in $Y$. More generally this argument implies that the image of any analytic $A' \subset A$ under $f$ is analytic in $Y$.

PROPOSITION 5.2. *Let* $X$ *and* $Y$ *be Polish spaces. Let* $A \subset X$ *be analytic* $f : A \to Y$ *injective and Borel mesurable.*

*Then* $f : A \to f(A)$ *is a Borel isomorphism.*

PROOF. Let $B \subset A$ Borel (i.e. there is a Borel set $C \subset X$ so that $B = C \cap A$). We need to show that $f(B)$ is Borel. Since $B$ and $A \setminus B$ are both analytic and $f$ is one to one, it follows that $f(B)$ and $f(A \setminus B)$ are disjoint and both analytic. Thus Theorem 4.1 yields that there is a Borel set $D \subset Y$ so that $f(B) \subset D$ and $f(A \setminus B) \subset Y \setminus D$. Since $f(B) = D \cap f(A)$ we deduce the claim. □

THEOREM 5.3. *Let* $X$ *and* $Y$ *be Polish spaces,* $A \subset X$ *analytic, and* $f : A \to Y$ *any map. Then the following are equivalent.*

a) $f$ *is Borel..*
b) $\mathrm{Graph}(f)$ *is Borel in* $A \times X$.
c) $\mathrm{Graph}(f)$ *is analytic.*

PROOF. (c)⇒(a). Let $U \subset Y$ be open. Since

$$f^{-1}(U) = \pi_1\big(\mathrm{Graph}(f) \cap (X \times U)\big),$$

is analytic, and since

$$f^{-1}(X \setminus U) = \pi_1\big(\mathrm{Graph}(f) \cap (X \times (Y \setminus U))\big),$$

is also analytic, Theorem 4.1 yields that there is a Borel set $B \subset X$ such that $f^{-1}(U) \subset B$ and $f^{-1}(Y \setminus U) \subset X \setminus B$. It follows that $f^{-1}(U) = A \cap B$, and, since $U \subset Y$ was an arbitrary open set, our claim follows.

□

THEOREM 5.4. *Let $X$ and $Y$ be Polish spaces, $A \subset X$ Borel and $A \to Y$ a onoe-to-one Borel map. Then $f(A)$ is Borel. In particular, if $f : X \to Y$ is Borel and one to one $f(B)$ is Borel for any Borel set $B \subset X$.*

PROOF. We can replace $X$ by $X \times Y$ $A$ by $\mathrm{Graph}(f)$ and $f$ by $\pi_Y|_{\mathrm{Graph}(f)}$ (note that $\pi_Y|_{\mathrm{Graph}(f)}(\mathrm{Graph}(f)) = f(A)$) and therefroe assume $f$ continuous. Since by Corollary 4.10 every Borel set is the one-to-one continuous image of a closed subset of $\mathbb{N}^{\mathbb{N}}$ we can assume that $X = \mathbb{N}^{\mathbb{N}}$ and that $A$ is closed.

We first claim that for any $s \in \mathbb{N}^{<\mathbb{N}}$ we can choose a Borel subset $B_s$ of $Y$ such that for any $s, t \in \mathbb{N}^{<\mathbb{N}}$

$$\text{(a)} \qquad f\big(\mathrm{Succ}_\infty(s) \cap A\big) \subset B_s \subset \overline{f\big(\mathrm{Succ}_\infty(s) \cap A\big)}$$

(Recall that $\mathrm{Succ}_\infty(s) = \{\alpha \in \mathbb{N}^{\mathbb{N}} : \alpha \succ s\}$)

$$\text{(b)} \qquad t \succ s \Rightarrow B_t \subset B_s \text{ and}$$

$$\text{(c)} \qquad s \neq t \text{ and } |s| = |t| \Rightarrow B_s \cap B_t = \emptyset$$

Indeed, choose $B_\emptyset = \overline{f(A)}$, and assume that $B_s$ has been defined for some $s \in \mathbb{N}^{<N}$. We need to choose $B_{(s,n)} \subset B_s$ pairwise disjoint. Since $f$ is Borel and injective it follows that the sets in $\big(f(\mathrm{Succ}_\infty(s,n) \cap A) : n \in \mathbb{N}\big)$ are pairwise disjoint and analytic. By Corollary 4.4 we can therefore choose a sequence of pairswise disjoint Borel subsets $(B'_n)$ of $Y$ so that $f(\mathrm{Succ}_\infty(s,n) \cap A) \subset B'_n$. We can take therefore for $n \in \mathbb{N}$

$$B_{(s,n)} = B_s \cap B'_n \cap \overline{f(\mathrm{Succ}_\infty(s,n) \cap A)}$$

and deduce that (a), (b) and (c) are satisfied.

Secondly we claim that

$$f(A) = D := \bigcap_{n \in \mathbb{N}} \bigcup_{|s|=n} B_s,$$

which would imply that $f(A)$ is Borel measurable.

If $\alpha \in A$ and $n \in \mathbb{N}$, then by (a) $f(\alpha) \in B_{\alpha|_n}$, which implies that $f(A) \subset D$. Conversely, if $y \in D$ we deduce from (b) and (c) that there is an $\alpha \in \mathbb{N}^{\mathbb{N}}$ so that $y \in B_{\alpha|_n}$ for all $n \in \mathbb{N}$. Since $B_{\alpha|_n} \subset \overline{f(\mathrm{Succ}_\infty(\alpha|_n) \cap A)}$, there is an $\alpha_n \in \mathrm{Succ}_\infty(\alpha|_n)$ such that $d(y, f(\alpha_n)) < 2^{-n}$ ($d(\cdot,\cdot)$ being a metric generating the topology on $\mathbb{N}^{\mathbb{N}}$). Since $\alpha_n$ converges to $\alpha$ and since $f$ was assumed to be continuous, it follows that $y = f(\alpha) \in f(A)$ □

We are now in the position to strengthen the Schröder Bernstein Theorem (Proposition 4.8) as follows.

Corollary 5.5. *Assume that $X$ and $Y$ are Polish spaces and that there are one-to-one Borel functions*

$$f : X \to Y \text{ and } g : Y \to X.$$

*Then there exists a Borel isomorphism $h : X \to Y$ so that*

$$\text{Graph}(h) \subset \text{Graph}(f) \cup \text{Graph}^{-1}(g).$$

## 6. Ranks of Pointclasses

Definition 6.1. A relation "$\leq$", on a set $S$ is called *pre-wellordering* if it is reflexive, transitive connected (i.e. $x \leq y$ or $y \leq x$ for any $x, y \in S$) and if there is no strictly decreasing sequence $(x_n)$. Here we mean by $x < y$, that $x \leq y$ but not $y \geq x$.

Definition 6.2. A *rank* or a *norm* on a set $S$ is a map $\phi : S \to \text{Ord}$, if $\phi(S)$ is an ordinal (i.e. if $\phi$ is a surjection onto $[0, \sup_{s\in S} \phi(s) + 1)$) we call $\phi$ a *regular rank*.

Remark. Note that a pre-wellordering does not need to be anti symmetric, and, thus, it does not need to be a wellordering. Nevertheless, if we consider the equivalence relation on $S$ defined by

$$x \sim y \iff x \leq y \text{ and } y \leq x,$$

then $\leq$ becomes a wellordering, which is order isomorphic to some ordinal $\alpha$, i.e. there is a *regular rank function* $\phi : S \to \alpha$, such that $\phi(x) \leq \phi(y)$ if and only if $x \leq y$.

Conversely, if $\phi : S \to \text{Ord}$ is a rank, then $\leq_\phi$ defined by

$$s \leq_\phi t \iff \phi(s) \leq \phi(t), \text{ whenever } s, t \in S,$$

is a pre-wellordering.

Definition 6.3. Let $\Gamma$ be a Pointclass, $\tilde{\Gamma}$ its complement class (see Definition 5.1) and put $\Delta = \Gamma \cap \tilde{\Gamma}$. Let $X$ be a Polish space, and $A \subset X$. We say that a rank $\phi : A \to \text{Ord}$ is a $\Gamma$-*rank* if there are two relations on $X \times X$, $\leq_\phi^\Gamma$ and $\leq_\phi^{\tilde{\Gamma}}$, so that

a) $\leq_\phi^\Gamma$ is a $\Gamma$-relation, and $\leq_\phi^{\tilde{\Gamma}}$ is a $\tilde{\Gamma}$-relation (see Definition 5.1)
b) For all $y \in A$:

$$\begin{aligned} \phi(x) \leq \phi(y) \big[ &\iff x \in A \;\&\; \phi(x) \leq \phi(y) \big] \\ &\iff x \leq_\phi^\Gamma y \\ &\iff x \leq_\phi^{\tilde{\Gamma}} y. \end{aligned}$$

Remark. Note that (b) implies that

c) for any $x \in A$ and any $y \in X \setminus A$ it follows that $x <_\phi^\Gamma y$ and $x <_\phi^{\tilde{\Gamma}} y$.

Indeed, assume for example that $y \leq_\phi^\Gamma x$ and $x \in A$. Then (b) would imply that $y \in A$.

PROPOSITION 6.4. *Assume that $\Gamma$ is closed under taking continuous preimages, and under taking finitely many intersections and unions. Let $X$ be a Polish space, $A \in \Gamma(X)$ and let $\phi : A \to \mathrm{Ord}$ be a rank and denote the corresponding strict order by $<_\phi$ (i.e. $x <_\phi y$ if and only if $\phi(x) < \phi(y)$).*

*Extend $\phi$ to all of $X$ by putting for $x \in X \setminus A$ $\phi(x) := \min\{\alpha \in \mathrm{Ord} : \forall y \in A \quad \alpha > \phi(y)\}$, and define for $x, y \in X$*

$$x \leq^* y \iff x \in A \,\&\, \phi(x) \leq \phi(y) \big[ \iff x \in A \,\&\, [y \notin A \text{ or } (y \in A \,\&\, \phi(x) \leq \phi(y))]\big]$$

*and*

$$x <^* y \iff x \in A \,\&\, \phi(x) < \phi(y).$$

*Then the following are equivalent:*

a) *$\phi$ is a $\Gamma$-rank,*

b) *there is a $\Gamma$-relation $<_\phi^\Gamma$ and a $\tilde{\Gamma}$-relation $<_\phi^{\tilde{\Gamma}}$ on $X$ so that for $x \in X$ and $y \in A$*

$$x <_\phi y \iff x \in A \;\&\; x <_\phi^\Gamma y \iff x \in A \;\&\; x <_\phi^{\tilde{\Gamma}} y,$$

c) *$\leq_\phi^*$ and $<_\phi^*$ are both $\Gamma$-relations,*

d) *there is a relation $\leq_\phi^{\tilde{\Gamma}}$, so that $\leq_\phi^{\tilde{\Gamma}}$, as well as its strict form $<_\phi^{\tilde{\Gamma}}$ are $\tilde{\Gamma}$-relations, and so that for $y \in A$*

$$\phi(x) \leq \phi(y) \big[ \iff x \in A \,\&\, \phi(x) \leq \phi(y)\big] \iff x \leq_\phi^{\tilde{\Gamma}} y$$

*and*

$$\phi(x) < \phi(y) \big[ \iff x \in A \,\&\, \phi(x) \leq \phi(y)\big] \iff x <_\phi^{\tilde{\Gamma}} y$$

PROOF. (a)$\Rightarrow$(b) define $<_\phi^\Gamma$ and $<_\phi^{\tilde{\Gamma}}$ by

$$x <_\phi^\Gamma y \iff \neg(x \notin A \,\text{or}\, y \leq_\phi^{\tilde{\Gamma}} x), \text{ and } x <_\phi^{\tilde{\Gamma}} y \iff \neg(x \notin A \text{ or } y \leq_\phi^{\tilde{\Gamma}} X)$$

Since $\Gamma$ (and thus also $\tilde{\Gamma}$) are closed under taking inverse images of continuous function (we will apply this assumption to the permutation $(x, y) \mapsto (y, x)$), we deduce that $<_\phi^\Gamma$ is a $\Gamma$- and $<_\phi^{\tilde{\Gamma}}$ is a $\tilde{\Gamma}$-relation. Moreover, for $y \in A$

$$\begin{aligned}
\phi(x) < \phi(y) &\iff x \in A \,\&\, \phi(x) < \phi(y)\\
&\iff x \in A \,\&\, \neg(\phi(y) \leq \phi(x)))\\
&\iff \neg(x \notin A \text{ or } \phi(y) \leq \phi(x))\\
&\iff \neg(x \notin A \text{ or } (x \in A \,\&\, \phi(y) \leq \phi(x))\\
&\iff \neg(x \notin A \text{ or } y \leq_\phi^{\tilde{\Gamma}} X) \iff x <_\phi^{\tilde{\Gamma}} y.
\end{aligned}$$

Similarly we show for $y \in A$ that $\phi(x) < \phi(y) \iff x <_\phi^{\tilde{\Gamma}} y$.

(b)$\Rightarrow$(a) Assume we are given relations $<_\phi^\Gamma$ and $<_\phi^{\tilde{\Gamma}}$ satisfying (b).

First we note that if $x \in A$ and $y \notin A$ then $x <_\phi^\Gamma y$. Indeed, assume that $x \in A$, $y \notin A$, and $\neg(x <_\phi^\Gamma y)$. Then the equivalences in (b) yield that $\not\phi(x) < \phi(y)$, which contradicts the definition of $\phi(\cdot)$ in that case. Simliarly we show that if $x \in A$ and $y \notin A$ then $x <^{\tilde{\Gamma}}_{\phi} y$.

Define $\leq^{\Gamma}_{\phi}$ and $\leq^{\tilde{\Gamma}}_{\phi}$ by

$$x \leq^{\Gamma}_{\phi} y \iff \neg\big(y <^{\tilde{\Gamma}}_{\phi} x\big) \text{ and } x \leq^{\tilde{\Gamma}}_{\phi} y \iff \neg\big(y <^{\Gamma}_{\phi} x\big)$$

Since $\Gamma$ and $\tilde{\Gamma}$ are closed under inverse images of continuous functions $\leq^{\Gamma}_{\phi}$ and $\leq^{\tilde{\Gamma}}_{\phi}$ are $\Gamma$- respectively $\tilde{\Gamma}$ relations. Moreover for $y \in A$

$$\begin{aligned}
\phi(x) \leq \phi(y) &\iff x \in A \text{ and } \neg(\phi(y) < \phi(x))\\
&\iff x \in A \text{ and } \neg(y <^{\Gamma}_{\phi} x)\\
&\iff \neg(y <^{\Gamma}_{\phi} x) \text{ (By above observation)}\\
&\iff x \leq^{\tilde{\Gamma}}_{\phi} y
\end{aligned}$$

and similarly

$$\phi(x) \leq \phi(y) \iff x \in A \text{ and } x \leq^{\Gamma}_{\phi} y.$$

(a) $\Rightarrow$(c) Note that

$$\begin{aligned}
x \leq^{*} y &\iff x \in A \,\&\, \phi(x) \leq \phi(y))\\
&\iff (x \in A \,\&\, y \notin A) \text{ or } (x \in A \,\&\, y \in A \,\&\, \phi(x) \leq \phi(y))\\
&\iff (x, y) \in A \times X \,\&\, x \leq^{\Gamma}_{\phi} y\\
&\quad \text{(By (b) of Definition 6.3 and remark thereafter)}
\end{aligned}$$

which shows by the assumpiton that $A \in \Gamma$ that $\leq^{*}$ is a $\Gamma$-relation. Secondly, we have

$$\begin{aligned}
x <^{*} y &\iff x \in A \,\&\, \phi(x) < \phi(y)\\
&\iff \big(x \in A \;\& y \notin A\big) \text{ or } \big(x \in A \;\& y \in A \,\&\, \phi(x) < \phi(y)\big)\\
&\iff \big(x \in A \;\& (x <^{\Gamma}_{\phi} y)\big) \text{ or } \big(x \in A \;\& y \in A \,\&\, \neg(\phi(y) \leq \phi(x))\big)\\
&\quad \text{(By remark after Definition 6.3)}
\end{aligned}$$

which shows that $<^{*}$ is a $\Gamma$-relation.
(c)$\Rightarrow$(d) define $\leq^{\tilde{\Gamma}}_{\phi}$ and $<_{\phi}$ by

$$x \leq^{\tilde{\Gamma}}_{\phi} y \iff x \in A \,\&\, \neg(y <^{*}_{\phi} x) \text{ and } x <^{\tilde{\Gamma}}_{\phi} y \iff x \in A \,\&\, \neg(y \leq^{*} x).$$

(d)$\Rightarrow$(a) define $\leq^{\Gamma}_{\phi}$ by

$$x \leq^{\Gamma}_{\phi} y \iff \neg(y \leq^{\tilde{\Gamma}}_{\phi} x).$$

□

EXAMPLE 6.5. In Example 3.4 it was shown that the set WF of wellfounded trees on $\mathbb{N}$ is $\Pi^{(1)}_1$-complete in the set of all trees $\text{Tr} \subset \big[\mathbb{N}^{<\mathbb{N}}\big] \equiv \{0,1\}^{\mathbb{N}^{<\mathbb{N}}}$. Recall that by Example 3.4 Tr is a $\mathcal{G}_\delta$-set in $\big[\mathbb{N}^{<\mathbb{N}}\big]$ and thus a Polish space itself by Theorem 1.2.

We claim that the ordinal tree $o(\cdot)$, as introduced in Subsection 2) is a $\Pi^{(1)}_1$-rank.

PROOF. Recall (Proposition 2.8) that for two trees $S$ and $T$

$$o(S)\le o(T) \iff \text{there is a map } f:S\to T \text{ so that}$$
$$\text{a) } f(\emptyset)=\emptyset$$
$$\text{b) } s,\tilde s\in S,\ \&\, s\prec\tilde s\Rightarrow f(s)\prec f(\tilde s)$$
$$\text{c) } t\in f(S)\,\&\,\tilde t\prec t\Rightarrow \exists s,\tilde s\in S\quad \tilde s\prec s \text{ and } t=f(s),\tilde t=f(\tilde s).$$

(we extend $f:S\to T$ as in Proposition 2.8) arbitrary to a map on all of $\mathbb{N}^{<\mathbb{N}}$ by putting $f(s)=\emptyset$ if $s\in\mathbb{N}^{<\mathbb{N}}\setminus S$).

We define

$$\begin{aligned}C&=\big\{(f,S,T): S,T\in\text{Tr and } f:\mathbb{N}^{<\mathbb{N}}\to\mathbb{N}^{<\mathbb{N}} \text{ satisfies (a),(b),(c)}\big\}\\
&=\bigcap_{s\in\mathbb{N}^{<\mathbb{N}}}\bigcup_{t\in\mathbb{N}^{<\mathbb{N}}}\left[\bigcap_{\tilde s\prec s}\bigcup_{\tilde t\prec t}\left\{(f,S,T): \begin{matrix} S,T\in\text{Tr and } f:\mathbb{N}^{<\mathbb{N}}\to\mathbb{N}^{<\mathbb{N}},\text{ so that}\\ s\notin S\text{ or}\\ \begin{bmatrix} s\in S,\ t\in T, f(s)=t\\ f(\emptyset)=\emptyset\text{ and } f(\tilde s)=f(\tilde t)\end{bmatrix}\end{matrix}\right\}\right.\\
&\qquad\left.\cap\bigcap_{\tilde t\prec t}\bigcup_{\tilde s\prec s}\left\{(f,S,T): \begin{matrix} S,T\in\text{Tr and } f:\mathbb{N}^{<\mathbb{N}}\to\mathbb{N}^{<\mathbb{N}},\text{ so that}\\ s\notin S\text{ or}\\ \begin{bmatrix} s\in S,\ t\in T, f(s)=t\\ f(\emptyset)=\emptyset\text{ and } f(\tilde s)=f(\tilde t)\end{bmatrix}\end{matrix}\right\}\right],\end{aligned}$$

which implies that $C$ is a Borel set in $\left[\mathbb{N}^{<\mathbb{N}}\right]^{\mathbb{N}^{<\mathbb{N}}}\times\text{Tr}\times\text{Tr}$. Therefore we conclude that the relation on Tr defined by

$$S\le_o^{\Sigma_1^{(1)}} T \iff \exists f\in\left[\mathbb{N}^{<\mathbb{N}}\right]^{\mathbb{N}^{<\mathbb{N}}}\quad (f,S,T)\in C,$$

is a $\Sigma_1$-relation, and for $T\in\text{WF}$, it follows from Proposition 2.8 that

$$S\in\text{WF}\,\&\, o(S)\le o(T) \iff \exists f\in\left[\mathbb{N}^{<\mathbb{N}}\right]^{\mathbb{N}^{<\mathbb{N}}}(f,S,T)\in C \iff S\le_o^{\Sigma_1^{(1)}} T.$$

From the second part of Proposition 2.8 we deduce that for two trees $S$ and $T$ we have $o(S)<o(T)$ if and only if there is an $n\in\mathbb{N}$ and a map $f:\mathbb{N}^{<\mathbb{N}}\to\mathbb{N}^{<\mathbb{N}}$ so that (a), (b) and (c) holds with $T$ replaced by $T(n)$. We let

$$\begin{aligned}C'&=\big\{(f,S,T): S,T\in\text{Tr}, f:\mathbb{N}^{<\mathbb{N}}\to\mathbb{N}^{<\mathbb{N}}\ \exists n\in\mathbb{N}\,(f,S,T(n)) \text{ satisfies (a),(b),(c)}\big\}\\
&=\bigcup_{n\in\mathbb{N}}\bigcap_{s\in\mathbb{N}^{<\mathbb{N}}}\bigcup_{t\in\mathbb{N}^{<\mathbb{N}}}\left[\bigcap_{\tilde s\prec s}\bigcup_{\tilde t\prec t}\left\{(f,S,T): \begin{matrix} S,T\in\text{Tr and } f:\mathbb{N}^{<\mathbb{N}}\to\mathbb{N}^{<\mathbb{N}},\text{ so that}\\ s\notin S\text{ or}\\ \begin{bmatrix} s\in S,\ (n,t)\in T, f(s)=f(n,t)\\ f(\emptyset)=\emptyset\text{ and } f(\tilde s)=f(n,\tilde t)\end{bmatrix}\end{matrix}\right\}\right.\\
&\qquad\left.\cap\bigcap_{\tilde t\prec t}\bigcup_{\tilde s\prec s}\left\{(f,S,T): \begin{matrix} S,T\in\text{Tr and } f:\mathbb{N}^{<\mathbb{N}}\to\mathbb{N}^{<\mathbb{N}},\text{ so that}\\ s\notin S\text{ or}\\ \begin{bmatrix} s\in S,\ t\in T, f(s)=f(n,t)\\ f(\emptyset)=\emptyset\text{ and } f(\tilde s)=f(n,\tilde t)\end{bmatrix}\end{matrix}\right\}\right],\end{aligned}$$

As before we conclude that the relation on Tr defined by

$$S<T \iff \exists f\in\left[\mathbb{N}^{<\mathbb{N}}\right]^{\mathbb{N}^{<\mathbb{N}}}\quad (f,S,T)\in C'$$

is a $\Sigma_1^{(1)}$-relation and by the second part of Proposition 2.8 for $T \in \mathrm{WF}$

$$S \in \mathrm{WF} \,\&\, o(S) < o(T) \iff \exists f \in \left[\mathbb{N}^{<\mathbb{N}}\right]^{\mathbb{N}^{<\mathbb{N}}} \quad (f, S, T) \in C' \iff S <_o^{\Sigma_1^{(1)}} T.$$

Now it follows from Proposition 6.4 (a) $\iff$ (d) that $o(\cdot)$ is a $\Pi_1^{(1)}$-rank. $\square$

COROLLARY 6.6. *For any co-analytic set in a Polish space $X$ there is a $\Pi_1^{(1)}$-rank with values in $\omega_1$.*

PROOF. Let $B \subset X$ be co-analytic. By Example 3.4 WF is $\Pi_1^{(1)}$-complete, and, thus there is a Borel function $F : X \to \mathrm{Tr}$ so that $B = f^{-1}(\mathrm{WF})$. Define

$$\phi : B \to \omega_1, \quad b \mapsto o(F(b)).$$

We claim that $\phi$ is a $\Pi_1^{(1)}$-rank on $B$.

Let $\leq_0^{\Sigma_1^{(1)}}$ and $<_0^{\Sigma_1^{(1)}}$ be defined as in the proof of Example 6.5 and define $\leq_\phi^{\Sigma_1^{(1)}}$ and $<_\phi^{\Sigma_1^{(1)}}$ by

$$x \leq_\phi^{\Sigma_1^{(1)}} y \iff F(x) \leq_o^{\Sigma_1^{(1)}} F(y)$$
$$x <_\phi^{\Sigma_1^{(1)}} y \iff F(x) <_o^{\Sigma_1^{(1)}} F(y).$$

Since $F$ is Borel it follows that $\leq_\phi^{\Sigma_1^{(1)}}$ and $<_\phi^{\Sigma_1^{(1)}}$ are $\Sigma_1^{(1)}$-relations.

Secondly it follows for $b, \tilde{b} \in B$, $\tilde{b} \in B$,

$$\phi(b) \leq \phi(\tilde{b}) \,\&\, b \in B \iff o(F(b)) \leq o(F(\tilde{b})) \,\&\, f(b) \in \mathrm{WF}$$
$$\iff F(b) \leq_o^{\Sigma_1^{(1)}} F(\tilde{b}) \iff b \leq_\phi^{\Sigma_1^{(1)}} \tilde{b}.$$

Similary,

$$\phi(b) < \phi(\tilde{b}) \,\&\, b \in B \iff b <_\phi^{\Sigma_1^{(1)}} \tilde{b},$$

which proves, again by using Proposition (a) $\iff$ (d), that $\phi$ is a $\Pi_1^{(1)}$-rank on $B$. $\square$

EXAMPLE 6.7. As in the Examples 3.6 and 3.7 let LO be the set of all linear orderings on $\mathbb{N}$, $\mathrm{LO}_1$ the set of all linear orderings on $\mathbb{N}$, for which 1 is the minimal element, $\mathrm{LO}^*$ he set of all linear orderings on subsets of $\mathbb{N}$, and $\mathrm{LO}^*$ he set of all linear orderings on subsets of $\mathbb{N}$, which contain 1 and for which 1 is the least element. As it was shown in Subsection 3 this are closed sets of $[\mathbb{N} \times \mathbb{N}]$ and the corresponding well orderings WO, $\mathrm{WO}_1$, $\mathrm{WO}^*$ and $WO_1^*$ are completely conalytic.

We now claim that the map which assigns to each element in WO, $\mathrm{WO}_1$, $\mathrm{WO}^*$ and $WO_1^*$ the ordinal $\alpha < \omega_1$ which is order isomorphic to it, is a rank function for WO, $\mathrm{WO}_1$, $\mathrm{WO}^*$ and $\mathrm{WO}_1^*$, respectively.

PROOF. Let

$$\| \cdot \| : \mathrm{WO}^* \to [0, \omega_1)$$

be the map assigning to each element of $\mathrm{WO}^*$ the (uniquely existing) countable ordinal to which it is order ismorphic. We only need to show that $\|\cdot\|$ is a conalytic rank for $\mathrm{WO}^*$ in $\mathrm{LO}^*$. Indeed, the restrictions of $\|\cdot\|$ to WO, $\mathrm{WO}_1$, and $\mathrm{WO}_1^*$ can be seen as compositions of $\|\cdot\|$ and the embeddings of LO, $\mathrm{LO}_1$ and $\mathrm{LO}_1^*$ into $\mathrm{LO}^*$. Thus, we can follow the argument of the proof of Corollary 6.6 (replacing $F$ by these embeddings) to show that the restrictions of $\|\cdot\|$ to WO, $\mathrm{WO}_1$, and $\mathrm{WO}_1^*$ are also co-analytic rank functions.

For a linear order $R \subset [\mathbb{N}\times\mathbb{N}]$ we put $||R|| = \omega_1$ if $R$ is not a well order, or equivalently, if there is a subset without any minimal elements.

Let $R_1$ and $R_2$ be linear orderings on a subset of $\mathbb{N}$ we note that $\|R_1\| \le \|R_2\|$ if and only if there is map $f : \mathrm{dom}(R_1) \to \mathrm{dom}(R_2)$ which is order preserving, and $\|R_1\| < \|R_2\|$ if and only there is an $n_0 \in \mathrm{dom}(R_2)$ and an order preseriving map $f : \mathrm{dom}(R_1) \to \mathrm{dom}(R_2) \setminus \{n \in \mathbb{N} : (n_0, n) \in R_2\}$.

From this point on the proof is similar to the proof in 6.5.

□

THEOREM 6.8. *Let $B$ a co-analytic set in a Polish space $X$, and let $\phi : B \to [0, \omega_1)$ be a co-analtyic rank for $B$. As in Proposition we put $\phi(x) = \omega_1$ if $x \in X \setminus B$.*

*Then*

a) *For all $\alpha < \omega_1$, the set $B_\alpha = \{x \in B : \phi(x) \le \alpha\}$ is Borel for all $\alpha < \omega_1$*
b) *If $B$ is not anlyitic and if $A \subset B$ is analytic, then $A \subset B_\alpha$ for some $\alpha < \omega_1$.*
c) *$f\ \psi : B \to [0, \omega_1)$ is another co-analtyic rank for $B$, then there is a map $\tau : [0, \omega_1) \to [0, \omega_1)$ so that $\psi(x) \le \tau(\phi(x))$ for all $x \in B$.*

PROOF. Assume that the relations $\le_\phi^{\Pi_1^{(1)}}$ and $\le_\phi^{\Sigma_1^{(1)}}$ on $X\times X$ are given as required by Defintion 6.3.

(a) Let $\alpha < \omega_1$. First assume that $\alpha = b_0$ for some $b_0 \in B$. Then

$$B_\alpha = \{b \in B : \phi(b) \le \phi(b_0)\} = \begin{cases} \{b \in X : b \le_\phi^{\Pi_1^{(1)}} b_0\} \in \Pi_1^{(1)}(X) \\ \{b \in X : b \le_\phi^{\Sigma_1^{(1)}} b_0\} \in \Sigma_1^{(1)}(X) \end{cases}$$

and thus $B_\alpha \in \Delta_1^{(1)}(X) = \mathcal{B}_X$ by Theorem 4.1. If $\alpha \notin \phi(B)$ then we can conclude our claim since $B_\alpha = \bigcup_{\beta<\alpha, \beta\in\phi(B)} B_\beta$.

(b) Assume that $A \subset B$ is analytic. Assume our claim is false and for any $\alpha < \omega_1$ there is an $a \in A$ for which $\phi(a) \ge \alpha$.

Then

$$\begin{aligned} b \in B &\iff \exists a \in A \quad \phi(b) \le \phi(a) \\ &\iff \exists a \in A \quad b \le_\phi^{\Sigma_1^{(1)}} a \\ &\iff b \in \pi_1\big(\{(x, y) : x \le_\phi^{\Sigma_1^{(1)}} y\} \cap X\times A\big), \end{aligned}$$

Thus, $B$ is analytic which is a contradiction.

(c) follows from (a) and (b). □

## 7. Derivatives and Ranks

The following notion generalizes the definition of the derivation of trees as introduced in Subsection 2.

DEFINITION 7.1. Let $X$ be a set and let $\mathcal{D} \subset [X]$ be closed under non empy intersections, a *derivative on* $\mathcal{D}$ is a map $D : \mathcal{D} \to \mathcal{D}$ such that $D(A) \subset A$ if $A \in \mathcal{D}$.

If $X$ is a Polish space and if $\mathcal{D} \subset \mathcal{F}(X)$ is Borel (with respect to the Effros-Borel $\sigma$-algebra) we call $D$ a *Borel derivative* if $D : \mathcal{D} \to \mathcal{D}$ is Borel measurable.

We can iterate and define for $A \in \mathcal{D}$ the set $D^{(\alpha)}(A)$ by transfinite induction:

$$D^{(0)}(A) = A$$

and, assuming that $D^{(\beta)}(A)$ has been defined for all $\beta < \alpha$,

$$D^{(\alpha)}(A) = D\big(D^{(\beta)}(A)\big) \text{ if } \alpha = \beta + 1, \text{ and}$$
$$D^{(\alpha)}(A) = \bigcap_{\beta<\alpha} D^{(\beta)}(A) \text{ if } \alpha, \text{ is a limit ordinal}$$

There must be some $\alpha \le \mathrm{card}(X)$ so that $D^{(\alpha)}(A) = D^{(\alpha+1)}(A)$ and we define

$$|A|_D = \min\{\alpha \le \mathrm{card}(X) : D^{(\alpha)}(A) = D^{(\alpha+1)}(A),$$

and write

$$D_\infty(A) = D^{(\alpha)}(A), \text{ where } \alpha \ge |A|_D.$$

THEOREM 7.2. [**Kech1**][Theorem 34.10, page 272]
*Let either $X$ be a a general Polish space and $\mathcal{D} = \mathcal{K}(X) \cup \{\emptyset\}$ or assume that $X$ is a Polish space which is a countable union of compact sets, and $\mathcal{D} = \mathcal{F}(X) \cup \{\emptyset\}$. Let $D : \mathcal{D} \to \mathcal{D}$ be a Borel derivative and put*

$$\Omega_D = \{F \in \mathcal{D} : D_\infty(F) = \emptyset\}.$$

*Then $\Omega_D$ is co-analyitc and the map $F \mapsto |F|_D$ is a co-analytic rank on $\Omega_D$.*

We first observe the following

LEMMA 7.3. *In either of the two cases of Theorem 7.2 the map*

$$\bigcap : \mathcal{D}^{\mathbb{N}} \to \mathcal{D}, (F_i) \mapsto \bigcup_{i=1}^{\infty} F_i$$

*is Borel measurable.*

PROOF. $\square$

CHAPTER 6

# The standard Borel space of separable Banach spaces

## 1. Introduction

Most of this subsection comes from [**Boss2**, Section 2]. We first introduce the following notations:

If $X$ and $Y$ are two Banach spaces and $C \geq 1$ we write $X \simeq_C Y$ if there is an isomorphism $T$ from $X$ onto $Y$, so that $\|T\| \cdot \|T^{-1}\| \leq C$ and we write $X \simeq Y$ if there is a $C \geq 1$ so that $X \simeq_C Y$. We write $X \hookrightarrow_C Y$ if $X$ there is an isomorphic embedding $T$ from $X$ into $Y$, so that $\|T\| \cdot \|T^{-1}|_{T(X)}\| \leq C$.

If $(x_i)$ and $(y_i)$ are two basic sequences we say $(x_i)$ $c$-dominates $(y_i)$ or $(y_i)$ is $c$-dominated by $(x_i)$, and write $(x_i) \succeq_c (y_i)$ or $(y_i) \preceq_c (x_i)$ if for all $(a_i) \in c_{00}$.

$$\Big\|\sum a_i x_i\Big\| \geq \frac{1}{c}\Big\|\sum a_i y_i\Big\|.$$

We say that $(x_i)$ and $(y_i)$ are *c-equivalent* and write $(x_i) \sim_c (y_i)$ if $(x_i) \succeq_c (y_i)$ and $(y_i) \succeq_c (y_i)$. We say that $(x_i)$ and $(y_i)$ are *equivalent* and write $(x_i) \sim (y_i)$ if for some $c < \infty$ $(x_i) \sim_c (y_i)$.

**Note:** $(x_i) \sim (y_i)$ implies $\overline{\text{span}}(x_i) \simeq \overline{\text{span}}(y_i)$ but the converse is not true.

The notation $\sim$ can be extended for all sequences: Let $(x_i)$ be any sequence in a Banach space $X$ and $(y_i)$ a sequence in a Banach space $Y$. We say that $(x_i)$ and $(y_i)$ are $c$-equivalent and write $(x_i) \sim_c (y_i)$ if for all $(a_i) \in c_{00}$

$$\frac{1}{c}\Big\|\sum a_i y_i\Big\| \leq \Big\|\sum a_i x_i\Big\| \leq c\Big\|\sum a_i y_i\Big\|.$$

Definition 1.1. Let $X$ be a separable (real) Banach space (thus $X$ is a Polish space). We put

$$\text{Subs}(X) := \{Y \subset X : Y \text{ is closed linear subspace of } X\}.$$

Proposition 1.2. *If $X$ is a Banach space then* Subs$(X)$ *is a Borel subset in $\mathcal{F}(X)$. Recall that we denoted by $\mathcal{F}(X)$ the Effors-Borel space of all closed subsets of a Polish space $X$ (see Subsection 3)*

Proof. Let $(d_n)$ be a sequence of selectors satifying the Selection Theorem of Kuratowski-Ryll-Nerdewiski (Theorem 3.7).

First note that for $F \in \mathcal{F}(X)$ it follows that

$$F \in \text{Subs}(X) \iff \forall m,n\in\mathbb{N}\,\forall p,q\in\mathbb{Q} \quad \big[pd_n(F) + qd_m(F) \in F\big].$$

Since for fixed $m,n\in\mathbb{N}$ and $p,q\in\mathbb{Q}$ the map $\mathcal{F} \to X$, $F \mapsto pd_n(F) + qd_m(F)$ is Borel measurable our claim follows from Proposition 3.4. $\square$

Definition 1.3. We abbreviate

$$\mathrm{SB} = \mathrm{Subs}\big(C(\Delta)\big).$$

Corollary 1.4. *By the Selection Theorem of Kuratowski-Ryll- Nerdewiski we can choose a sequence of measurable selectors $d_n$ $d_n : \mathrm{SB} \to C(\Delta)$, so that $(d_n(X)$ is dense in $X$ for all $X \in \mathrm{SB} \setminus \big\{\{0\}\big\}$.*

*Replacing $d_n$ by*

$$\tilde{d}_n(X) = \begin{cases} d_n(X) & \text{if } d_n(X) \neq 0\} \\ d_{\min\{m\in\mathbb{N}: d_m(X)\neq 0)\}}(X) & \text{if } d_n(X) = 0\}, \end{cases}$$

*we can assume that $d_n(X) \neq 0$ and thus the sequence $(s_n) = (d_n(X)/\|d_n(X)\| : n \in \mathbb{N})$ is dense in $S_X$ for each $X \in \mathrm{Subs}$.*

In the following Propositions and Lemmas we collect some elementary results about the measurability of certain subsets of $\mathrm{Subs}(X)$, SB, $X^{\mathbb{N}}$ and $C(\Delta)^{\mathbb{N}}$.

Proposition 1.5. *If $X \in SB$ then $\mathrm{Subs}(X)$ is a Borel subset of SB and the relation $\{(Y, X) \in \mathrm{SB}\times\mathrm{SB} : Y \subset X\}$ is Borel in $\mathrm{SB}\times SB$.*

Proof. First claim follows from Proposition 5.4 (6) the second one from (1). $\square$

Proposition 1.6. *The finite dimensional subspaces of $\mathrm{Subs}_f(X)$ of a Banach space are measurable in $\mathrm{Subs}(X)$.*

*Thus the following sets are also measurable*

$$\mathrm{SB}_\infty = \{X \in \mathrm{SB} : \dim(X) = \infty\}$$
$$\mathrm{Subs}_\infty(X) = \{Y \in \mathrm{Subs}(X) : \dim(Y) = \infty\}$$

Proof. Note that Let $D \subset B_X$ be dense and countable

$$\{F \in \mathrm{Subs}(X) : \dim(F) < \infty\} = \{F \in \mathrm{Subs}(X) : B_F \text{ is totally bounded}\}$$
$$= \bigcap_{\varepsilon>0,\varepsilon\in\mathbb{Q}} \bigcup_{D'\subset D,\text{finite}} \{F \in \mathrm{Subs}(X) : F \cap B_X^\circ \subset \overline{D}'_\varepsilon\}.$$

Since for $\varepsilon > 0, \varepsilon \in \mathbb{Q}$ and $D' \subset D$, finite

$$\{F \in \mathrm{Subs}(X) : F\cap B_X^\circ \subset \overline{D}'_\varepsilon\} = \mathrm{Subs}(X)\setminus\big\{Y \in \mathrm{Subs}(X) : Y\cap B_X^\circ\cap(X\setminus\overline{D}'_\varepsilon) \neq \emptyset\big\},$$

our claim follows. $\square$

Lemma 1.7. *Let $X$ be a Banach space, $(w_i)_{i\in\mathbb{N}}$ a basic sequence and $c \geq 1$. The following sets are closed in the Polish space $S_X^{\mathbb{N}}$.*

a) $\big\{(x_n) \subset S_X^{\mathbb{N}} : (x_n)$ *is basic with basis-constant* $\leq c\big\}$,
b) $\big\{(x_n) \subset S_X^{\mathbb{N}} : (x_n)$ *is basic and* $(x_n) \sim_c (w_i)\big\}$.
c) $\big\{(x_n) \subset S_X^{\mathbb{N}} : (x_n)$ *is* $c_1$ *basic and* $(x_n) \succeq_{c_2} (w_i)\big\}$.
d) $\big\{(x_n) \subset S_X^{\mathbb{N}} : (x_n)$ *is* $c_1$ *basic and* $(x_n) \preceq_{c_2} (w_i)\big\}$.

PROOF. First note that for $\overline{a} = (a_i) \in c_{00}$ then the map

$$f_a : S_X^{\mathbb{N}} \to \mathbb{R}, \quad (x_i) \mapsto \Big\| \sum a_i x_i \Big\|,$$

is continuous. In order to verify (a) we write

$$\{(x_n) \subset S_X^{\mathbb{N}} : (x_n) \text{ is basic with basis-constant } \le c\}$$
$$= \bigcap_{(q_1,\dots q_n)|\in\mathbb{Q}^{<\mathbb{N}}} \bigcap_{m\le n} \Big\{ (x_n) \subset S_X^{\mathbb{N}} : \Big\| \sum_{i=1}^m q_i x_i \Big\| \le c \Big\| \sum_{i=1}^n q_i x_i \Big\| \Big\}.$$

and (b) follows from

$$\{(x_n) \subset S_X^{\mathbb{N}} : (x_n) \text{ is basic and } c\text{-equivalent to } (w_i)\}$$
$$= \bigcap_{(q_1,\dots q_n)\in\mathbb{Q}^{<\mathbb{N}}} \Big\{ (x_n) \subset S_X^{\mathbb{N}} : \frac{1}{c} \Big\| \sum_{i=1}^n q_i w_i \Big\| \le \Big\| \sum_{i=1}^n q_i x_i \Big\| \le c \Big\| \sum_{i=1}^n q_i w_i \Big\| \Big\}.$$

The claims (c) and (d) can be verified similarly. ☐

LEMMA 1.8. *Let $X$ be a Banach space. The map:*

$$\Psi : X^{\mathbb{N}} \to \mathrm{Subs}(X), \quad (x_n) \mapsto \overline{\mathrm{span}(x_n :\in \mathbb{N})}$$

*is Borel measurable.*

PROOF. Note that for an open set $U \subset X$ it follows that

$$\Psi^{-1}\big(\{Y \in \mathrm{Subs}(X) : Y \cap U \ne \emptyset\}\big) = \big\{ (y_n) \in X^{\mathbb{N}} : \overline{\mathrm{span}(y_n : n \in \mathbb{N})} \cap U \ne \emptyset \big\}$$
$$= \bigcup_{(q_1,q_2,\dots q_n)\in\mathbb{Q}^{<\mathbb{N}}} f^{-1}_{(q_1,\dots q_n)}(U),$$

with

$$f_{(q_1,\dots q_n)} : X^{\mathbb{N}} \to X, \quad (x_i) \mapsto \sum_{i=1}^n q_i x_i.$$

☐

LEMMA 1.9. *Let $X$ be a separable Banach space.*

a) $\{(Y, y) \in \mathrm{Subs}(X) \times X \;:\; y \in Y\}$ *is Borel in* $\mathrm{Subs}(X) \times X$.

b) $\big\{\big(Y, (y_n)\big) \in \mathrm{Subs}(X) \times X^{\mathbb{N}} : \overline{\mathrm{span}(y_i : i \in \mathbb{N})} = Y\big\}$ *is Borel in* $\mathrm{Subs}(X) \times X^{\mathbb{N}}$.

c) $\big\{\big((x_n), (y_n)\big) \in X^{\mathbb{N}} \times X^{\mathbb{N}} \;:\; (x_n) \sim (y_n)\big\}$ *is Borel in* $X^{\mathbb{N}} \times X^{\mathbb{N}}$.

d) $\big\{(W, Z) \in \mathrm{Subs}(X) \times \mathrm{Subs}(X) : W \subset Z\big\}$ *is Borel in* $\mathrm{Subs}(X) \times \mathrm{Subs}(X)$.

PROOF. (a) and (d) follows from the corresponding observations for general Effros-Borel spaces (Propositions 3.4 and 3.3 (1)).

(b) We let $\Psi : X^{\mathbb{N}} \to \mathrm{Subs}(X)$ be defined as in Lemma 1.8 and note that

$$\big\{ \big(Y, (y_n)\big) \in \mathrm{Subs}(X) \times X^{\mathbb{N}} : \overline{\mathrm{span}(y_i : i \in \mathbb{N})} = Y \big\}$$
$$= (Id, \Psi)^{-1}(\big\{ \big(Y, Z\big) \in \mathrm{Subs}(X) \times \mathrm{Subs}(X) : Y = Z \big\}.$$

(c) Note that

$$\begin{aligned}\big\{\big((x_n),(y_n)\big) \in X^{\mathbb{N}}\times X^{\mathbb{N}} \,:\, (x_n)\sim(y_n)\big\}\\ = \bigcup_{k\in\mathbb{N}}\bigcap_{(q_1,q_2,\dots q_n)\in\mathbb{Q}^{<\mathbb{N}}}\Big\{\big((x_n),(y_n)\big)\in X^{\mathbb{N}}\times X^{\mathbb{N}} : \frac{1}{k}\le\frac{\|\sum_{i=1}^n q_i x_i\|}{\|\sum_{i=1}^n q_i y_i\|}\le k\Big\}.\end{aligned}$$

(0/0 := 1 in above quotient). and note that for $(q_1, q_2, \dots q_n) \in \mathbb{Q}^{<\mathbb{N}}$ the map

$$g_{(q_1,q_2,\dots q_n)} : X^{\mathbb{N}} \to X, \quad (x_n) \mapsto \Big\|\sum_{i=1}^n q_i x_i\Big\|,$$

is continuous.

□

Lemma 1.10. *Let $X$ be a Banach space Then the set*

$$B = \big\{(\overline{y},\overline{z}) \in X^{\mathbb{N}}\times X^{\mathbb{N}} : \overline{\text{span}(\overline{y})} \text{ and } \overline{\text{span}(\overline{z})} \text{ form a complemented sum}\big\}$$

*is Borel in $X^{\mathbb{N}}\times X^{\mathbb{N}}$ We say two subspaces $Y$ and $Z$ of $X$* form a complemented sum *if $Y + Z$ is closed and $Y \cap Z = \{0\}$.*

Proof. (by Francisco Torres-Ayala)

First we claim that two closed subspaces $Y$ and $Z$ form a complemented sum if and only if there is a $c \ge 1$ so that for any $y \in Y$ and and $z \in Z$

$$\|y + z\| \le \|y\| + \|z\| \le c\|y + z\| \tag{8}$$

Indeed, assume (8) for all $y \in Y$ and $z \in Z$. In order to show that $Y + Z$ is closed assume that $(y_n) \subset$ and $(z_n) \subset Z$ and assume that $w = \lim_{n\to\infty} y_n + z_n$ exists. Since for $m, n \in \mathbb{N}$

$$\|y_m - y_n\| + \|z_m - z_n\| \le c\|y_m + z_m - (y_n + z_n)\|$$

$(y_m)$ and $(z_n)$ converge, say to $y$ and $z$, respectively. It folllows that $w = y + z$, and thus that $w \in Y + Z$. If $w \in Y \cap Z$ then

$$2\|w\| \le c\|w - w\| = 0,$$

thus, $w = 0$.

Conversely if $Y + Z$ is closed and $Y \cap Z = \{0\}$ we define the Banach space $Y \oplus Z$ to be the complemented sum of $Y$ and $Z$, i.e. $Y \oplus Z = Y \times Z$ with $\|(y, z)\| = \|y\| + \|z\|$. Then the mapping

$$Y \oplus Z \ni (y, z) \mapsto y + z \in Y + Z,$$

is bounded (triangle inequality) linear, injective ($Y \cap Z = \{0\}$) and is surjective. Since its range is a Banach space it follows from the Closed Graph Theorem that it is an isomorphism, which implies (8) for some $c \ge 1$ and all $y \in Y$ and $z \in Z$.

Hence it follows that (define $0/0=0$)

$$\begin{aligned} B &= \Big\{(\overline{y},\overline{z})\in X^{\mathbb{N}}\times X^{\mathbb{N}} : \exists c>0 \text{ (8) holds for } Y=\overline{\operatorname{span}((\overline{y})} \text{ and } Z=\overline{\operatorname{span}((\overline{z})} \Big\}\\ &= \bigcup_{k\in\mathbb{N}} \bigcap_{\substack{a_1,a_2,\dots a_m\\ b_1,b_2,\dots b_n\in\mathbb{Q}}} \Big\{(\overline{y},\overline{z})\in X^{\mathbb{N}}\times X^{\mathbb{N}} : \frac{\big\|\sum_{i=1}^m a_i y_i\big\| + \big\|\sum_{i=1}^n b_i z_i\big\|}{\big\|\sum_{i=1}^m a_i y_i + \sum_{i=1}^n b_i z_i\big\|} \le k\Big\}. \end{aligned}$$

Since for $a_1,a_2,\dots a_m, b_1,b_2,\dots b_n\in\mathbb{Q}$ the maps

$$X^{\mathbb{N}}\to\mathbb{R},\quad \overline{x}\mapsto \Big\|\sum_{i=1}^n a_i x_i\Big\|, \text{ and}$$
$$X^{\mathbb{N}}\times X^{\mathbb{N}}\to\mathbb{R}, (\overline{y},\overline{z})\mapsto \Big\|\sum_{i=1}^m a_i y_i + \sum_{i=1}^n b_i z_i\Big\|$$

are continuous, our claim follows. □

LEMMA 1.11. *If $X$ is a Banach space, then the set of reflexive subspaces* Ref($X$) *is co analytic in* Subs($X$)

Before proving Lemma 1.11 we need to recall the following fact

REMARK. Note that a Banach space $X$ is not reflexive if and only its unit ball $B_X$ is not weakly compact, which is equivalent of $B_X$ not being weakly sequentially compact.

So a space $X$ is not reflexive if and only if there is a sequence $(x_n)$ in the sphere $S_X$ which does not have any weakly convergent sub sequence. By Rosenthal's $\ell_1$ Theorem this implies that $(x_n)$ either has a subsequence which is equivalent to the $\ell_1$-basis, or it has a non-trivial (i.e not weakly converging) weak Cauchy subsequence. In the second case this implies (see cf. [**AGR**, Proposition II.1.5]) that it has an *s-subsequence*, which implies that it dominates the summing basis of $c_0$. Thus in both cases we obtain from the non reflexivity of $X$ a normalized basic sequence which dominates the summing basis of $c_0$. Conversely, if $X$ contains a normalized basic sequence $(x_n)$ which dominates the summing basis of $c_0$, then no convex block of $(x_n)$ is norm convergent, and thus $(x_n)$ has no subsequence converging weakly to 0. But this means (since $(x_n)$ is basic) that $(x_n)$ cannot have any weak converging subsequence, which implies that $X$ cannot be reflexive.

PROOF OF LEMMA 1.11. We will show that the set NRef($X$) of nonreflexive subspaces of $X$ is analytic. Note that $Y\in$ Subs($X$) is non reflexive if and only if there is a normalized basic sequence $(y_n)$ which dominates the summing basis $(s_n)$ of $c_0$.

Let $A$ be the subset of $S_X^{\mathbb{N}}$ consisting of all basic sequences which dominates the summing basis. By Lemma 1.7 $A$ is an $\mathcal{G}_\delta$-set in $S_X^{\mathbb{N}}$ and, thus Polish. Let $D\subset S_X$

be countable and dense in $S_X$. We first note that the set

$$\begin{aligned}\Big\{\big((y_n),Y\big) \in A\times\mathrm{Subs}(X) : (y_n)\subset Y\Big\} \\ = \bigcap_{\substack{m\in\mathbb{N}\\ \varepsilon>0,\varepsilon\in\mathbb{Q}}} \bigcup_{d_1,\dots d_m\in D} \big\{(y_n)\in A : \|y_n-d_n\|<\varepsilon,\ \text{if } n\le m\big\} \\ \times\big\{Y\in\mathrm{Subs}(X) : B_\varepsilon(d_i)\cap Y\neq\emptyset,\ \text{for } i\le m\big\}\end{aligned}$$

is Borel in $A\times\mathrm{Subs}(X)$, then observe that

$$\mathrm{NRef}(X) = \exists^A\Big\{\big((y_n),Y\big)\in A\times\mathrm{Subs}(X) : (y_n)\subset Y\Big\}.$$

□

## 2. Universal spaces with bases

The following Theorem is due to Pełczyński, the much shorter proof to Schechtman.

THEOREM 2.1. [**Pelc**] and [**Schech**]

1) *There exists a separable Banach space $U_u$ with a normalized 1-unconditional basis $(u_i)$ so that for every normalized unconditional basic sequence $(x_i)$ and $\varepsilon>0$ there is a subsequence $(u_{n_i})$ of $(u_i)$ which is $(1+\varepsilon)K$-equivalent to $(x_i)$, where $K$ is the unconditionality constant of $(x_i)$.*
2) *There exists a separable Banach space $U_c$ with a monotone basis $(v_i)$ so that for every normalized basic sequence $(x_i)$ and $\varepsilon>0$ there is a subsequence $(u_{n_i})$ of $(u_i)$ which is $(1+\varepsilon)K$-equivalent to $(x_i)$, where $K$ is the monotonicity constant of $(x_i)$. Moreover the closed linear span of $(u_{n_i})$ is 1-complemented in $U_c$.*

*Moreover the spaces $U_u$ and $U_c$ are determined uniquely (up to isomorphism) by the properties (1) and (2) respectively. In (2) we can replace monotone by bimonotone.*

PROOF. We choose a dense countable sequence $(d_n)$ in $S_{C(\Delta)}$.
(1) We define for $(a_i)\in c_{00}$

$$\Big\|\sum a_iu_i\Big\|_u = \sup\Big\{\Big\|\sum\sigma_ia_id_i\Big\| : \sigma_i=\pm1 \text{ for } i\in\mathbb{N}\Big\}.$$

Clearly, $(u_i)$ is an unconditional basis for the completion of $c_{00}$ under $\|\cdot\|_u$.

If $(x_i)$ is any normalized unconditional basic sequence of a Banach space $X$, we can assume w.l.o.g. that $X$ is a subspace. of $C(\Delta)$ and therefore we can choose a subsequence $(d_{n_i})$ of $(d_n)$ so that $\|x_i-d_{n_i}\|_{C(\Delta)}<\varepsilon 2^{-i}$, it follows that for $(a_i)\in c_{00}\cap S_{U_u}$ that

$$\Big\|\sum a_ix_i\Big\| \le \sup_{\sigma=\pm1}\Big\|\sum\sigma_ia_ix_i\Big\| \le \varepsilon + \Big\|\sum\sigma_1a_id_{n_i}\Big\|,$$

and

$$\Big\|\sum a_i x_i\Big\| \geq \frac{1}{K}\sup_{\sigma=\pm 1}\Big\|\sum \sigma_i a_i x_i\Big\| \geq \frac{1}{K}\sup_{\sigma=\pm 1}\Big\|\sum \sigma_i a_i x_i\Big\| \geq \frac{1}{K}\Big[-\varepsilon + \Big\|\sum \sigma_i a_i d_{n_i}\Big\|\Big],$$

which implies our claim.
(2) We define on $c_{00}(\mathbb{N}^{<\mathbb{N}})$ (where the usual basis is denoted by $(e_s : s \in \mathbb{N}^{<\mathbb{N}})$) For $\big(a(s) : s \in \mathbb{N}^{<\mathbb{N}}\big)$ we put

$$\Big\|\sum a(\alpha_1,\alpha_2,\dots\alpha_i)e_{\alpha_i}\Big\|_c = \max\Big\{\Big\|\sum_{i=1}^{j} a(\alpha_1,\alpha_2,\dots\alpha_i)d_{\alpha_i}\Big\|_{C(\Delta)} : \alpha = (\alpha_i) \in \mathbb{N}^{\mathbb{N}}, j \in \mathbb{N}\Big\}.$$

If we want to achieve a bimonotne basis we put for $\big(a(s) : s \in \mathbb{N}^{<\mathbb{N}}\big)$ we put

$$\Big\|\sum a(\alpha_1,\alpha_2,\dots\alpha_i)e_{\alpha_i}\Big\|_{bc} = \max\Big\{\Big\|\sum_{i=j_1}^{j_2} a(\alpha_1,\alpha_2,\dots\alpha_i)d_{\alpha_i}\Big\|_{C(\Delta)} : \alpha = (\alpha_i) \in \mathbb{N}^{\mathbb{N}}, j_1 \leq j_2 \in \mathbb{N}\Big\}.$$

Then we order $(e_s : s \in \mathbb{N}^{<\mathbb{N}})$ linearly in a way compatible with $\prec$ and define $(v_i)$ to be the basis $(e_s : s \in \mathbb{N}^{<\mathbb{N}})$ in this linear ordering. Similar arguments as in the proof of (1) show that $(v_s)$ is a (bi)monotone basis that for every normalized (bi)monotnone basic sequence $(x_i)$ and $\varepsilon > 0$ there is a subsequence $(v_{n_i})$ of $(v_i)$ which is $(1+\varepsilon)K$-equivalent to $(x_i)$, where $K$ is the monotonicity constant of $(x_i)$. An important point is that in this case every subsequence $v_{n_i}$ appears as a branch of $(e_s : s \in \mathbb{N}^{<\mathbb{N}})$ in its linear span is thus complemented.

In order to show the statement starting with "moreover" we use Pełczyński's decomposition method. We first observe that that for two spaces $U$ and $V$ which both satisfy (1) or (2) respectively, it follows that for some Banach space $X$, $Y$ and $Z$

$$U \sim V \oplus X, \quad V \sim U \oplus Y \text{ and } U \sim (U \oplus U \oplus \dots)_{\ell_2} \oplus Z,$$

and, thus,

$$U \oplus U \sim U \oplus (U \oplus U \oplus \dots)_{\ell_2} \oplus Z \sim U.$$

Similarly we show that $V \oplus V \sim V$, and, thus,

$$U \sim V \oplus X \sim V \oplus V \oplus X \sim V \oplus U \sim Y \oplus U \oplus U \sim Y \oplus U \sim V.$$

□

Let $\overline{w} = (w_i)$ be a normalized basic sequence. To each tree $T$ on $\mathbb{N}$ we assign Banach spaces $X_{(\overline{w},T,p)}$, $1 \leq p \leq \infty$, in the following way. We call two intervals $I, J$ in $T$ are called *incomparable* if for any $s \in I$ and any $t \in J$ $s$ and $t$ are not comparable (i.e. extensions of one another).

For $(a_s)_{s\in T}\in c_{00}(T)$ we put for $1\le p<\infty$

(9) $$\Big\|\sum_{s\in T}a_s e_s\Big\|_{(\overline{w},T,p)}=\max\left\{\Big(\sum_{i=1}^n\Big\|\sum_{s\in J_i}a_s w_{|s|}\Big\|^p\Big)^{1/p}:\ \begin{matrix}n\in\mathbb{N},\ J_1,J_2,\dots J_n\subset T\\ \text{incomparable intervals}\end{matrix}\right\}$$

and

(10)
$$\Big\|\sum_{s\in T}a_s e_s\Big\|_{(\overline{w},T,\infty)}=\max\left\{\Big\|\sum_{s\in J}a_s w_{|s|}\Big\|:\ J\subset T\text{ interval}\right\}$$

We denote the completion of $c_{00}$ with respect to $\|\cdot\|_{(\overline{w},T,p)}$ by $X_{(\overline{w},T,p)}$ respectively. It is easy to see that if $(e_s:s\in T)$ is linear ordered in a compatible way to $\succ$, it is a monotone basis of these spaces.

Proposition 2.2. *If $T$ is a well founded tree then the basis $(e_s:s\in T)$ (ordered compatibly with $\succ$) is*

a) *boundedly complete in $X_{(\overline{w},T,1)}$.*
b) *boundedly complete and shrinking in $X_{(\overline{w},T,p)}$, $1<p<\infty$.*
c) *shrinking in $X_{(\overline{w},T,\infty)}$.*

*If $\overline{w}$ is the universal basis $(u_i)$ or $(v_i)$ constructed in Theorem 2.1 and if $T$ is not well founded then $X_{(\overline{w},T,p)}$ is isomorphic to $U_u$ or $U_c$, respectively.*

Proof. We will only show (a) of the first part of our claim, (b) and (c) can be shown similarly.

We will show $(a)$ by tranfinite induction on $\alpha=o(T)$. For $\alpha=1$ (and thus $T=\{\emptyset\}$) our claim is trivially true.

Assume that the claim is true for all $\beta<\alpha$ and that $o(T)\le\alpha$. If $(a_s)_{s\in T}\in c_{00}(T)$ and $a_\emptyset=0$ we observe that (recall that by Proposition 2.5 it follows for $k\in\mathbb{N}$ and $T(k)=\{s\in\mathbb{N}^{<\mathbb{N}}:(k,s)\in T\}$ hat $o(T(k))<\alpha$)

$$\begin{aligned}\Big\|\sum_{s\in T}a_s e_s\Big\|_{(\overline{w},T,1)}&=\max\left\{\sum_{i=1}^n\Big\|\sum_{s\in J_i}a_s w_{|s|}\Big\|:\ \begin{matrix}n\in\mathbb{N},\ J_1,J_2,\dots J_n\subset T\\ \text{incomparable intervals}\end{matrix}\right\}\\ &=\sum_{k\in\mathbb{N}}\max\left\{\sum_{i=1}^n\Big\|\sum_{s\in J_i}a_{(k,s)}w_{|(k,s)|}\Big\|:\ \begin{matrix}n\in\mathbb{N},\ J_1,J_2,\dots J_n\subset T(k)\\ \text{incomparable intervals}\end{matrix}\right\}\\ &=\sum_{k\in\mathbb{N}}\Big\|\sum_{s\in T(k)}a_{(k,s)}e_s\Big\|_{(\overline{w}',T(n),1)}\text{ where }\overline{w}'=(w_{i+1})_{i=1}^\infty,\end{aligned}$$

which implies that $X_{(\overline{w},T,1)}$ is a 1-dimensional extension of the space

$$\big(\oplus_{k\in\mathbb{N}}X_{(\overline{w}',T(k),1)}\big)_{\ell_1},$$

From our induction hypothesis it follows that for each $k\in N$ the basis $(e_s:s\in T(k))$ is boundedly complete in $X_{(\overline{w}',T(k),1)}$. From this it is easy to see that any compatible reordering of $\{e_\emptyset\}\bigcup_{k\in\mathbb{N}}\{e_{(k,s)}:s\in T(k)\}$ is a boundedly complete basis for $X_{(\overline{w},T,1)}$.

If $T \in \mathrm{Tr}$ is not well founded then the body $[T]$ of $T$ is not empty, and thus $T$ contains a branch $(\alpha|_n)_{n\in\mathbb{N}}$ which implies that in $X_{(\overline{v},T,p)}$, $(e_{\alpha)_n:n\in\mathbb{N}}$ is isometric to $U_c$ and furthermore its span is complemented. By Theorem 2.1 it follows that $X_{(\overline{v},T,p)}$ is isomorphic to $X_c$. □

## 3. Isomorphism classes of Banach spaces and their complexity

PROPOSITION 3.1. *Let $\overline{w} = (w_i)$, be a basic sequence $1 \le p \le \infty$. For a tree $T \subset \mathbb{N}^{<\mathbb{N}}$ we consider the space $X_{(\overline{w},T,p)}$ to be a subspace of $X_{(\overline{w},\mathbb{N}^{<\mathbb{N}},p)}$ . The map*

$$\Phi : \mathrm{Tr} \to \mathrm{SB}, \quad T \mapsto X_{(\overline{w},T,p)},$$

*is Borel measurable.*

PROOF. Let $U \subset X_{(\overline{w},\mathbb{N}^{<\mathbb{N}},p)}$ be open. We need to show that $\{T \in \mathrm{Tr} : \Phi(T) \cap U \neq \emptyset\}$ is Borel measurable in Tr. Note that

$$\begin{aligned} C &= \{T \in \mathrm{Tr} : \Phi(T) \cap U \neq \emptyset\} \\ &= \{T \in \mathrm{Tr} : U \cap X_{(\overline{w},T,p)} \neq \emptyset\} \\ &= \{T \in \mathrm{Tr} : \exists n \in \mathbb{N}\ \exists s_1, s_2, \dots s_n \in T \exists q_1, q_2, \dots q_n \in \mathbb{Q} \quad \sum_{i=1}^n q_i e_{s_i} \in U\} \\ &\text{[Since } U \text{ is open]} \\ &= \bigcup_{n\in\mathbb{N}} \bigcup_{q_1,q_2,\dots q_n \in\mathbb{Q}} \bigcup_{\substack{s_1,\dots s_n \in \mathbb{N}^{<\mathbb{N}} \\ \text{so that } \sum_{i=1}^n q_i e_{s_i} \in U}} \bigcap_{i=1}^n \{T \in Tr : s_i \in T\}, \end{aligned}$$

which implies our claim. □

Using now Proposition 3.2 (2), Lemma 1.11 and Proposition 3.1 we deduce the following Corollary.

COROLLARY 3.2. Ref *is completely co-analytic in $SB$.*

PROPOSITION 3.3. *Let $U_c$ be the Banach space defined (up to isomorphism uniquely) in Theorem 2.1. Then*

$$\mathcal{I}s(U_c) = \{X \in \mathrm{SB} : X \text{ is isomorphic to } U_c\}$$

*is analytic but not a Borel set.*

PROOF. Let $\tilde{v} = (v_i)$ be the universal basis of Theorem 2.1 and

$$\Psi : \mathrm{Tr} \to \mathrm{SB}, \quad T \mapsto X_{(\tilde{v},T,2)}.$$

By Proposition 3.1, $\Psi$ is measurable and t $\Phi^{-1}(\mathcal{I}s(U_c)) = \mathrm{Tr} \setminus \mathrm{WF}$. It follows that $\mathcal{I}s(U_c)$ cannot be Borel since WF is not Borel (Example 3.4).

On the other hand, let $(u_i)$ be the universal basis of $U_c$. Then the set

$$A = \{(x_n) \in C(\Delta)^{\mathbb{N}} : (x_n) \sim (u_n)\}$$

is by Lemma 1.7 closed in $C(\Delta)^{\mathbb{N}}$, and $\mathcal{I}s(U_c)$ is image of $A$ under the Borel measurable map $\Psi$ defined in Lemma 1.8. □

THEOREM 3.4. [**Boss2**, Theorem 2.3] *The isomorphism relation $\simeq$ is analytic but not Borel, i.e. the set* $\mathrm{SpEq} = \{(X,Y) : X, Y \in \mathrm{SB}, X \simeq Y\}$ *is analytic in* $\mathrm{SB}\times\mathrm{SB}$ *but not Borel.*

*Moreover, the isomorphism relation has no analytic section.*

*Recall that a* section *of an equivalence relation $\sim$ on a set $A$ is a subset $S \subset A$ which meets every equivalence class exactly once.*

PROOF. Let $\pi_1, \pi_2 : \mathrm{SB}\times\mathrm{SB} \to \mathrm{SB}$, be the projection on the first coordinate and second coordinate respectively.

From Lemma 1.9 we know that the set

$$\mathrm{SeqEq} = \big\{\big((x_n),(y_n)\big) \in C(\Delta)^{\mathbb{N}}\times C(\Delta)^{\mathbb{N}} : (x_n) \sim (y_n)\big\},$$

is Borel in $C(\Delta)^{\mathbb{N}}\times C(\Delta)^{\mathbb{N}}$. Note that SpEq is the image of SeqEq under the map

$$(\Psi,\Psi) : C(\Delta)^{\mathbb{N}}\times C(\Delta)^{\mathbb{N}} \to \mathrm{SB}\times \mathrm{SB}, \quad \big((x_n),(y_n)\big) \mapsto \Big(\overline{\mathrm{span}(x_n)}, \overline{\mathrm{span}(y_n)}\Big),$$

which by Lemma 1.8 is measurable.This implies the claim since images of Borel sets under Borel measurable maps are analytic.

On the other hand SpEq cannot be Borel in $\mathrm{SB}\times\mathrm{SB}$ since by Proposition $\mathcal{I}s(U_c)$ is not Borel. Indeed

$$\mathcal{I}s(U_s) = \pi_1^{-1}\Big(\mathrm{SpEq} \cap \big(\mathrm{SB} \times \{U_c\}\big)\Big).$$

In order to show the second part assume that $S \subset \mathrm{SB}$ is an analytic section of the equivalence relation. Let $U'$ the unique element in $\mathcal{I}s(U_c) \cap S$. Then also $S \setminus \{U'\}$ is analytic, so is the set

$$\begin{aligned}
&\big\{(X,Y) \in \mathrm{SB}\times\mathrm{SB} : Y \in S, X \simeq Y, X \notin \mathcal{I}s(U_c)\big\}\\
&\quad = \big\{(X,Y) \in \mathrm{SB}\times\mathrm{SB} : Y \in S \setminus \{U'\}, X \simeq Y\big\}\\
&\quad = (\Psi,\Psi)\big\{((x_n),(y_n)) \in C(\Delta)^{\mathbb{N}}\times C(\Delta)^{\mathbb{N}} : \Psi((y_n)) \in S \setminus \{U'\}, (x_n) \sim (y_n)\big\}.
\end{aligned}$$

Thus the $\pi_1$-image of that set should also be analytic. However that image is the set $\mathrm{SB} \setminus \mathcal{I}s(U_c)$, which by Proposition 3.3 co-analytic but not Borel, and thus by the Separation Theorem 4.1 not analytic. □

THEOREM 3.5. [**Boss2**, Theorem 2.3]
*The following relations are analytic but not Borel in their respective Polish spaces:*

a) $\mathrm{Emb} = \{(X,Y) \in \mathrm{SB}\times\mathrm{SB} : X \hookrightarrow Y\}$,
b) $\mathrm{CompEmb} = \{(X,Y) \in \mathrm{SB}\times\mathrm{SB} : \exists Z \in \mathrm{SB} \quad X \simeq Z \oplus Y\}$
c) $\mathrm{CompEq} = \{(X,Y,Z) \in \mathrm{SB}\times\mathrm{SB}\times \mathrm{SB} : \quad Z \simeq X \oplus Y\}$

REMARK. Theorem 3.5 implies that the following sets are analytic:

a) $\mathrm{Emb}(X) = \{Y \in \mathrm{Subs}(X) : Y \hookrightarrow X\}$
a) $\mathrm{CompEmb}(X) = \{Y \in \mathrm{Subs}(X) : Y \hookrightarrow X \text{ complemented}\}$

Proof of Theorem 3.5. We first show that all three relations are not Borel. For $(a)$ consider the the equivalence relation $\sim_{\text{Emb}}$ on SB given by

$$X \sim_{\text{Emb}} Y \iff X \hookrightarrow Y \text{ and } Y \hookrightarrow X.$$

If Emb where Borel measurable, then also $B = \{(X,Y) \in \text{SB}\times\text{SB} : X \sim_{\text{Emb}} Y\}$ would be Borel measurable. Let $\tilde{v} = (v_i)$ be the universal basis of Theorem 2.1 and

$$\Psi : \text{Tr} \to \text{SB}, \quad T \mapsto X_{(\tilde{v},T,2)}.$$

By Proposition 3.1, $\Psi$ is measurable and $\Phi^{-1}(\{X \in \text{SB} : X \sim_{\text{Emb}} U_c\}) = \text{Tr} \setminus \text{WF}$. Which implies that $\{X \in \text{SB} : X \sim_{\text{Emb}} U_c\}$ is not Borel, and thus Emb cannot be Borel either.

Replacing above introduced equivalence relation $\sim$ on SB given, by

$$X \sim_{\text{comp}} Y \iff X \hookrightarrow Y \text{complemented, and } Y \hookrightarrow X \text{ complemented.}$$

we can argue similarly as in part (a) to deduce that CompEmb is not Borel.

In order to show that CompEq is not Borel we consider the map

$$\Phi : \text{Tr}^2 \to \text{SB}^3, \quad (S,T) \mapsto (\Psi(X), \Psi(Y), U_c)$$

where $\Psi$ is defined as above and note that

$$\Phi^{-1}\big(\text{CompEq} \cap (\text{SB}^2 \times \{U_c\}\big) = \text{Tr}^2 \setminus \text{WF}^2,$$

(note that for two well founded trees $S$ and $T$ the space $X_{(\tilde{v},S,2)} \oplus X_{(\tilde{v},T,2)}$ cannot be isomorphic to $U_c$) which implies that CompEmb cannot be Borel.

We now prove that the sets Emb, CompEmb, CompEq are analytic.

(a) (by Alejandro Chavez Dominguez)

By Lemma 1.9 (d) the set

$$B = \{(X,Y) \in \text{SB}^2 : X \subset Y\}$$

is Borel in $\text{SB}^2$. Using Theorem 3.4 we deduce that

$$A = \big\{(X,Y,Z) \in \text{SB}^3 X \sim Y, Y \subset Z\big\} = \big(\text{SpEq} \times SB\big) \cap \big(\text{SB} \times B\big)$$

is analytic. Since Emb is the projection of $A$ onto the first and third coordinate we deduce the claim.

(c) (by Francisco Torres-Ayala) First note that the map

$$C(\Delta)^{\mathbb{N}} \times C(\Delta)^{\mathbb{N}} \ni (\overline{x}, \overline{y}) \mapsto \overline{x} * \overline{y} := (x_1, y_1, x_2, y_2, \ldots) \in C(\Delta)^{\mathbb{N}}$$

is continuous and thus Borel measurable. This, together with Lemma 1.10 yields that

$$B' = \left\{ \big((\overline{x}, \overline{y}, \overline{z}) \in \Big(C(\Delta)^{\mathbb{N}}\Big)^3 : \begin{matrix} \overline{\text{span}(\overline{x})} \text{ and } \overline{\text{span}(\overline{y})} \text{ form a complemented sum} \\ \text{and } \overline{z} = \overline{x} * \overline{y} \end{matrix} \right\},$$

is Borel in $\big(C(\Delta)^{\mathbb{N}}\big)^3$.

Secondly, consider the map

$$(\Psi, \Psi, \Psi) : \Big(C(\Delta)^{\mathbb{N}}\Big)^3 \to \text{SB}^3, \quad (\overline{x}, \overline{y}, \overline{z}) \mapsto \big(\overline{\text{span}(\overline{x})}, \overline{\text{span}(\overline{y})}, \overline{\text{span}(\overline{z})}\big),$$

which is by Lemma 1.8 Borel measurable. Since images of Borel sets under Borel measurable maps are analytc we deduce that the set

$$A = \{(X, Y, Z) \in SB^3 : X \cap Y = \{0\}, X + Y = Z\}$$

analytic. Further more the following sets are analytic by Theorem 3.4

$$I_1 = \{(X, Y, Z, U, V, W) \in \mathrm{SB}^6 : X \sim U\}$$
$$I_2 = \{(X, Y, Z, U, V, W) \in \mathrm{SB}^6 : Y \sim V\}$$
$$I_3 = \{(X, Y, Z, U, V, W) \in \mathrm{SB}^6 : Z \sim W\}$$

are analytic. Finally note that our set is the projection of

$$\big(A \times \mathrm{SB}^3\big) \cap I_1 \cap I_2 \cap I_3$$

on the last three coordinates.
(b) follows from (c) by projection onto the first and third coordinate. □

Corollary 3.6. *Assume* $\mathcal{A} \subset \mathrm{SB}$ *analytic. Then the* isomorphic hull of $\mathcal{A}$, *defined by*

$$\tilde{\mathcal{A}} = \big\{Y \in \mathrm{SB} : \exists X \in \mathcal{A}\ Y \simeq X\big\}$$

*is also analyitc in* SB.

Proof. Note that

$$\tilde{\mathcal{A}} = \pi_1\big(\mathrm{SpEq} \cap \mathrm{SB} \times \mathcal{A}\big).$$

□

## 4. Universality Theorems

Definition 4.1. Let $\mathcal{A} \subset$ Subs and $X$ a separable Banach space. We say $X$ *is* $\mathcal{A}$*-universal* or *universal for* $\mathcal{A}$ if every $Y \in \mathcal{A}$ (isomorphically) embeds into $X$.

If $\mathcal{A} =$ Subs we simply say that $X$ is universal if it is Subs-universal.

Theorem 4.2. [**Boss2**] *If* $\mathcal{A} \subset \mathrm{SB}$ *is analytic and contains all separable reflexive spaces, it must contain a universal element.*

Proof. Let $\mathcal{A} \subset \mathrm{SB}$, so that Ref $\subset \mathcal{A}$. Since Ref is completely co-analytic (by Corollary 3.2) and, thus, it cannot be analytic.

Let $\Phi : \mathrm{Tr} \to \mathrm{SB}$ the map defined in Proposition 3.1 for $p = 2$ and $\overline{w} = (u_i)$, the basis of the universal space $U_c$ defined in Theorem 2.1. Since $\Phi$ is measurable $\Phi^{-1}(\mathcal{A})$ is analytic in Tr and contains by Proposition 2.2 the set WF. Since WF is completely co-analytic, and thus not analytic $\Phi^{-1}(\mathcal{A}) \setminus \mathrm{WF}$ cannot be empty.

By Proposition 2.2, second part, it follows now for any $T \in \Phi^{-1}(\mathcal{A}) \setminus \mathrm{WF}$ that $\Phi(T) = U_c$. □

Corollary 4.3. [**Bour**] *Every separable space which is universal for the class of separable reflexive spaces is universal.*

PROOF. Let $X \in \mathrm{SB}$ be universal for Ref. By Theorem 3.5 and the remark following Theorem 3.5 the set

$$\mathrm{Emb}(X) = \{Y \in \mathrm{SB} : Y \hookrightarrow X\}$$

is analytic, and thus, must contain a universal element. $\square$

COROLLARY 4.4. [**Sz**] *There is no separable reflexive space which contains isomorphically all separable reflexive spaces.*

## 5. A Problem

DEFINITION 5.1. Let $X$ be a Banach space. We call a family $(x_t : t \in \mathbb{N}^{<\mathbb{N}})$ *a tree in* $X$. If $(x_t : t \in \mathbb{N}^{<\mathbb{N}})$ is a tree in $X$ we call sequences of the form $(x_{(t,n)} : n \in \mathbb{N})$ *nodes of* $(x_t : t \in \mathbb{N}^{<\mathbb{N}})$, and we call sequence $(y_k : k \in \mathbb{N}_0)$, where $y_k = x_{(n_1,n_2\dots,n_k)}$, $k \in \mathbb{N}_0$, for some sequence $(n_i) \subset \mathbb{N}$, a *branch of* $(x_t : t \in \mathbb{N}^{<\mathbb{N}})$.

*A normalized weakly null tree in* $X$ is a tree $(x_t : t \in \mathbb{N}^{<\mathbb{N}})$ so that $\|x_t\| = 1$, and the node $(x_{(t,n)})_{n\in\mathbb{N}}$ is weakly null, for all $t \in \mathbb{N}^{<\mathbb{N}}$.

We say $X$ has the *$C$-$\ell_p$-tree property*, for some $C \geq 1$, if every normalized weakly null tree has a branch which is equivalent to the $\ell_p$-unit vector basis.

REMARK. As shown in [**OS3**] if every normalized weakly null tree in $X$ has a branch equivalent to the $\ell_p$ unit vector basis, then there is a $C$ so that $X$ has the $C$-$\ell_p$-tree property (i.e. one can find a uniform $C$ which "works" for all normalized weakly null trees).

In [**OS1**] the following result was shown.

THEOREM 5.2. [**OS3**]
*If $X$ is separable, reflexive and has the $\ell_p$-tree property, then there is a sequence of finite dimensional spaces $(F_n)$ so that $X$ embeds (isomorphically) into $(\oplus F_n)_{\ell_p}$.*

An easy Corollary to that result is the following

COROLLARY 5.3. *Among the separable reflexive Banach spaces which have the $\ell_p$-tree property, there are universal ones.*

PROOF. Recall that for $n \in \mathbb{N}$ the set of all finite-dimensional Banach spaces, with the *Banach Mazur distance $d_{BM}$* is separable.

$d_{BM}$ is the metric on $\{(\mathbb{R}^n, \|\cdot\|) : \|\cdot\| \text{ is norm on } \mathbb{R}^n\}$ defined by

$$d_{BM}\big((\mathbb{R}^n, \|\cdot\|), (\mathbb{R}^m, \|\!|\cdot|\!\|)\big)$$
$$= \begin{cases} 1 & \text{if } n \neq m \\ \log\inf\big\{\|T\|\cdot\|T^{-1}\| : T : (R^n, \|\cdot\|) \to (R^n, \|\!|\cdot|\!\|) \text{ isom.}\big\} & \text{if } n = m \end{cases}$$

(We identify spaces which are isometric images of each other).

Choose a dense sequence $(F_n)$ of finite dimensional spaces and take $U = (\oplus F_n)_{\ell_p}$. $\square$

Theorem 3.5 and the Remark following that theorem yield:

COROLLARY 5.4. *The set of of all reflexive spaces (as subset of* SB*) which have the $\ell_p$-tree property are analytic.*

PROBLEMS 5.5. *Assume* $1 < p < \infty$.

a) *Is there a direct proof of Corollary 5.4 ?*
b) *Is it true that the set of all spaces with separable dual having the $\ell_p$-tree property are analytic?*
c) *Is it true that the set of* **all** *separable spaces having the $\ell_p$-tree property is analytic.*

Maybe the following result (which is part of the proof of Theorem 5.2) might help:

Let $X$ be a space with separable dual. By a theorem of Zippin (see Section 8) $X$ embeds into a space $Z$ with a shrinking bases $(z_i)$ (i.e. the coordinate functionals of that basis are a basis of $Z^*$). $Z$ can be chosen to be reflexive if $X$ is reflexive.

Let us define for $m \le n$ in $\mathbb{N}$ the following projection:

$$P_{[m,n]} : Z \to Z, \quad \sum a_i z_i \mapsto \sum_{i=m}^{n} a_i z_i.$$

Let $\overline{\delta} = (\delta_i)$ be a null sequence in $(0,1)$. We call a normalized sequence $(y_i)$ in $Z$ a *$\overline{\delta}$-skipped block sequence in $Z$,* if there are $1 \le m_1 < n_1 < m_2 < n_2 \dots$ in $\mathbb{N}$ so that

$$\|P_{[m_i+1,n_i]}(y_i) - y_i\| < \delta_i.$$

(i.e. $(y_i)$ is close to a *skipped block sequence*, a block sequence in which at least one coordinate is skipped in between the vectors).

Let $K = (k_i)$ be strictly increasing. We call a normalized sequence $(y_i)$ in $Z$ a $(\overline{\delta}, K)$-skipped block sequence in $Z$, if there are $1 \le m_1 < n_1 < m_2 < n_2 \dots$ in $K$ so that

$$\|P_{[m_i+1,n_i]}(y_i) - y_i\| < \delta_i.$$

Using now a *game interpretation* of the tree conditions we can show the following equivalence:

PROPOSITION 5.6. *Let $\delta_i = 2^{-i}$ (or your favored summable sequence in* $(0,1)$*). The following are equivalent.*

a) *$X$ has the $\ell_p$ tree property.*
b) *There exists a $C \ge 1$ and a strictly increasing sequence $K = (k_i)$ in $\mathbb{N}$ so that every $(\overline{\delta}, K)$-skipped block $(z_n)$ which is in $X$ is $C$ equivalent to the $\ell_p$-unit vector basis.*

*(Note that (b)⇒(a) is easy since every weakly null sequence has a subsequence which is a $(\overline{\delta}, K)$-skipped block)*

REMARK. In general, i.e. if the super space $Z$ has not a shrinking basis, (a) and (b) in Proposition 5.6 might not be equivalent. Note that $\ell_1$ satisfiesthe $\ell_p$-tree condition because of trivial reasons. But the following question makes sense.

PROBLEM 5.7. *Assume that $Z$ is a Banach space with a normalized basis $(z_i)$ (for example $Z = C(\Delta)$).*

*Is the following set analytic in* $\mathrm{Subs}(Z)$ *($\overline{\delta} = (\delta_i) \subset (0,1)$ is a fixed summable sequence):*

$$\left\{ X \in \mathrm{Subs}(Z) : \begin{array}{c} \exists K \subset \mathbb{N} \text{ infinite, so that all} \\ (\overline{\delta}, K)\text{-skipped blocks are equivalent to } \ell_p\text{-unit basis} \end{array} \right\}.$$

CHAPTER 7

# Amalgamations of Schauder trees

This section presents part of the paper [**AD**]

Definition 0.8. Let $\Gamma$ be a set. We call a set $T \subset \Gamma^{<\mathbb{N}} \setminus \{\emptyset\}$ an *unrooted subtree on* $\Gamma$ if $T \cup \{\emptyset\}$ is a tree on $\Gamma$ and in this section notion "tree" will also include unrooted trees.

A family $(x_t)_{t\in T}$ in a Banach space $X$ indexed over a tree $T$ on a countable set $\Gamma$ (i.e. either $\Gamma$ is finite or $\Gamma = \mathbb{N}$) called *Schauder tree in* $X$ if there is a $c > 0$ so that for every $\sigma \in [T]$ $(x_{\sigma|_n})_{n\in\mathbb{N}}$, respectively $(x_{\sigma|_n})_{n\in\mathbb{N}_0}$, if $T$ is rooted, is a basic sequence whose basis constant does not exceed $c$.

In that smallest $c$ so that so that above condition holds the *Schauder tree constant of* $\overline{x}$. We call a Schauder tree monotone if $c = 1$, and we call it bimonotone, if for all $\sigma \in [T]$ $(x_{\sigma|_n})_{n\in\mathbb{N}}$ is bimonotone.

An *amalgamation of a Schauder tree* $(x_t)_{t\in T}$ is a completion $X$ of $c_{00}$ under some norm $\|\cdot\|$ so that the family $(e_t)_{t\in T}$ constitutes a basis of $X$, when ordered compatibly with the order on $T$, an so that for all branches $\sigma$ in $T$ the closed subspace $X_\sigma$ spanned by $(e_s : s \in \sigma)$ is complemented and isomorphic to $(x_s : s \in \sigma)$.

## 1. The $\ell_p$-Baire sum of Schauder tree bases

Definition 1.1. If $\overline{x} = (x_t)$ is a Schauder tree and $1 \le p \le \infty$ we can define the following two norms on $c_{00}(T)$. For $z = (z_t)_{t\in T} \in c_{00}(T)$ we define in case that $1 \le p < \infty$

$$\|z\|_{\overline{x},T,p} = \sup\left\{ \left[\sum_{i=1}^{\ell} \left\| \sum_{t\in S_i} z_t x_t \right\|_X^p \right]^{1/p} : \begin{matrix} \ell \in \mathbb{N},\ S_i \subset T \text{ incomparable} \\ \text{segments for } i = 1, \dots \ell \end{matrix} \right\},$$

$$\|\!|z|\!\|_{\overline{x},T,p} = \sup\left\{ \left[\sum_{i=1}^{\ell} \left\| \sum_{t\in S_i} z_t x_t \right\|_X^p \right]^{1/p} : \begin{matrix} \ell \in \mathbb{N},\ S_i \subset T \text{ disjoint} \\ \text{segments for } i = 1, \dots \ell \end{matrix} \right\},$$

and if $p = \infty$, we define $\|z\|_{\overline{x},T,\infty}$. We denote the completion of $c_{00}(T)$ under $\|\cdot\|_{\overline{x},T,p}$ by $X_{(\overline{x},T,p)}$ and the completion of $c_{00}(T)$ under $\|\!|\cdot|\!\|_{\overline{x},T,p}$ by $Y_{(\overline{x},T,p)}$.

A sequence $(x_n)$ in $c_{00}(T)$ is called *block sequence* if for all $m < n$ in $\mathbb{N}$ and any choice of $s \in \operatorname{supp}(x_m)$ and $t \in \operatorname{supp}(x_n)$, either $s \prec t$ or $s$ and $t$ are not comparable.

Note that the spaces $X_{(\overline{w},T,p)}$ as defined in Subsection 2 are special examples of the spaces $X_{(\overline{x},T,p)}$ introduced in Definition 1.1.

PROPOSITION 1.2. *A sequence $(x_n)$ in $c_{00}(T)$ are a block sequence if and only if there is a linear order $<$ on $T$ which is compatible with the order on $T$ and with respect to which $(x_n)$ is a block sequence of $(e_t)$.*

PROPOSITION 1.3. *Every sequence $(x_n)$ in $c_{00}$ which is coordinate wise null (i.e. $\lim_{n\to\infty} e_t^*(x_n) = 0$ for $t \in T$) has subsequence which is a block sequence $(y_k)$ in.*

PROOF. Note that for any $x \in c_{00}$ $\{s \in T : \exists t \in \text{supp}(x) \quad s \preceq t\}$ is finite. $\square$

PROPOSITION 1.4. *Let $T$ be a tree on $\Gamma$, $\Gamma$ countable, $1 \le p \le \infty$, and let $\overline{x} = (x_t : t \in T)$ be a Schauder tree in a Banach space $X$.*

a) *Any reordering of the unit basis $(e_t)_{t\in T}$ which is compatible with the order on $T$, is a basis of $X_{(\overline{x},T,p)}$ and of $Y_{(\overline{x},T,p)}$, whose basis constant does not exceed the Schauder tree constant $c$ of $\overline{x}$.*

b) *Let $\sigma$ be a branch of $T$. This means that $\sigma = (\sigma_i)_{i\in\mathbb{N}}$ or $\sigma = (\sigma_i)_{i=1}^{\ell}$ is a maximal linear ordered sequence in $T$ $\sigma = (t_1, \dots t_i) : i = 1, \dots \ell$, i.e. $t_i = (\gamma_1, \dots \gamma_i)$ for some sequence $(\gamma_i) \subset \Gamma$ which is either inifinte or of length $\ell$. Let $X_\sigma = X_{(\sigma,\overline{x},T,p)}$ be the closed linear subspace of $X_{(\overline{x},T,p)}$ which is spanned by $(e_t : t \in \sigma)$.*

*Then $X_\sigma$ is isometric to $\overline{span(x_t : t \in \sigma)}$, via the canonical map and $1-$complemented in $X_{(\overline{x},T,p)}$ via the canonincal projection*

$$P_\sigma : X_{(\overline{x},T,p)} \to X_{(\sigma,\overline{x},T,p)}, \quad \sum_{t\in T} a_t e_t \mapsto \sum_{t\in\sigma} a_t e_t.$$

*For $Y_{(\overline{x},T,p)}$ the analogous statement is true, up to a constant $c \ge 1$ if $(x_t : t \in \sigma)$ $c$-dominates the $\ell_p$-unit vector basis.*

PROPOSITION 1.5. *Let $T$ be a tree on $\Gamma$, $\Gamma$ countable, $1 \le p < \infty$, and let $\overline{x} = (x_t : t \in T)$ be a Schauder tree in a Banach space $X$.*
*Assume that $T$ has the following property:*

(*) *There exists a sequences $(N_k) \subset \mathbb{N}$, with $N_k \nearrow \infty$, for $k \nearrow \infty$, and for each*

$$\overline{n} = (n_1, n_2 \dots n_k) \in \bigcup_{\ell=1}^{\infty} \prod_{i=1}^{\ell} \{1, 2, \dots N_i\},$$

*there is a $t(\overline{n}) \in T$, so that the map $\overline{n} \to t(\overline{n})$ is strictly order preserving, i.e.*

$$t(m_1, m_2, \dots m_k) \prec t(n_1, n_2, \dots n_\ell) \iff k < \ell \text{ and } m_i = n_i \text{ for } i = 1, 2 \dots k.$$

*Then $X(\overline{x}, T, p)$ contains copies of $c_0$*

PROOF. W.l.o.g. can we assume that for a given sequence $(\varepsilon_k) \subset (0, 1)$, with $\varepsilon_k \searrow 0$ it follows that $N_{k+1} \ge \frac{1}{\varepsilon_k} N_1 \cdot N_2, \dots N_k$.
for $k \in \mathbb{N}$ choose

$$x_k = \frac{1}{(N_1 \cdot N_2 \cdots N_k)^{1/p}} \sum_{(n_1,n_2,\dots n_k)\in\prod_{i=1}^{k}\{1,2,\dots N_i\}} e_{t(n_1,n_2,\dots n_k)}.$$

Then $\|x_k\| = 1$ for $k \in \mathbb{N}$ (one segement in $T$ can not contain $t(\overline{m})$ and $t(\overline{n})$ with $\overline{m} \neq \overline{n}$ in $\prod_{i=1}^{k}\{1, 2, \dots N_i\}$).

Since an segement going an element of the support of $x_k$ can only contain at most one element of the support of $x_{k+1}$, for $k \in \mathbb{N}$, the sequence $(x_k)$ will be equivalent the $c_0$-unit vectro basis if $(\varepsilon_k)$ decreases fast enough to 0.

□

DEFINITION 1.6. Let $T$ be a tree on $\Gamma$, $\Gamma$ countable, $1 \leq p \leq \infty$, and let $\overline{x} = (x_t : t \in T)$ be a Schauder tree in a Banach space $X$.

For a closed linear and infinite dimensional subspace $Y$ of $X_{(\overline{x},T,p)}$ we say that $Y$ is *$X$-singular* or *$X$-compact* if for all $\sigma \in [T]$ the map $P_\sigma|_Y : Y \to X_{(\sigma,\overline{x},T,p)}$ is strictly singular or compact, respectively.

THEOREM 1.7. *Let $\overline{x} = (x_t)_{t\in T}$ be a bimonotone Schauder tree in a separable Banach space $X$ and let $1 < p < \infty$. Then there is a separable Banach space $Z$ and a $c > 1$ with the following property.*

a) *$Z$ is the completion of some norm on $c_{00}(T)$ and the unit vector basis $(e_t)_{t\in T}$ forms a basis of $Z$ if one linearly orders by $T$ in a way which is compatible with the order $\prec$ on $T$ (i.e $s \prec t \Rightarrow e_s < e_t$).*

b) *For every $\sigma \in [T]$ the space $Z_\sigma = \overline{[e_{\sigma|_n} : n \in \mathbb{N}]}$ is, via the mapping $e_{\sigma|_n} \mapsto x_{\sigma|_n}/\|x_{\sigma|_n}\|$, $c$-isomorphic to the space $X_\sigma = \overline{[x_{\sigma|_n} : n \in \mathbb{N}]}$.*

c) *For every $\sigma \in [T]$ the space $Z_\sigma$ is $c$-complemented in $Z$. We denote the projection by $P_\sigma$.*

d) *For every infinite dimensional subspace $Y$ at least one of the following cases holds true*

*Case 1: Either $Y$ contains an infinite dimensional subspace $W$ so that for every $\sigma \in [T]$ $P_\sigma|_Y$ is strictly singular. Then $W$ contains $\ell_p$.*

*Case 2: Or $Y$ does not contain an infinite dimensional subspace $W$ so that for every $\sigma \in [T]$ $P_\sigma|_Y$ is strictly singular. Then there is a finite set $A \subset [T]$ so that the map*

$$P_A : Y \to \overline{[e_s : s \prec \sigma, \sigma \in A]}, \quad \sum_{s\in T} a_s x_s \mapsto \sum_{s\in\{\sigma|_n : n\in\mathbb{N}, \sigma\in A\}} a_s x_s$$

*is an isomorphic embedding.*

CHAPTER 8

# Zippin's Theorem on embedding separable spaces into spaces with bases

## 1. The Davis, Figiel, Johnson and Pelczynski Factorization

We want to recall the crucial Lemma in [**DFJP**]. We need the following notations:

Let $X$ be a Banach space and $W \subset X$ convex, symmetric, and bounded. For $n \in \mathbb{N}$ define $U_n = 2^n W + 2^{-n} B_X$ and for $x \in X$

$$\|x\|_n = \inf \Big\{ r : \frac{x}{r} \in U_n \Big\} = \inf \big\{ r : \exists w \in W, z \in B_X \quad x = r2^n w + r2^{-n} z \big\}$$
$$|||x||| = \Big( \sum_{n\in\mathbb{N}} \|x\|_n^2 \Big)^{1/2}$$

Put

$$Y = \{ x \in X : |||x||| < \infty \}$$
$$Z = \big( \oplus_{n\in\mathbb{N}} (X, \|\cdot\|_n) \big)_{\ell_2}$$
$$\phi : Y \to Z, \quad y \mapsto (y, y, y \dots). \text{ and}$$
$$j : Y \to X, \text{ formal identity.}$$

Notation. For a vector space $V$ and a set $L$ of linear functionals on $V$, we denote the topology on $V$ generated by $L$ by $\sigma(V, L)$.

Definition 1.1. We call a sequence $(S_n)$ of projections on a Banach space $V$ *a Schauder decomposition of* $V$, if $S_m \circ S_n = S_{\min(m,n)}$, for all $m, n \in N$ and if for all $v \in V$

$$v = \lim_{n\to\infty} S_n(v) = \sum_{n=1}^{\infty} (S_n - S_{n-1})(v) \text{ with } S_0 \equiv 0.$$

Note that in this case the representation of $V$ as $v = \sum_{n=1}^{\infty} (S_n - S_{n-1})(v)$ is the unique representation of $v$ as a sum of $x_n$'s with $x_n \in (S_n - S_{n-1})(V)$, for $n \in \mathbb{N}$.

The (complemented) subspaces $(S_n - S_{n-1})(V)$ are called *the summands of the decomposition.*

A Schauder decomposition $(S_n)$ is called *shrinking* if $(S_n^*)$ is a Schauder decomposition of $V^*$, or, equivalently, if for all $z^{**} \in V^{**}$ $w^* - \lim_{n\to\infty} S_n^{**}(z^{**}) = z^{**}$. It is called *boundedly complete* if any bounded sequence $(z_n)$ which has the property that $z_n = S_n(z_n) = S_n(z_{n+1})$ (and thus $S_n(z_k) = S_n \circ S_{k-1}(z_k) = S_n \circ z_{k-1} + \dots z_n$ whenever $k > n$) it follows that $(z_n)$ converges.

Remark. If $(S_n)$ is a Schauder decomposition of a Banach space $V$, and $\dim((S_n - S_{n-1})(V)) = 1$ for all $n \in \mathbb{N}$, then any sequence $(v_n)$ with $v_n \in (S_n - S_{n-1})(V) \setminus \{0\}$, for $n \in \mathbb{N}$, is a basis of $V$.

If $\dim((S_n - S_{n-1})(V)) < \infty$, then the sequence $(F_n) = ((S_n - S_{n-1})(V))$ is called *finite dimensional decomposition of* $V$ (FDD).

Lemma 1.2. [**DFJP**, Lemma 1]

1) *$W \subset B_Y$.*
2) *$Y$ is, via the isometric embedding $\phi$, isometric to a closed subspace of $Z$, thus, in particular a Banach space, and $j : Y \to X$ is continuous.*
3) *$j^{**} : Y^{**} \to X^{**}$, is injective and $(j^{**})^{-1}(X) = Y$.*
4) *$Y$ is reflexive if and only if $W$ is weakly relatively compact.*
5) *If $\Gamma \subset X^*$ separates the points in $X$ then*
$$\overline{B_Y}^{\sigma(X,\Gamma)} \subset \overline{\operatorname{span}\big(\overline{W}^{\sigma(X,\Gamma)}\big)}^{\|\cdot\|}.$$
6) *If $\Gamma \subset X^*$ norms $X$ and $W$ is $\sigma(X,\Gamma)$-compact so is $B_Y$.*
7) *The topologies $\sigma(X^{**}, X^*)$ and $\sigma(Y^{**}, Y^*)$ coincide on $\overline{B_Y}^{\sigma(X^{**},X^*)}$.*
8) *If $S : X \to X$ is linear and $S(W) \subset aW$ for some $a \in \mathbb{R}$, then $S(j(W)) \subset j(W)$, then $S(j(Y)) \subset j(Y)$ and*
$$\|j^{-1} \circ S \circ j\| \le \max(\|S\|, |a|).$$
9) *If $(S_n)_{n\in\mathbb{N}}$ is a Schauder decomposition for $X$ with $S_n(W) \subset aW$, for some $a \in \mathbb{R}$, for all $n \in \mathbb{N}$ then $(j^{-1} \circ S_n \circ j)_{n\in\mathbb{N}}$ is a Schauder decomposition of $Y$.*
10) *If $(x_n)$ is an unconditional basis for $X$ with $P_I(W) \subset aW$, for some $a \in \mathbb{R}$, for all finite $I \subset \mathbb{N}$ (($P_I$ denotes the canonical projection onto the subspace spanned by $(x_i : i \in I)$, then $\{x_i : i \in \mathbb{N} \,\&\, x_i \in Y\}$ is an unconditional basis of $Y$.*

## 2. Slicings and Selections

This section follows very closely the beginning of a paper by Ghousoub, Maurey, and Schachermayer [**GMS**], which leads to an alternate proof of Zippin's theorem [**Z**], that reflexive spaces, or spaces whith separable duals, embed into reflexive space or spaces whith separable duals, respectively, with a basis (see Subsection 3).

Theorem 2.1. "The Dessert Selection" [**GMS**, Theorem A] *Assume that $X$ is a topological space that is fragmentable by a metric $d(\cdot,\cdot)$ (see Definition 4.3) Then there is a mapping $S : \mathcal{K}(X) \to X$ (recall that $\mathcal{K}(X)$ denotes the set of non empty compact subsets of $X$) so that*

a) *$S$ is a selection, i.e. $S(K) \in K$, for $K \in \mathcal{K}(X)$.*
b) *If $K \subset \tilde{K}$ are in $\mathcal{K}$ and if $S(\tilde{K}) \in K$, then $S(K) = S(\tilde{K})$*
c) *If $(K_i)$ is a decreasing net in $\mathcal{K}(X)$ and if $K = \bigcap_i K_i$ then*
$$\lim_i d(S(K_i), S(K)) = 0.$$
d) *If $\Gamma : (Z, d_Z) \to \mathcal{K}(X)$ is a* slice-upper-semi-continuous multivalued mapping *from a metric space $(Z, d_Z)$ into $\mathcal{K}(X)$, then the map $S \circ \Gamma : Z \to X$ is a Baire-1 function from $Z$ into $(X, \Delta)$.*

The notions necessary to understand above statements will be introduced in the following definitions.

Definition 2.2. Let $X$ be a topological space:

A map $f : X \to A$, where $A$ is totally ordered is called a *slicing* if the sets

$$X_\alpha = \{x \in X : f(x) \geq \alpha\}, \quad \text{for } \alpha \in A$$

are closed. A slicing is called a *discrete slicing* if for all $\alpha \in A$ the set

$$X_{\alpha+} = \{x \in X : f(x) > \alpha\}$$

is also closed.

We say that a slicing is well ordered if $A$ is well ordered.

Proposition 2.3. *Assume $X$ is a topological space, $A$ is a well ordered set and $f : X \to A$ is a slicing.*

a) *Either $\bigcap_{\alpha\in f(X)} X_\alpha = \emptyset$ or $f(X)$ has a maximal element.*
b) *If $K \subset X$ is non empty and compact then $f(K)$ has a maximal element.*

Proof. (a) If $x \in \bigcap_{\alpha\in f(X)} X_\alpha$ then it follows that $f(x) = \max f(X)$. On the other hand if $\bigcap_{\alpha\in f(X)} X_\alpha = \emptyset$ and if $f(x) \in f(X)$, then there is an $\alpha \in f(X)$ so that $f(x) \notin X_\alpha$ and thus $f(x) < \alpha$. Thus, $f(x)$ is not maximal in $f(X)$.

(b) The set

$$\{K \cap X_\alpha : \alpha \in f(K)\}$$

is a nested collection of compact sets which enjoys the finite intersection property (if $\alpha_1 < \alpha_2 < \ldots < \alpha_n$ are in $f(K)$ then $\bigcap_{i=1}^n K \cap X_{\alpha_i} = K \cap X_{\alpha_n} \neq \emptyset$). Thus

$$K_0 = \bigcap\{K \cap X_\alpha : \alpha \in f(K)\} \neq \emptyset$$

and, as observed in (a), for every $x \in K_0$, $f(x)$ is the maximum of $f(K)$. □

Definition 2.4. A class $\mathcal{C}$ of subsets of a topological space $X$ is called *slice generating* if for every non empty closed set $F \subset X$ there exists a closed subset $G \subset X$ so that $F \setminus G$ is in $\mathcal{C}$.

Proposition 2.5. *For a hereditary class $\mathcal{C}$ of subsets of $X$ the following are equivalent.*

a) *$\mathcal{C}$ is slice generating.*
b) *There is a well ordered slicing $f : X \to \alpha$, $\alpha \in \mathbf{ON}$, so that the difference sets $(D_\gamma)_{\gamma<\alpha}$ belong to $\mathcal{C}$.*

Proof. "(a)⇒(b)" By transfinite induction we choose for $\alpha \in \mathbf{ON}$ a closed set $X_\alpha \subset X$, so that $X_\alpha \setminus X_{\alpha+1}$ is in $\mathcal{C}$ and not empty if $X_\alpha$ is not empty in the following way:

$$
\begin{aligned}
&X_0 = X \\
&X_\alpha = \emptyset,\ \text{if } \alpha = \beta+1, \text{ and } X_\beta = \emptyset \\
&X_\alpha = G,\ \text{if } \alpha = \beta+1, \text{ and } X_\beta \neq \emptyset \text{ and } G \subset X \text{ closed is chosen such that} \\
&\qquad\qquad X_\beta \setminus G \in \mathcal{C} \setminus \{\emptyset\} \text{ (such a } G \text{ exists by assumption)}, \\
&X_\alpha = \bigcap_{\beta<\alpha} X_\beta,\ \text{if } \alpha \text{ is limit ordinal.}
\end{aligned}
$$

There is an $\alpha \in \mathbf{ON}$ so that $X_\alpha = \emptyset$.

We can define $f : X \to \alpha$ as follows: For $x \in X$ let $\beta(x) = \min\{\beta \le \alpha : x \notin X_\beta\}$. Note that, from the definition of the $X_\gamma$'s we deduce that $\beta(x)$ cannot be a limit ordinal. Thus $\beta(x) = \gamma(x) + 1$, and we put $f(x) = \gamma(x)$.

"(b)⇒(a)" Let $f : X \to \alpha$, $\alpha \in \mathbf{ON}$, be a slicing with $D_\alpha = X_\alpha \setminus X_{\alpha+1} \in \mathcal{C}$, and let $F \subset X$ be closed and not empty. Choose $\gamma_0 = \min f(F)$ and let $G = X_{\gamma_0+1}$, then $F \setminus G \neq \emptyset$ (since there is an $x \in F$ so that $f(x) = \gamma_0$) and $F \setminus G \subset X_{\gamma_0} \setminus X_{\gamma_0+1} = D_{\gamma_0} \in \mathcal{C}$. □

Proposition 2.6. *Assume that $X$ is a topological space and that $f : X \to \alpha$, $\alpha \in \mathbf{ON}$, is a well ordered slicing. For $K \in \mathcal{K}(X) = \{F \subset X, F \text{ is compact}\} \setminus \{\emptyset\}$ we define (using Proposition 2.3 (b))*

$$o(K) = o(K, f) = \max f(K) \text{ and } L(K) = K \cap X_{o(K)}.$$

*We deduce the following properties:*

a) *For $K \in \mathcal{K}(X)$ the set $L(K)$ is a non empty subset of $K$ and $L(K) \cap X_{o(K)+1} = \emptyset$. Thus, $L(K) = K \cap (X_{o(K)} \setminus X_{o(K)+1})$.*
b) *If $K_1 \subset K_2$ are in $\mathcal{K}(X)$ then $o(K_1) \le o(K_2)$.*
c) *If $K_1 \subset K_2$ are in $\mathcal{K}(X)$ and $L(K_2) \cap K_1 \neq \emptyset$, then $o(K_1) = o(K_2)$ and $L(K_1) \subset L(K_2)$.*
d) *If $(K_i : i \in I)$ is a decreasing net in $\mathcal{K}(X)$ with $K = \bigcap_{i \in I} K_i$ then $o(K_i)$ becomes eventually constant and there exists an $i_0 \in I$ so that $L(\bigcap_{i\in I} K_i) = \bigcap_{i \ge i_0} L(K_i)$.*

Proof. (a) Note that for $x \in K$ such that $f(x) = \max f(K)$ it follows that $x \in K \cap X_{o(K)}$, but that $K \cap X_{o(K)+1} = \emptyset$ thus $x \in L(K) = K \cap (X_{o(K)} \setminus X_{o(K)+1})$.
(b) obvious.
(c) Assume $K_1 \subset K_2$ and $L(K_2) \cap K_1 \neq \emptyset$, Then

$$o(K_1) \ge o(K_1 \cap L(K_2)) = \max f(K_1 \cap L(K_2)) = \max f(K_2).$$

(d) Assume $(K_i : i \in I)$ is a decreasing net in $\mathcal{K}(X)$ with $K = \bigcap_{i\in I} K_i$. Define $\alpha_i = o(K_i)$ for $i \in I$ then $(\alpha_i)$ is a decreasing net, and, thus, there is an $i_0 \in I$ so that $\alpha_{i_0} = \min_{i\in I} \alpha_i = \alpha_j$ for all $j \ge i_0$. It follows that $L(K_i) = K_i \cap X_{\alpha_{i_0}} \neq \emptyset$, whenever

$i \ge i_0$, and thus for all $j \ge i_0$ we have

$$\max f(K_j) = \max f\Big(\bigcap_{j\ge i_0} K_j\Big), \text{ and, thus,}$$

$$\bigcap_{i\ge i_0} L(K_i) = \bigcap_{i\ge i_0} K_i \cap X_{\alpha_{i_0}} = L\Big(\bigcap_{i\ge i_0} K_i\Big).$$

□

Proof of Theorem 2.1 (a),(b) and (c). For $n \in \mathbb{N}$ let $\mathcal{C}_n$ be the subsets of $X$ whose $d$-diameter is not larger than $2^{-n}$.

By assumption $\mathcal{C}_n$ is slice generating and, thus, by Proposition 2.5, we can choose for every $n \in \mathbb{N}$ a well ordered slicing $f_n : X \to \alpha_n$ with $\alpha_n \in \mathbf{ON}$ and $D^{(n)}_\beta = X^{(n)}_\beta \setminus X^{(n)}_{\beta+1} \in \mathcal{C}_n$, where $X^{(n)}_\beta = \{x \in X : f_n(x) \ge \beta\}$ for $\beta < \alpha_n$ and $n \in \mathbb{N}$.

For $n \in \mathbb{N}$, we define $L_n : \mathcal{K}(X) \to \mathcal{K}(X)$ as in Proposition 2.6. Then we define by induction $S_n : \mathcal{K}(X) \to \mathcal{K}(X)$ as follows:

$$S_0(K) = K \text{ for } K \in \mathcal{K}(X)$$

and, assuming $S_n$ has been defined,

$$S_{n+1}(K) = L_{n+1}(S_n(K)) \text{ for } K \in \mathcal{K}(X).$$

Note that for $K \in \mathcal{K}(X)$ and $n \in \mathbb{N}$, $S_{n+1}(K) \in \mathcal{C}_{n+1}$ and $S_{n+1}(K) \subset S_n(K)$. Thus $\bigcap_{n\in\mathbb{N}} S_n(K)$ has a unique element. We define $S(K)$ to be this unique element and verify (a), (b) and (c).

(a) obvious.

(b) Assume $K \subset \tilde K$ and $S(\tilde K) \in K$. We claim that $S_n(K) \subset S_n(\tilde K)$, for all $n \in \mathbb{N}_0$. For $n = 0$ this is obvious, and assuming $S_n(K) \subset S_n(\tilde K)$ we observe that $S(\tilde K) \in S_n(K) \cap L_{n+1}(S_n(\tilde K))$ (thus in particular $S_n(K) \cap L_{n+1}(S_n(\tilde K)) \ne \emptyset$), which implies by Proposition 2.6 (c) $S_{n+1}(K) = L_{n+1}(S_n(K)) \subset L_{n+1}(S_n(\tilde K)) = S_{n+1}(\tilde K)$, which finishes the induction step. Condition (b) follows now from the uniqueness of $S(K) \in \bigcap S_n(K)$ and $S(\tilde K) \in \bigcap S_n(\tilde K)$.

(c) Assume that $(K_i)_{i\in I}$ is a decreasing net in $\mathcal{K}(X)$ and let $K = \bigcap_{i\in I} K_i$.

By induction on $n \in \mathbb{N}_0$ we prove, using Proposition 2.6 (d), that there is an $i_n \in I$, and an $\beta_n \le \alpha_n$, so that $o(S_{n-1}(K_i), f_n) = o(S_{n-1}(K), f_n) = \beta_n$ for all $i \ge i_n$ and $\bigcap_{i\ge i_n} S_n(K_i) = S_n(K)$. This implies that for $i \ge i_n$

$$S(K_i) \in S_n(K_i) = L_n(S_{n-1}(K_i)) \text{ and } S(K) \in S_n(K) \subset S_n(K_i) = L_n(S_{n-1}(K_i))$$

and since $d - \mathrm{diam}(L_n(S_{n-1}(K_i))) \le 2^{-n}$ it follows that $d(S(K), S(K_i)) \le 2^{-n}$ for all $i \ge i_n$. □

Definition 2.7. $\Gamma : Z \to [X]$ is called *upper/lower semi-continuous* if for all closed $F \subset X$ the set $\{z \in Z : \Gamma(z) \cap F \ne \emptyset\}$ respectively the set $\{z \in Z : \Gamma(z) \subset F\}$ is closed.

A map $\Gamma : Z, \to [X]$ is called *slice-upper semi-continuous* or *slice-lower semi-continuous* if there exists a discrete slicing $g : Z \to A$ such that for all $\alpha \in A$ the restriction $\Gamma|_{Z_{\alpha+}\setminus Z_\alpha} \to [X]$ is upper, respectively lower semi-continuous.

## 3. Ghoussoub, Maurey, and Schachermayer's proof of Zippin's Theorem

For $\sigma = (\sigma_1, \sigma_2, \dots, \sigma_\ell) \in \bigcup_{n=0}^\infty \{0,1\}^n$, put

$$\Delta_\sigma = \{(\alpha_i) \in \Delta : \alpha_i = \sigma_i, \text{ whenever } 1 \le i \le \ell\}.$$

Since for $\ell \in \mathbb{N}_0$

$$\pi_\ell = \big(\Delta_\sigma : \sigma \in \{0,1\}^\ell\big),$$

is a partition of $\Delta$ it follows that the map

$$F_\ell : \Delta \to \pi_\ell, \quad \text{with } F_\ell(t) = A \iff t \in A, \text{ for } A \in \pi_\ell, \tag{11}$$

is well defined.

Note that the set $\mathcal{V} = \{\Delta_\sigma : \sigma \in \bigcup_{n=0}^\infty \{0,1\}^n\}$ is a basis of the topology on $\Delta$ and consists of clopen sets and that therefore by the Stone Weierstraß Theorem

$$C(\Delta) = \overline{\text{span}(1_V : V \in \mathcal{V})}.$$

Let us order the set $\mathcal{V}$ linearly:

$$\Delta_\emptyset, \Delta_0, \Delta_1, \Delta_{00}, \Delta_{01}, \dots$$

and denote this ordering by $(A_n)_{n=1}^\infty$. Note that $A_n$ is the disjoint union of $A_{2n}$ and $A_{2n+1}$, for $n \in \mathbb{N}$.

Let $S : \bigcup_{\ell=0}^\infty \pi_\ell \to \Delta$, be a selector (i.e. $S(A) \in A$) so that if $t \prec s$ and $S(A_t) \in A_s$ then $S(A_s) = S(A_t)$. Such a selection is easily obtainable, without using Theorem 2.1. Later we will require more properties of $S$ (which will require the use of at least part of that theorem).

For each $n \in \mathbb{N}$ we choose $t_n = S(A_n)$.

Finally we choose a subsequence $(B_n)$ of $(A_n)$ in the following way:

a) $B_1 = A_1 = \Delta$.
b) If $n > 1$, then there is a maximal $m < n$ for which $A_n \subsetneq A_m$ (meaning $n = 2m$ or $n = 2m+1$). If $t_m \in A_n$ (thus $t_n = t_m$) then $A_n$ will not be element of the sequence $(B_i)$, otherwise it will.

By induction on the length of $\sigma \in \bigcup_{n=0}^\infty \{0,1\}^n$, it is now easy to observe that

$$\forall \sigma \in \bigcup_{n=0}^\infty \{0,1\}^n \quad \Delta_{(\sigma,0)} \in \{B_n : n \in \mathbb{N}\} \iff \Delta_{(\sigma,1)} \notin \{B_n : n \in \mathbb{N}\} \tag{12}$$

and, thus, $\{1_{B_n} : n \in \mathbb{N}\}$ is a linear independent set whose closed linear span is all of $C(\Delta)$.

For $n \in \mathbb{N}$ there is a unique $k \in \mathbb{N}$, so that $B_n = A_k$, define $s_n := t_k = S(A_k) \in B_n$. Note also that if $n > 1$ there is a maximal $m \in \{1, 2, \dots n\}$ so that $B_n \subsetneq B_m$ and define $s'_n := s_m$ (by the way we have chosen $B_n$ it follows that $s'_n \ne s_n$).

We claim that $(1_{B_n} : n \in \mathbb{N}_0)$ is a monotone basic sequence and, thus, a monotone basis of $C(\Delta)$. Moreover we claim that for every $f \in C(\Delta)$

$$f = \sum_{n=1}^{\infty} \big(f(s_n) - f(s'_n)\big) \cdot 1_{B_n} \text{ where we put } f(s'_1) := 0. \tag{13}$$

First we note that for $\ell \in \mathbb{N}$ and $f \in C(\Delta)$

$$\begin{aligned}
P_\ell(f) :&= \sum_{A \in \pi_\ell} f(S(A)) 1_A \\
&= \sum_{n \in \mathbb{N}, A_n \in \pi_\ell} f(t_n) 1_{A_n} \\
&= \sum_{j=0}^{\ell} \Big[ \sum_{n \in \mathbb{N}, A_n \in \pi_j} f(t_n) 1_{A_n} - \sum_{n \in \mathbb{N}, A_n \in \pi_{j-1}} f(t_n) 1_{A_n} \Big] \\
&\text{[where } \pi_{-1} := \emptyset] \\
&= f(t_1) 1_{A_1} + \sum_{j=1}^{\ell-1} \Big[ \sum_{n \in \mathbb{N}, A_n \in \pi_{j-1}} f(t_{2n}) 1_{A_{2n}} + f(t_{2n+1}) 1_{A_{2n+1}} - f(t_n) 1_{A_n} \Big] \\
&= f(t_1) 1_{B_1} + \sum_{j=2}^{\ell} \sum_{n \in \mathbb{N}, B_n \in \pi_j} [f(s_n) - f(s'_n)] 1_{B_n} \\
&= \sum_{n=1}^{m_\ell} [f(s_n) - f(s'_n)] 1_{B_n}.
\end{aligned} \tag{14}$$

where $m_\ell \in \mathbb{N}$ is chosen so that $\{B_i : i = 1, 2, \ldots, m_\ell\} = \{B_i : i \in \mathbb{N}\} \cap \bigcup_{j=0}^{\ell} \pi_j$.

The uniform continuity of $f \in C(\Delta)$ yields

$$\begin{aligned}
f &= \|\cdot\|_\infty - \lim_{\ell \to \infty} P_\ell(f) \\
&= \|\cdot\|_\infty - \lim_{\ell \to \infty} \sum_{n=1}^{m_\ell} [f(s_n) - f(s'_n)] 1_{B_n} \\
&= \|\cdot\|_\infty - \lim_{m \to \infty} \sum_{n=1}^{m} [f(s_n) - f(s'_n)] 1_{B_n}.
\end{aligned} \tag{15}$$

Moreover if $n \in \mathbb{N}$, and $m < n$ maximal so that $B_n \subset B_m$ we deduce for $(a_i)_{i=1}^n \subset \mathbb{R}$

$$\max_{t \in \Delta \setminus B_m} \Big| \sum_{i=1}^{n} a_i 1_{B_i}(t) \Big| = \max_{t \in \Delta \setminus B_m} \Big| \sum_{i=1}^{n-1} a_i 1_{B_i}(t) \Big|$$

$$\max_{t \in B_m \setminus B_n} \Big| \sum_{i=1}^{n} a_i 1_{B_i}(t) \Big| = \max_{t \in B_m} \Big| \sum_{i=1}^{n-1} a_i 1_{B_i}(t) \Big|$$

and thus

$$\max_{t\in\Delta}\Big|\sum_{i=1}^{n} a_i 1_{B_i}(t)\Big| \geq \max_{t\in\Delta\setminus B_n}\Big|\sum_{i=1}^{n} a_i 1_{B_i}(t)\Big| = \max_{t\in\Delta}\Big|\sum_{i=1}^{n-1} a_i 1_{B_i}(t)\Big|,$$

which implies that $(1_{B_n})$ is a monotone basis for $C(\Delta)$ and $(\delta_{s_n} - \delta_{s'_n})$ is the sequence of coordinate functionals.

Let $W \subset C(\Delta)$ be a bounded, closed, convex and symmetric set and define

$$W_0 = \bigcup_{n\in\mathbb{N}} Q_n(W),$$

where $Q_n$ is the canonical projection onto $\mathrm{span}(1_{B_i} : i \leq n)$, i.e.

$$Q_n : C(\Delta) \to C(\Delta), x = \sum_{i\in\mathbb{N}} x_i 1_{B_i} \mapsto \sum_{i=1}^{n} x_i 1_{B_i}.$$

Proposition 3.1. *Define the following metric on* $\Delta$

$$d_W(\xi,\eta) := \sup_{f\in W} |f(\xi) - f(\eta)| \text{ whenever } \xi,\eta \in \Delta,$$

*and assume that for all* $t \in \Delta$

$$\lim_{k\to\infty} d_W\big(S(F_k(t)), t\big) = 0.$$

*For all* $\mu \in M(\Delta) = C^*(\Delta)$ *we have*

$$\lim_{k\to\infty} \sup_{w\in W_0} \langle (Id - Q_k)(w), \mu\rangle = 0. \tag{16}$$

*Moreover if* $W$ *is relatively weakly compact so is* $W_0$.

Remark. Note that the assumption of Proposition 3.1 is satisfied if $\Delta$ is $d_W$-fragmentable and $S$ is chosen like in Theorem 2.1.

Proof of Proposition 3.1. For $k, n \in \mathbb{N}$ and $w \in W$ it follows that

$$(Id - Q_k)\big(Q_n(w)\big) = \begin{cases} 0 & \text{if } n \leq k \\ (Q_n - Q_k)(w) = (Id - Q_k)(w) - (Id - Q_n)(w) & \text{if } n > k, \end{cases}$$

and thus

$$\begin{aligned}
\sup_{w\in W_0} |\langle (Id-Q_k)(w),\mu\rangle| &\le 2 \sup_{w\in W, n\ge k} |\langle (Id-Q_n)(w),\mu\rangle| \\
&= 2\sup_{w\in W_0}\Big|\sum_{A\in\pi_k}\int_A w(t)-w(S(F_k(t)))d\mu\Big| \\
&\quad \text{[Recall (11) for definition of } F_k] \\
&\le 2\sup_{w\in W}\sum_{A\in\pi_k}\int_A |w(t)-w(S(F_k(t)))| d|\mu| \\
&\le \int d_W\big(t, S(F_k(t))\big) d|\mu| \\
&\to 0 \text{ if } k\to\infty. \\
&\quad \text{[Using assumption and Dominated Convergence Theorem]}
\end{aligned}$$

Now assume that $W$ is relatively weakly compact let $Q_{n_k}(w_k)\in W_0$ with $w_k\in W$ and $n_k\in\mathbb{N}$, for $k\in\mathbb{N}$. W.l.o.g we can assume that $n_k\nearrow\infty$ and that $w_k$ converges weakly to some $w\in W$.

It follows from the first part of our claim, that for any $\mu\in M(\Delta)$

$$|\langle w-Q_{n_k}w_k,\mu\rangle| \le |\langle w-w_k,\mu\rangle| + |\langle (Id-Q_{n_k})(w_k),\mu\rangle| \to 0 \text{ if } k\nearrow\infty.$$

which implies that $Q_{n_k}(w_k)$ converges weakly to $w$. □

Now we apply the interpolation scheme of [**DFJP**].

For $n\in\mathbb{N}$, let $U_n = 2^k\overline{W_0} + 2^{-k}B_{C(\Delta)}$ and

$$\|\cdot\|_n : C(\Delta)\to[0,\infty), \quad \|x\|_n := \inf\Big\{r : \frac{x}{r}\in U_n\Big\}.$$

Then define

$$|\!|\!|\cdot|\!|\!| : C(\Delta)\to[0,\infty], \quad |\!|\!|x|\!|\!| := \Big(\sum_{n\in\mathbb{N}}\|x\|_n^2\Big)^{1/2}.$$

and

$$Y = \big\{y\in C(\Delta) : |\!|\!|y|\!|\!| < \infty\big\}.$$

PROPOSITION 3.2. [**DFJP**]
*$Y$ with $|\!|\!|\cdot|\!|\!|$ is a Banach space and $\overline{W_0}\subset B_Y$.*
*If $\overline{W_0}$ is relatively weakly compact, then $Y$ is reflexive.*

PROPOSITION 3.3. [**GMS**]
*Assume that $W$ contains the constant function 1 and a point separating function $f_0$. Then $(1_{B_n})\subset Y$ and $(y_n)$ with $y_n = 1_{B_n}/|\!|\!|y_n|\!|\!|$, for $n\in\mathbb{N}$, is a normalized basis for $Y$.*

*If the conclusion* (16) *of Proposition 3.1 is satisfied then $(y_n)$ is a shrinking basis.*

PROOF. First note that for $w\in\overline{W_0}$ and $n\in\mathbb{N}$ $Q_n(w)\in\overline{W_0}$ and by (15) $f_0 = \sum_{i\in\mathbb{N}}(f_0(s_i)-f_0(s_i'))1_{B_i}$, and thus $(Q_i-Q_{i-1})(f_0) = (f_0(s_i)-f_0(s_i'))1_{B_i}\in Y\setminus\{0\}$ for $i>1$, and $1_{B_1} = Q_1(1)\in Y\setminus\{0\}$.

In order to show that $(y_n)$ is a normalized basis of $Y$ we will show that $Q_n : Y \to Y$, is well defined and $\|Q_n\| = 1$.

Let $y \in Y$, $k, n \in \mathbb{N}$, and $\eta > 0$ arbitrary then there are $w_k \in \overline{W_0}$ and $x_k \in B_{C(\Delta)}$ so that $y = (\|y\|_k + \eta 2^{-k})(2^k w_k + 2^{-k} x_k)$, and thus

$$Q_n(y) = (\|y\|_k + \eta 2^{-k})(2^k Q_n(w_k) + 2^{-k} Q_n(x_k)) \in (\|y\|_k + \eta 2^{-k})\big(2^k \overline{W_0} + 2^{-k} B_{C(\Delta)}\big),$$

which implies, since $\eta > 0$ was arbitrary, that $\|Q_n(y)\|_n \le \|y\|_n$ and thus $|||Q_n(y)||| \le |||y|||$.

Now assume that for all $\mu \in C^*(\Delta)$

$$\lim_{k\to\infty} \sup_{w \in W_0} \langle (Id - Q_k)(w), \mu \rangle = 0.$$

We need to show that for any $y^* \in Y^*$

$$\lim_{n\to\infty} |||(I - Q_n)^*(y^*)||| = \lim_{n\to\infty} \sup_{y \in B_Y} |\langle y^*, (I - Q_n)(y) \rangle| = 0. \tag{17}$$

Let $j : Y \to C(\Delta)$ be the formal identity. Lemma 1.2 (3) implies that $j^* : C^*(\Delta) \to Y^*$ has dense range. Indeed, if this were wrong we could choose (by the Theorem of Hahn Banach) $y^{**} \in S_{Y^{**}}$ so that $\langle y^{**}, j^*(\mu) \rangle = 0$ for all $\mu \in C^*(\Delta)$, but this would yield $\langle j^{**}(y^{**}), \mu \rangle = 0$ for all $\mu \in C^*(\Delta)$, and thus $y^{**} = 0$, by injectivity of $j^{**}$.

Thus, we only need to verify (17) for $y^* = j^*(\mu^*)$ for some $\mu \in C(\Delta)$. However for any $\eta > 0$, we can choose $k \in \mathbb{N}$ so that $2^{1-k} < \eta$ and for any $y \in B_Y$ there is a $w \in W_0$ so that $y - 2^k w \in 2^{1-k} B_{C(\Delta)}$ and thus, by our assumption,

$$\begin{aligned}\lim_{n\to\infty} \sup_{y \in B_Y} |\langle j^*(\mu), (I - Q_n)(y) \rangle| &= \lim_{n\to\infty} \sup_{y \in B_Y} |\langle \mu, (I - Q_n)(y) \rangle| \\ &\le 2\eta\|\mu\| + \lim_{n\to\infty} \sup_{w \in 2^k W_0} \langle \mu, (I - Q_n)(y) \rangle| = 2\eta\|\mu\|,\end{aligned}$$

which implies the claim since $\eta > 0$ was arbitrary. □

CHAPTER 9

# Coding separable Banach spaces as quotients of $\ell_1$

## 1. Introduction

This section is based in parts on [**Boss1**].
Since every separable Banach space $X$ is a quotient of $\ell_1$ we can think of it being coded by linear closed subspaces $H$ of $\ell_1$. This might be important if we also want to consider the dual $X^*$ of $X$. Note that if $X = \ell_1/H$, for some $H \subset \ell_1$, the map

$$I_H : (\ell_1/H)^* \to H^\perp = \big\{\xi \in \ell_\infty \,:\, \forall h \in H \quad \langle \xi, h\rangle = 0\big\} \hookrightarrow \ell_\infty$$

where for $x^* \in (\ell_1/H)^*$ $I_H(x^*) \in \ell_\infty \equiv \ell_1^*$ is defined by

$$I_H(x^*) : \ell_1 \to \mathbb{R},\ a \mapsto \langle x^*, Q_H(a)\rangle,\ \text{where } Q_H : \ell_1 \to \ell_1/H \text{ is the quotient map,}$$

is an isometric isomorphism onto $H^\perp$. Moreover, since $I_H$ is the adjoint of the quotient map $Q_H$, it is as operator from $(\ell_1/H)^*$ into $\ell_\infty$ also $w^*$ continuous.

We introduce the following notations and conventions. We denote the unit ball of $\ell_\infty$ endowed with the $\sigma(\ell_\infty, \ell_1)$ topology by $B_\infty$. Note that $B_\infty$ is a compact metric space and, thus, Polish. The set $\mathcal{K}(B_\infty)$ of all non empty compact subsets is equipped with the Vietoris topology (see Definition 3.1).

Proposition 1.1. *The Borel sets of $\mathcal{K}(B_\infty)$ are the same as the Borel sets of $\mathcal{K}(B_\infty)$ with respect to the Hausdorff topology $\mathcal{T}_H$ on $\mathcal{K}(B_\infty)$ (see Proposition 3.3), which is generated by the following Basis $\mathcal{H}_{\mathcal{K}(B_\infty)}$.*

$$\mathcal{H}_{\mathcal{K}(B_\infty)} = \big\{\{K \in \mathcal{K} : K \subset U\} : U \subset B_\infty \text{ open}\big\}.$$

Proof. Let $U_0, U_1, \dots U_n$ be open subsets of $B_\infty$, then

$$\begin{aligned}
V(U_0,&U_1, U_2, \dots U_n)\\
&= \big\{K \in \mathcal{K}(B_\infty) \,:\, K \subset U_0, K \cap U_i \neq \emptyset, \text{ for } i = 1, 2 \dots n\big\}\\
&= \big\{K \in \mathcal{K}(B_\infty) \,:\, K \subset U_0\}\\
&\qquad \cap \bigcap_{i=1}^n \mathcal{K}(B_\infty) \setminus \{K \in \mathcal{K}(B_\infty) : K \subset B_\infty \setminus U_i\}\\
&= \big\{K \in \mathcal{K}(B_\infty) \,:\, K \subset U_0\}\\
&\qquad \cap \bigcap_{i=1}^n \Big[\mathcal{K}(B_\infty) \setminus \bigcap_{\ell\in\mathbb{N}} \big\{K \in \mathcal{K}(B_\infty) : K \subset A_{(i,\ell)}\big\}\Big],
\end{aligned}$$

where $A_{(i,\ell)} \subset B_\infty$ is open for $i, \ell \in \mathbb{N}$ with $B_\infty \setminus U_i = \bigcap_{\ell=1}^\infty A_{(i,\ell)}$. $\square$

PROPOSITION 1.2. *The map:*

$$\Psi : \mathrm{Subs}(\ell_1) \to \mathcal{K}(B_\infty), \quad H \mapsto H^\perp \cap B_\infty,$$

*is Borel mesurable.*

PROOF. First consider the map

$$\tilde{\Psi} : S_{\ell_1}^{\mathbb{N}} \to \mathcal{K}(B_\infty), \quad (x_n) \mapsto \bigcap_{n\in\mathbb{N}} x_n^\perp \cap B_\infty.$$

Since for a closed $F \subset B_\infty$ the set

$$\big\{(x_n) \in S_{\ell_1}^{\mathbb{N}} : \bigcap x_n^\perp \cap F \neq \emptyset\big\} = \bigcap_{n\in\mathbb{N}} \big\{(x_i) \in S_{\ell_1}^{\mathbb{N}} : \bigcap_{i=1}^{n} x_i^\perp \cap F \neq \emptyset\big\}$$

is closed, it follows that $\tilde{\Psi}$ is measurable. Let

$$D : \mathrm{Subs}(\ell_1) \to S_{\ell_1}^{\mathbb{N}}, \quad Y \mapsto (d_n(X))$$

be Borel measurable so that for any $Y \hookrightarrow \ell_1$ the sequence $(d_n(Y))$ is dense in $S_Y$ (Selection Theorem 3.7 of Kuratowski, Ryll and Nerdewiski). It follows that $\Psi = \tilde{\Psi} \circ D$, and thus we deduce the claim. □

PROPOSITION 1.3. *For $K \in \mathcal{K}(B_\infty)$ the following two conditions are equivalent:*

a) *There exists an $H \in \mathrm{Subs}(\ell_1)$ so that $K = H^\perp \cap B_\infty = I_H(B_{(\ell_1/H)^*})$.*
b) *$K$ is convex, symetric and if $\xi \in K$ and $a \in \mathbb{R}$ so that $\|a\xi\|_{\ell_\infty} \leq 1$, then $a\xi \in K$.*

*Moreover the set of all* dual balls in $\ell_\infty$

$$\mathrm{DB} = \{H^\perp \cap B_\infty : H \in \mathrm{Subs}(\ell_1)\} = \{K \in \mathcal{K}(B_\infty) : K \textit{ satisfies (b)}\}.$$

*is Borel measurable in $\mathcal{K}(B_\infty)$.*

PROOF. If

$$K = H^\perp \cap B_\infty = \{\xi \in B_\infty : \forall h \in H \quad \langle \xi, h\rangle = 0\},$$

then $K$ clearly satisfies (b).

Conversely, if $K \in \mathcal{K}(B_\infty)$ satisfies (b) we define

$$H = {}^\perp K = \{h \in \ell_1 : \forall \xi \in K \quad \langle \xi, h\rangle = 0\}.$$

It follows that $K \subset H^\perp \cap B_\infty$. We need to show that $K \supset H^\perp \cap B_\infty$.

We first note that (b) implies that

$$K = \overline{\mathrm{span}(K)}^{\sigma(\ell_\infty,\ell_1)} \cap B_{\ell_\infty}.$$

Indeed, if $\xi = \sigma(\ell_\infty, \ell_1) - \lim_{i\in I} \xi_i$, with $\xi_i = \sum_{j=1}^{n_i} a_j^{(i)} \eta(i,j) \in B_\infty$, so that $n_i \in \mathbb{N}$, $a_j^{(i)} \in \mathbb{R}$ and $\eta(i,j) \in K$, for $i \in I$, and $j \leq n(i)$, then by convexity and symmetry of $K$ we obtain for $i \in I$

$$\tilde{\xi}_i = \sum_{j=1}^{n_i} \frac{a_j^{(i)}}{\sum_{j=1}^{n_i} |a_j^{(i)}|} \eta(i,j) \in K$$

(W.l.o.g. we can assume that above fraction is well defined). Since $\xi_i = \tilde{\xi}_i \sum_{j=1}^{n_i} |a_j^{(i)}|$ and $\|\xi_i\|_{\ell_\infty} \le 1$, it follows from (b) that $\xi_i \in K$ for all $i \in I$ and, thus, since $K$ is $\sigma(\ell_\infty, \ell_1)$-closed, also $\xi \in K$.

Assume now that there is a $\xi_0 \in H^\perp \cap B_\infty \setminus \overline{\text{span}(K)}^{\sigma(\ell_\infty,\ell_1)}$. Then by a Corollary of the Theorem of Hahn Banach (see [**FHHMPZ**, Theorem 4.25]) there is an $f \in \ell_1$, so that $\langle f, \xi\rangle = 0$ for all $\xi \in K$, but $f(\xi_0) = 1$. The first condition implies that $f \in H = {}^\perp K$, which contradicts the assumption that $\xi_0 \in H^\perp$, since $f(\xi_0) = 1$.

In order to show the "moreover" part we let $D \subset B_\infty$ be countable and dense. We observe that

$$\text{DB} = \bigcap_{\substack{d_1,d_2 \in D \\ \varepsilon>0, q_1,q_2 \text{ in } \mathbb{Q}\cap[-1,1]}} \left\{ K \in \mathcal{K}(B_\infty) : \begin{array}{c} \overline{B_\varepsilon(d_1)} \cap K = \emptyset \text{ or } \overline{B_\varepsilon(d_2)} \cap K = \emptyset \text{ or} \\ \|q_1 d_1 + q_2 d_2\|_\infty > 1 \text{ or } \overline{B_{2\varepsilon}(q_1 d_1 + q_2 d_2)} \cap K \neq \emptyset \end{array} \right\}.$$

Secondly we note that if $C \subset S_{\ell_1}$ is countable and dense we can write for $d \in B_\infty$ and $\varepsilon > 0$

$$\{K \in \mathcal{K}(B_\infty) \,:\, K \cap \overline{B_\varepsilon(d)} = \emptyset\} = \big\{K \in \mathcal{K}(B_\infty) : K \subset B_\infty \setminus \overline{B_\varepsilon(d)}\big\}$$

and

$$\begin{aligned} &\{K \in \mathcal{K}(B_\infty) \,:\, K \cap \overline{B_\varepsilon(d)} \neq \emptyset\} \\ &\qquad = \bigcap_{F \in [C]^{<\mathbb{N}}} \big\{K \in \mathcal{K}(B_\infty) \,:\, K \cap \bigcap_{f \in F} \{\xi \in B_\infty : |\langle f, \xi - d\rangle| \le \varepsilon\} \neq \emptyset\big\}. \end{aligned}$$

□

We denote the unit vector basis of $\ell_1$ by $(e_n)$. If $H \in \text{Subs}(\ell_1)$, we denote the image of $(e_n)$ under the canonical quotient map $Q_H : \ell_1 \searrow \ell_1/H$ by $(e_n^H)$, thus

$$\Big\| \sum_{n\in\mathbb{N}} a_n e_n^H \Big\|_{\ell_1/H} = \inf_{h \in H} \Big\| h + \sum_{n \in \mathbb{N}} a_n e_n \Big\| \text{ whenever } (a_i) \in c_{00}.$$

PROPOSITION 1.4. *The set*

$$\mathcal{S} = \left\{ (H, \overline{h}, X, \overline{x}) : \begin{array}{c} H \in \text{Subs}(\ell_1), X \in \text{SB}, \overline{h} = (h_n) \in \ell_1^{\mathbb{N}}, \overline{x} \in C(\Delta)^{\mathbb{N}} \\ \overline{\text{span}(x_n)} = X, \overline{\text{span}(h_n)} = H, \text{ and } (x_n) \sim_1 (e_n^H) \end{array} \right\}$$

is Borel measurable in $\text{Subs}(\ell_1) \times \ell_1^{\mathbb{N}} \times \text{SB} \times C(\Delta)^{\mathbb{N}}$.

PROOF. Usual arguments. □

## 2. Equivalence of Coding of separable Banach spaces via subspaces of $C(\Delta)$ and quotients of $\ell_1$

The following result was provided to us by Christian Rosendal.

THEOREM 2.1. [**Ro**] *There is a Borel isomorphism $I : \mathrm{SB} \to \mathrm{Subs}(\ell_1)$, so that for any $X \in \mathrm{SB}$, $X$ is isometrically isomorphic to $\ell_1/I(X)$, and, thus, for any $Y \hookrightarrow \ell_1$ it follows that $\ell_1/Y$ is isomorphic to $I^{-1}(Y)$.*

PROOF. It is enough to construct injective Borel maps

$$\Phi : \mathrm{SB} \to \mathrm{Subs}(\ell_1), \text{ and } \Psi : \mathrm{Subs}(\ell_1) \to \mathrm{SB},$$

so that each $X \hookrightarrow C(\Delta)$ is isometrically isomorphic to $\ell_1/\Phi(X)$ and so that for any $Y \hookrightarrow \ell_1$ $\Psi(Y)$ is isometrically isomorphic to $\ell_1/Y$. Indeed, we can then conclude from Corollary 5.5 that there is a Borel isomorphism $I$ whose graph lies in the union of the graph of $\Phi$ and the inverse of the graph of $\Psi$. Thus, if $X \in C(\Delta)$ then either $(X, I(X))$ is in the graph of $\Phi$, which yields that $X$ is isometrically isomorphic to $\ell_1/I(X)$, or $(I(X), X)$ is in the graph of $\Psi$ which implies that $\Psi(I(X)) = I^{-1}(I(X)) = X$ is also isometrically isomorphic to $\ell_1/I(X)$.
Construction of $\Phi : \mathrm{SB} \to \mathrm{Subs}(\ell_1)$:

Let $D : \mathrm{SB} \to S^{\mathbb{N}}_{C(\Delta)}$, $X \mapsto (d_n(X))$, be Borel measurable and $(d_n(Y))$ be a dense sequence in $S_X$ for any $X \hookrightarrow C(\Delta)$ (Selection Theorem 3.7 of Kuratowski, Ryll and Nerdewiski). Secondly let

$$\tilde{\Phi} : S^{\mathbb{N}}_{C(\Delta)} \to \mathrm{Subs}(\ell_1), \quad \overline{x} = (x_i) \mapsto \mathrm{Ker}\big(Q^{\overline{x}}\big),$$

where for $\overline{x} = (x_n) \in S_{C(\Delta)}$

$$Q^{\overline{x}} : \ell_1 \to C(\Delta), \quad (\xi_i) \mapsto \sum \xi_i x_i.$$

Once we have shown that $\tilde{\Phi}$ is measurable we put $\Phi = \tilde{\Phi} \circ D$, and note that $\Phi$ has the wanted properties. Indeed, for $X \hookrightarrow C(\Delta)$ $\Phi(X)$ is the Kernel of the normalized quotient map $Q^{\overline{x}} : \ell_1 \to X$, where $\overline{x} \subset S_X$ is dense. Thus, $\ell_1/\Phi(X)$ and $X$ are isometric.

In order to show that $\Phi$ is measurable we fix $\varepsilon > 0$ and $y = (\eta_n) \in \ell_1$ and need to show that the set

$$A_{(y,\varepsilon)} = \{\overline{x} \in S^N_{C(\Delta)} : \mathrm{Ker}(Q^{\overline{x}}) \cap B_\varepsilon(y) \neq \emptyset\}$$

is Borel in $S^N_{C(\Delta)}$.

Note that for $\overline{x} = (x_n) \in S^N_{C(\Delta)}$ we have

$$\begin{aligned} &\mathrm{Ker}(Q^{\overline{x}}) \cap B_\varepsilon(y) \neq \emptyset \\ &\quad \iff \exists \delta \in (0,\varepsilon) \cap \mathbb{Q} \quad \mathrm{Ker}(Q^{\overline{x}}) \cap \overline{B_\delta(y)} \neq \emptyset \\ &\quad \iff \exists \delta \in (0,\varepsilon) \cap \mathbb{Q}\, \exists z = (\zeta_n) \in [-1,1]^{\mathbb{N}} \quad \|z\|_{\ell_1} \leq \delta \text{ and } \sum_{n=1}^{\infty} (\eta_n + \zeta_n) x_n = 0. \end{aligned}$$

Note that

$$\Big\{\big((x_n),(\zeta_n)\big)\in S^N_{C(\Delta)}\times[-1,1]^{\mathbb{N}} : \|(\zeta_n)\|_{\ell_1}\le\delta \,\&\, \sum_{n=1}^{\infty}(\eta_n+\zeta_n)x_n=0\Big\},$$

is closed and since $[-1,1]^{\mathbb{N}}$ is compact it follows that

$$\Big\{(x_n)\in S^N_{C(\Delta)} : \exists(\zeta_n)\in[-1,1]^{\mathbb{N}} \quad \|(\zeta_n)\|_{\ell_1}\le\delta \,\&\, \sum_{n=1}^{\infty}(\eta_n+\zeta_n)x_n=0\Big\},$$

is also closed, and we obtain the claim (see Remark after Proposition 1.2).

Construction of $\Psi:\text{Subs}(\ell_1)\to\text{SB}$:

Put:

$$\Psi_1:\text{Subs}(\ell_1)\to\mathcal{K}(B_\infty),\quad Y\mapsto Y^\perp\cap B_\infty,$$

which according is to Proposition 1.2 Borel measurable.

Secondly we define a measurable function

$$H:\mathcal{K}(B_\infty)\to C(\Delta,B_\infty)\equiv B^{\mathbb{N}}_{C(\Delta)},\quad K\mapsto h_K$$

which has the property, that for each $K\in\mathcal{K}(B_\infty)$ the map $h_K$ is a surjection onto $K$.

By Theorem 2.11 there is a continuous surjection $h:\Delta\to B_\infty$. For $K\in\mathcal{K}(B_\infty)$ and $\sigma\in\Delta$ we choose $\tilde\sigma\in h^{-1}(K)$ as follows. If $h(\sigma)\in K$ then $\tilde\sigma=\sigma$. Otherwise we choose

$$n=n(K,\sigma)=\max\big\{n\in\mathbb{N}:\Delta_{\sigma|_n}\cap h^{-1}(K)\neq\emptyset\big\}.$$

(Recall $\Delta_s=\{\sigma\in\Delta:\sigma\succ s\}$ for $s\in\{0,1,\}^{<\mathbb{N}}$) Then choose $\tilde\sigma=\tilde\sigma_K$ to be the element in $\Delta_{\sigma|_{n(K,\sigma)}}\cap h^{-1}(K)$ which is minimal with respect to the lexicographical order on $\Delta$ (i.e. $\tau<_{\text{lex}}\tau'$ if there is an $\ell\in\mathbb{N}$, so that $\tau_i=\tau'_i$ if $i<\ell$, $\tau_\ell=0$ and $\tau'_\ell=0$).

Define $h_k(\sigma):=h(\tilde\sigma)$. Note that $h_K$ is a retract on $K$. In order to show that $h_K$ is continuous let $\sigma^{(k)}\to\sigma$ in $\Delta$.

If $\sigma^{(k)}\in h^{-1}(K)$, for infinitely many $k\in\mathbb{N}$ then also $\sigma\in h^{-1}(K)$, and after passing to a subsequence we assume $\sigma^{(k)}\in h^{-1}(K)$ for all $k\in\mathbb{N}$, and then

$$h_K(\sigma)=h(\sigma)=\lim_{k\to\infty}h(\sigma^{(k)})=\lim_{k\to\infty}h_K(\sigma^{(k)}).$$

If $\sigma^{(k)}\notin h^{-1}(K)$, for infinitely many $k$'s, but $\sigma\in h^{-1}(K)$, we assume, after passing to a subsequence that $\sigma^{(k)}\notin h^{-1}(K)$ for all $k\in\mathbb{N}$ and proceed as follows.

Put $N_k:=\max\{n\in\mathbb{N}:\sigma^{(k)}|_n=\sigma|_n\}$. It follows that $N_k\to\infty$ if $k\nearrow\infty$, and thus $\lim_{k\to\infty}\text{diam}-h(\{\tau:\tau|_{N_k}=\sigma|_{N_k}\})=0$ Since $\tilde\sigma^{(k)}|_{N_k}=\sigma|_{N_k}$ it follows that

$$\lim_{k\to\infty}h_K(\sigma^{(k)})=\lim_{k\to\infty}h(\tilde\sigma^{(k)})=h(\sigma).$$

Finally if $\sigma\notin h^{-1}(K)$ and, thus w.l.o.g $\sigma^{(k)}\notin h^{-1}(K)$, for $k\in\mathbb{N}$. Let $n=n(K,\sigma)$ and choose $k_0$ large enough so that $\sigma^{(k)}\in\Delta_{\sigma|_n}$ for $k\ge k_0$ and thus $n(K,\sigma^{(k)})=n$ and $\tilde\sigma^{(k)}=\tilde\sigma$, which implies our claim.

Now we need to show that the map $H : \mathcal{K}(B_\infty) \to C(\Delta, B_\infty)$ is measurable. We write $H$ as $(H^{(n)} : n \in \mathbb{N})$ with

$$H^{(n)} : \mathcal{K}(B_\infty) \to C(\Delta, [-1,1]) = S_{C(\Delta)}, \quad K \mapsto (h_K^{(m)}) : m \in \mathbb{N}).$$

In order to do so, it is enough to show that for $a < b$, $m \in \mathbb{N}$ and $\sigma \in \Delta$ the set

$$A_{(a,b,m,\sigma)} = \left\{K \in \mathcal{K}(B_\infty) : h_K^{(m)}(\sigma) \in (a,b)\right\}$$

is Borel measurable in $\mathcal{K}(B_\infty)$. First note that for $n \in \mathbb{N}$ the set

$$\begin{aligned}\{K \in &\mathcal{K}(B_\infty) : n(K,\sigma) = n\} \\ &= \{K \in \mathcal{K}(B_\infty) : h(\Delta_{\sigma|_n}) \cap K \neq \emptyset\} \cap \{K \in \mathcal{K}(B_\infty) : h(\Delta_{\sigma|n+1}) \cap K = \emptyset\},\end{aligned}$$

is Borel. For $s \in \{0,1\}^{<\mathbb{N}}$, with $s \geq n$ and $s \succeq \sigma|_n$

$$\begin{aligned}\{K \in \mathcal{K}(B_\infty) : n(K,\sigma) = n \,\&\, \tilde{\sigma} \succeq s\} &= \\ &\{K \in \mathcal{K}(B_\infty) : n(K,\sigma) = n \,\&\, h(\Delta_s) \cap K \neq \emptyset\} \\ &\cap \bigcap_{s' \succeq \sigma|_n, s' <_{\mathrm{lex}} s} \{K \in \mathcal{K}(B_\infty) : n(K,\sigma) = n \,\&\, h(\Delta_{s'}) \cap K = \emptyset\},\end{aligned}$$

which implies that map $\mathcal{K}(B_\infty) \to \Delta$, $K \mapsto \tilde{\sigma}_K$ is measurable, and thus that

$$A_{(a,b,m,\sigma)} = \{K \in \mathcal{K}(B_\infty) : h^{(m)}(\tilde{\sigma}_K) \in (a,b)\}$$

is Borel measurable.

The third part and last step towards defining $\Psi : \mathrm{Subs}(\ell_1) \to \mathrm{SB}$, is to consider the map

$$\Psi_2 : C(\Delta, B_\infty) \to \mathrm{SB}, \quad h \mapsto \overline{\{x \circ h = \sum_{k \in \mathbb{N}} x_n h^{(k)} : x \in \ell_1\}}$$

(note that $\Psi_2$ is well defined since $\{x \circ h_K = \sum_{k \in \mathbb{N}} x_n h_K^{(k)} : x \in \ell_1\}$ is a linear subspace of $C(\Delta)$) and prove that it is Borel measurable.

Let $U \in C(\Delta)$ be open. Since every open set is a countable union of countable intersections of sets of the form

$$U_{(\sigma,a,b)} = \{f \in C(\Delta) : f(\sigma) \in (a,b)\}$$

with $\sigma \in \Delta$ and $a < b$, we need to show that

$$\{h \in C(\Delta, B_\infty) : \Psi_2 \cap U_{(\sigma,a,b)} \neq \emptyset\}$$

is Borel measurable. But this follows since

$$\begin{aligned}\{h \in C(\Delta, B_\infty) :&\Psi_2 \cap U_{(\sigma,a,b)} \neq \emptyset\} \\ &= \bigcup_{x \in D} \left\{h \in C(\Delta, B_\infty) : \sum_{k=1}^\infty x_i h^{(k)}(\sigma) \in (a,b)\right\}\end{aligned}$$

where $D \subset c_{00} \cap \ell_1$ is countable and dense in $\ell_1$.

Finally we define $\Psi : \mathrm{Subs}(\ell_1) \to \mathrm{SB}$ to be $\Psi = \Psi_2 \circ H \circ \Psi_1$ and note that for $Y \hookrightarrow \ell_1$ we have

$$X = \Psi_2 \circ H \circ \Psi_1(Y) = \Psi_2(h_{Y^\perp \cap B_\infty}) = \overline{\{x \circ h_{Y^\perp \cap B_\infty} : x \in \ell_1\}}^{C(\Delta)}.$$

Since $(\ell_1/Y)^* \equiv Y^\perp$ we deduce that for $x \in \ell_1$

$$\|x \circ h_{Y^\perp \cap B_\infty}\|_{C(\Delta)} = \sup_{\sigma \in \Delta} \|x \circ h_{Y^\perp \cap B_\infty}(\sigma)\| = \sup_{y \in Y^\perp \cap B_\infty} \|x(y)\| = \|x\|_{\ell_1/Y}.$$

This yields that the map

$$T : \{x \circ h_{Y^\perp \cap B_\infty} : x \in \ell_1\} \to \ell_1/Y, \quad x \circ h_{Y^\perp \cap B_\infty} \mapsto \overline{x},$$

is well defined and an isometry. Thus, the linear space $\{x \circ h_{Y^\perp \cap B_\infty} : x \in \ell_1\}$ is closed in $C(\Delta)$ and coincides with $\Psi(Y)$. Therefore $X$ is isometrically isomorphic to $\ell_1/Y$. ☐

We want to observe a Corollary from Theorem 2.1 and its proof which will be use later. We will need the following easy observation.

LEMMA 2.2. *Assume that $K$ and $\tilde{K}$ are two compact metrizable spaces, and $h : K \to \tilde{K}$ continuous and onto. Let $g : \tilde{K} \to \mathbb{R}$ and put $f = g \circ h$. Then $f$ is continuous if and only $g$ is continuous.*

COROLLARY 2.3. *Assume that $\Phi$ and $Psi$ are defined as in the proof of Theorem 2.1. Let $X \in \mathrm{SB}$, let $Y = \Phi(X) \in \mathrm{Subs}(\ell_1)$, and let $\tilde{X} = \Psi(Y) = \Psi \circ \Phi(Y)$. By Theorem 2.1 the space $\tilde{X}$ is a subspace of $C(\Delta)$ which is isometrically isomorphic to $X$ which has the following suplementary property.*

(18) *If $(x_n) \subset \tilde{X}$ is a sequence which converges pointwise to a continuous function $f : \Delta \to \mathbb{R}$. Then $f \in \tilde{X}$.*

PROOF. Assume that $(f^{(n)})_{n \in \mathbb{N}} \in \Psi(Y)$ converges pointwise to $f$, and assume that $f \in C(\Delta)$. As shown in the proof of Theorem 2.1, there is a surjective and continuous map $h = (h_i) : \Delta \to Y^\perp \cap B_\infty$ so that

$$\tilde{X} = \overline{\{x \circ h : x \in \ell_1\}}^{C(\Delta)} = \{x \circ h : x \in \ell_1\}.$$

and we can therefore find a sequence $x^{(n)} = (x_i^{(n)}) \in \ell_1$ so that $f^{(n)} = \langle x^{(n)}, h(\cdot)\rangle$. for $n \in \mathbb{N}$.

The dual space $\tilde{X}^*$ can be identified with $Y^\perp$ via the pairing

$$\langle x \circ h, z\rangle = \sum x_i z_i \text{ for } x = (x_i) \in \ell_1 \text{ and } y = (y_i) \in Y^\perp \subset \ell_\infty.$$

Since $h$ is surjective $g(z) = \lim_{n\to\infty} \langle x^{(n)}, z\rangle$, exists for every $z \in Y^\perp$. Secondly $g$ is linear and, since $\sup_{z \in B_{Y^\perp}} |g(z)| = \|f\|_{C(\Delta)} < \infty$, it follows that $g \in \big(Y^\perp\big)^* = X^{**}$.

Since $g \circ h = f$ it follows from Lemma 2.2 and the continuity of $f$ that the restriction of $g$ onto $B_{X^*} = Y^\perp \cap B_\infty$ is $w^*$-continuous. This implies (cf. [**FHHMPZ**, Corollary 4.46]) that $g \in \tilde{X}$. ☐

CHAPTER 10

# On the Szlenk index

## 1. Introduction

As in the previous section we define $B_\infty$ to be $B_{\ell_\infty}$ endowed with the topology $\sigma(\ell_\infty, \ell_1)$. For any $K \in \mathcal{K}(B_\infty), \tilde{K} \subset K$ and $\varepsilon > 0$ define:

$$\begin{aligned} K'_\varepsilon &= \{\xi \in K \,:\, \forall\, V \subset B_\infty \ w^*\text{-open, } \operatorname{diam}(V \cap K) > \varepsilon\} \\ &= K \setminus \bigcup \{V \subset \ w^*\text{-open, } \operatorname{diam}(V \cap K) \leq \varepsilon\}. \end{aligned}$$

Here $w^*$-topology is the $w^*$-topology on $\ell_\infty$ and "diam" refers to the $\ell_\infty$-norm. Nevertheless if $K \subset \tilde{K} = H^\perp \cap B_\infty \in \text{DB}$ then the $w^*$-topology on $H^\perp \equiv (\ell_1/H)^*$ is the restriction of the $w^*$-topology on $\ell_\infty$, thus, the above term "$w^*$-open" can mean both topologies. Similarly the dual norm on $(\ell_1/H)^* \equiv H^\perp$ coincides with the $\ell_\infty$-norm and, thus the above notion can also refer to the dual norm on $(\ell_1/H)^*$.

For $\varepsilon > 0$ and $\alpha < \omega_1$ define $K_\varepsilon^{(\alpha)}$ by transfinite induction.

$$K_\varepsilon^{(0)} = K$$

and if $K_\varepsilon^\beta$ has been defined for all $\beta < \alpha$ we put

$$\begin{aligned} K_\varepsilon^\alpha &= \left(K_\varepsilon^\beta\right)'_\varepsilon \text{ if } \alpha = \beta + 1 \\ K_\varepsilon^\alpha &= \bigcap_{\beta < \alpha} K_\varepsilon^\beta \text{ if } \alpha = \sup_{\beta < \alpha} \beta. \end{aligned}$$

We define the following *fragmentation index* for elements of $\mathcal{K}(B_\infty)$:

$$o_F(\varepsilon, K) = \begin{cases} \min\{\alpha : K^\alpha = \emptyset\} & \text{if that exists} \\ \omega_1 & \text{if not.} \end{cases}$$

and

$$o_F(K) = \sup_{\varepsilon > 0} o_F(\varepsilon, K).$$

We can now define the *Szlenk index of a separable Banach space* $X$ using its representation as quotient of $\ell_1$ and the fragmentation index of its dual ball. Thus, let $X$ be isometrically isomorphic to $\ell_1/H$ with $H \in \text{Subs}(\ell_1)$. Put

$$K = H^\perp \cap B_\infty = I_H(B_{(\ell_1/H)^*}) \in \text{DB}.$$

Then put

$$\mathrm{Sz}_\varepsilon(X) = o_F(\varepsilon, K) = \begin{cases} \min\{\alpha : K^\alpha = \emptyset\} & \text{if that exists} \\ \omega_1 & \text{if not.} \end{cases}$$

Finally we define the *Szlenk index* to be

$$\mathrm{Sz}(X) = \sup_{\varepsilon>0} \mathrm{Sz}_\varepsilon(X).$$

THEOREM 1.1. [**Sz**] *Let $X$ and $Y$ be separable Banach spaces.*

a) *If $X$ embeds (isomorphically) into $Y$, then $\mathrm{Sz}(X) \le \mathrm{Sz}(Y)$. In particular, if $X$ and $Y$ are isomorphic it follows that $\mathrm{Sz}(X) = \mathrm{Sz}(Y)$.*
b) *$X^*$ is separable if and only if $\mathrm{Sz}(X) < \omega_1$ and for any $\alpha < \omega_1$.*
c) *For any $\alpha < \omega_1$ there are separable reflexive spaces $X$ for which $\mathrm{Sz}(X) \ge \alpha$.*

PROOF. Assume that $X$ embeds $c$-isomorphically into $Y$, $c \ge 1$,i .e. There exists a linear bounded $T : X \to \tilde{X}$, so that

$$\frac{1}{c}\|x\| \le \|T(x)\| \le c\|x\|.$$

Write $X$ and $\tilde{X}$ as $X = \ell_1/G$ and $\tilde{X} = \ell_1/H$, with $G, H \subset \ell_1$ are closed linear subspaces. For the adjoint $T^* : Y^* \searrow X^*$ his implies that $\frac{1}{c}B_{X^*} \subset T^*(B_{Y^*}) \subset cB_{Y^*}$

We first show that for compact sets $A \subset Y^*$ and $B \subset X_*$ for which $B \subset T^*(A)$, and $\varepsilon > 0$ it follows that

$$B'_{c\varepsilon} \subset T^*(A'_\varepsilon).$$

Indeed if $x^* \in B'_{c\varepsilon}$ then there is an $y^* \in B$ with $x^* = T^*(y^*)$. for any $w^*$ open $U \subset Y^*$ with $y^* \in U$ it follows that $x^* \in T^*(U)$ and thus, since $T^*(U)$ is $w^*$ open in $X^*$ it follows that $\mathrm{diam}(T^*(U) \cap B) > c\varepsilon$. But this implies that $\mathrm{diam}(U^* \cap A) > \varepsilon$, which means that $y^* \in A'_\varepsilon$.

Applying this observation we can show by transfinite induction, that for all $\alpha < \omega_1$ (the equality follows from a simple similarity argument)

$$\frac{1}{c}\Big[B_{X^*}\Big]^\alpha_{\varepsilon c^2} = \Big[\frac{1}{c}B_{X^*}\Big]^\alpha_{\varepsilon c} \subset T^*\big((B_{Z^*})^\alpha_\varepsilon\big),$$

which implies the claim.
(b) follows from Theorem 4.4 and the remark following that result.

□

In order to show that certain maps defined on $\mathcal{K}(B_\infty)$ are measurable we define for $K \in \mathcal{K}(B_\infty)$ *finer fragmentations* we introduce the following sets $K_m^{(\alpha,n)}$, with $m, n \in \mathbb{N}$ and $\alpha < \omega_1$. We first fix a countable basis $\mathcal{U}$ of $B_\infty$ and write it as $\mathcal{U} = \{U_n : n \in \mathbb{N}\}$.

We let

$$K_m^{(0,0)} = K$$

assuming $K_m^{(\alpha,n)}$ has been defined for $\alpha < \omega_1$ and $n \geq 0$ we put

$$K_m^{(\alpha,n+1)} = \begin{cases} K_m^{(\alpha,n)} \setminus U_{n+1} & \text{if } \operatorname{diam}(U_n \cap K_m^{(\alpha,n)}) \leq 2^{-m} \\ K_m^{(\alpha,n)} & \text{if } \operatorname{diam}(U_n \cap K_m^{(\alpha,n)}) > 2^{-m} \end{cases}$$

assuming $K_m^{(\alpha,n)}$ has been defined for all $n \in \mathbb{N}$ we put

$$K_m^{(\alpha+1,0)} = \bigcap_{n\in\mathbb{N}} K_m^{(\alpha,n)}$$

and assuming $\beta$ is a limit ordinal

$$K_m^{(\beta,0)} = \bigcap_{\alpha<\beta} K_m^{(\alpha,0)}.$$

Remark. By transfinite induction we can easily observe that for all $K \in \mathcal{K}(X)$ and $\alpha < \omega_1$

$$K_{2^{-m}}^{\alpha} = K_m^{(\alpha,0)}.$$

Lemma 1.2. *Let $m, n \in \mathbb{N}$ and $\alpha < \omega_1$. Then*

$$D_m^{(\alpha,n)} : \mathcal{K}(B_\infty) \to \mathcal{K}(B_\infty) \cup \{\emptyset\}, \quad K \mapsto K_m^{(\alpha,n)},$$

*is Borel measurable. (on $\mathcal{K}(B_\infty) \cup \{\emptyset\}$ is a disjoint union of Polish spaces also Polish).*

Proof. We will show the following two claims:
Claim 1: Assume $U \subset B_\infty$ is open and $\varepsilon > 0$. Then the map:

$$D_U : \mathcal{K}(B_\infty) \to \mathcal{K}(B_\infty) \cup \{\emptyset\}, K \mapsto \begin{cases} K \setminus U & \text{if } \operatorname{diam}(K \setminus U) \leq \varepsilon \\ K & \text{if } \operatorname{diam}(K \setminus U) > \varepsilon \end{cases},$$

is Borel measurable.
Claim 2: The map

$$I : \mathcal{K}(B_\infty)^{\mathbb{N}} \to \mathcal{K}(B_\infty), (K_n) \mapsto \bigcap K_n,$$

is Borel measurable.

The claim of the Lemma, the follows easily by tranfinite induction on $\alpha < \omega_1$ and an induction on $n \in \mathbb{N}$. □

Corollary 1.3. *For $\alpha < \omega_1$ the set*

$$\mathcal{K}_\alpha(B_\infty) = \{K \in B_\infty : o_F(K) \leq \alpha\}$$

*is measurable.*

Proof. Note that

$$\mathcal{K}_\alpha(B_\infty) = \bigcap_{m\in\mathbb{N}} \{K \in B_\infty : o_F(2^{-m}, K) \leq \alpha\} = \bigcap_{m\in\mathbb{N}} \left(D_m^{(\alpha,0)}\right)^{-1}(\{\emptyset\})$$

□

## 2. The index $\mathrm{Sz}(X^*)$

Let us first recall that the Szlenk index of a separable Banach $X$ can be defined as follows.

We identify $X$ with a quotient space $\ell_1/Y$ where $Y$ is a closed linear subspace of $\ell_1$ With this identification, its dual $(\ell_1/Y)^*$ is $Y^\perp$, via the duality $\langle z, \overline{x}\rangle = \sum z_i x_i$, for $z = (z_i) \in Y^\perp$ and $\overline{x} \in \ell_1/Y$ (being the equivalence class of $x = (x_i) \in \ell_1$). Also note that with this identification and the $w^*$-topology on $Y^*$ concides with restriction of the $w^*$-topology on $\ell_\infty$.

As in the previous section we define $B_\infty$ to be $B_{\ell_\infty}$ endowed with the topology $\sigma(\ell_\infty, \ell_1)$.

For any $K \in \mathcal{K}(B_\infty)$, and $\delta > 0$ define:

$$\begin{aligned} D_\delta(K) &= \{\xi \in K \,:\, \forall\, V \subset B_\infty \ w^*\text{-open, } \operatorname{diam}(V \cap K) > \delta\} \\ &= K \setminus \bigcup \{V \subset \ w^*\text{-open, } \operatorname{diam}(V \cap K) \le \delta\}. \end{aligned}$$

Here the $w^*$-topology is the $w^*$-topology on $\ell_\infty$ and "diam" refers to the $\ell_\infty$-norm. Nevertheless, if $K \subset \tilde{K} = Y^\perp \cap B_\infty$ then the $w^*$-topology on $Y^\perp \equiv (\ell_1/Y)^*$ is the restriction of the $w^*$-topology of $\ell_\infty$, thus, the above term "$w^*$-open" can refer to both topologies. Similarly the dual norm on $(\ell_1/Y)^* \equiv Y^\perp$ coincides with the $\ell_\infty$-norm and, thus the above notion can also refer to the dual norm on $(\ell_1/Y)^*$.

For $\delta > 0$, $K \in \mathcal{K}(B_\infty)$ and $\alpha < \omega_1$ define $K^{(\alpha)}$ by transfinite induction.

$$K^{(0)} = K$$

and if $K^{(\beta)}$ has been defined for all $\beta < \alpha$ we put

$$\begin{aligned} K^{(\alpha)} &= D_\delta\big(K^{(\beta)}\big) \text{ if } \alpha = \beta + 1 \\ K^{(\alpha)} &= \bigcap_{\beta < \alpha} K^{(\beta)} \text{ if } \alpha = \sup_{\beta < \alpha} \beta. \end{aligned}$$

By the Lindelöf property of $K$ there exists an $\alpha < \omega_1$ so that $K^{(\alpha)} = K^{(\alpha+1)}$ and we define

$$o_\delta(K) = \min\{\alpha : K^{(\alpha)} = K^{(\alpha+1)}\} \text{ and } D_\delta^{(\infty)}(K) = K^{(o_\delta(K))}.$$

We can now define the *Szlenk index of a separable Banach space* $X$ using its representation as quotient of $\ell_1$ and the fragmentation index of its dual ball. Thus, let $X$ be isometrically isomorphic to $\ell_1/H$ with $H \in \mathrm{Subs}(\ell_1)$. Put

$$K = H^\perp \cap B_\infty = I_H(B_{(\ell_1/H)^*}) \in \mathrm{DB}.$$

Then put

$$\mathrm{Sz}_\delta(X) = \begin{cases} \min\{\alpha : K^\alpha = \emptyset\} & \text{if that exists (i.e. if } D_\delta^{(\infty)}(K) = \emptyset) \\ \omega_1 & \text{if not.} \end{cases}$$

Finally we define the *Szlenk index* to be

$$\mathrm{Sz}(X) = \sup_{\delta > 0} \mathrm{Sz}_\delta(X).$$

Our next step is to introduce a "dual index" for a separable Banach space $X$ which will turn out to be Sz($X^*$) in the case that $X$ is reflexive. Let $\Delta' = \{\xi_n : n \in \mathbb{N}\}$ be dense in $\Delta$ and consider the isometric embedding

$$E : C(\Delta) \to \ell_\infty, \quad f \mapsto (f(\xi_n) : n \in \mathbb{N}).$$

LEMMA 2.1. *The map*

$$\Phi : \mathrm{SB} \to \mathcal{K}(B_\infty), \quad X \mapsto \overline{E(B_X)}^{\sigma(\ell_\infty,\ell_1)}$$

*is Borel measurable.*

PROOF. Let $a < b$ and $n_0 \in \mathbb{N}$ and let $U = \{(y_n) \in B_\infty : a < y_{n_0} < b\}$. Not that such sets $U$ form a sub basis of the $w^*$-topology on $B_\infty$. Secondly note that

$$\begin{aligned}\big\{X \in \mathrm{SB} :&\Phi(X) \cap U \neq \emptyset\big\}\\ &= \big\{X \in \mathrm{SB} : \overline{E(B_X)}^{\sigma(\ell_\infty,\ell_1)} \cap U \neq \emptyset\big\}\\ &= \big\{X \in \mathrm{SB} : X \cap \{f \in C(\Delta) f(\xi_{n_0}) \in (a,b)\} \neq \emptyset\big\}\end{aligned}$$

which implies the claim. $\square$

From the proof of Theorem 2.1 and Corollary 2.3 we deduce the following

COROLLARY 2.2. *There is a Borel function* $\Theta : \mathrm{SB} \to \mathrm{SB}$ *so that*

a) *For every $X \hookrightarrow C(\Delta)$, $\Theta(X)$ is isometrically isomorphic to $X$.*
b) *For every $X \hookrightarrow C(\Delta)$ and every bounded sequence $(f_n) \subset \Theta(X)$ which is point-wise converging to a continuous function $f$ it follows that $f \in \Theta(X)$.*

We now choose a sequence $(f_n) \subset B_{C(\Delta)}$ which is dense in $B_{C(\Delta)}$ with respect to the sup-norm. For $\delta > 0$ and $n \in \mathbb{N}$ we define

$$B_{(\delta,n)} = \big\{y \in B_\infty : \|E(f_n) - y\|_\infty \leq \delta\big\}.$$

Since $B_{(\delta,n)}$ compact with respect to $\sigma(\ell_\infty, \ell_1)$ we deduce that the set

$$\begin{aligned}\mathcal{B}_\delta &= \big\{C \in \mathcal{K}(B_\infty) : \exists n \in \mathbb{N} \quad C \subset B_{(\delta,n)}\big\}\\ &= \bigcup_{n\in\mathbb{N}} \big[\mathcal{K}(B_\infty) \setminus \big\{C \in \mathcal{K}(B_\infty) : C \cap B^c_{(\delta,n)} \neq \infty\big\}\big]\end{aligned} \tag{19}$$

is Borel in $\mathcal{K}(B_\infty)$.

For $\delta > 0$, we consider now *the derivative with respect to* $\mathcal{B}_\delta$. For $F \in \mathcal{K}(B_\infty)$ we put

$$\begin{aligned}D^*_\delta(F) &= \big\{x \in F : \forall U \subset B_\infty \text{ open}, x \in U \quad \overline{U \cap F} \notin \mathcal{B}_\delta\big\}\\ &= F \setminus \bigcup \big\{U : U \subset B_\infty \text{ open}, \overline{U \cap F}^{\sigma(\ell_\infty,\ell_1)} \in \mathcal{B}_\delta\big\}.\end{aligned}$$

LEMMA 2.3. (cf. [Exercise 34.12]Kech1)
*For $\delta > 0$ the map $D^*_\delta(\cdot) : \mathcal{K}(B_\infty) \to \mathcal{K}(B_\infty)$, is Borel measurable.*

PROOF. Let $\{U_n : n \in \mathbb{N}\}$ be a countable basis of the $w^*$ topology on $B_\infty$. Let $V \subset B_\infty$ be open and write it as $V = \bigcup_{n=1}^\infty A_n$ where $A_n$ is closed. We note that (abbreviate $F' = D^*_\delta(F)$)

$$\{F \in \mathcal{K}(B_\infty) : F' \cap V = \emptyset\} = \bigcup_{n\in\mathbb{N}} \{F \in \mathcal{K}(B_\infty) : F' \cap A_n = \emptyset\}$$

and compactness yields for $n \in \mathbb{N}$ and $F \in \mathcal{K}(B_\infty)$

$$\begin{aligned} F' \cap A_n = \emptyset &\iff \forall x \in F \cap A_n, \exists m \in \mathbb{N},\, x \in U_m \quad \overline{U_m \cap F} \in \mathcal{B}_\delta \\ &\iff F \cap A_n \in \bigcup_{\mathcal{B}' \subset \mathcal{B}_\delta,\text{ finite}} \{C \in \mathcal{K}_\infty : C \subset \bigcup \mathcal{B}'\}. \end{aligned}$$

Together with the fact that the map $\mathcal{K}(B_\infty) \ni F \mapsto F \cap A_n \in \mathcal{K}(B_\infty)$ is Borel measurable this implies the claim. $\square$

For $\alpha < \omega_1$, and $\delta > 0$ we define $F^{(\alpha)} = F^{(\alpha)}_\delta$ by transfinite induction as usual:

$$F^{(0)} = F$$

and if $F^{(\gamma)}$ has been defined for all $\gamma < \alpha$ we put

$$\begin{aligned} F^{(\alpha)} &= \big(F^{(\beta)}\big)' \text{ if } \alpha = \beta + 1, \text{ and} \\ F^{(\alpha)} &= \bigcap_{\beta<\alpha} F^{(\beta)} \text{ if } \alpha = \sup_{\beta<\alpha} \beta \end{aligned}$$

There is a least ordinal $\alpha < \omega_1$ so that $F^{(\alpha)} = F^{(\alpha+1)}$ and we put $\tilde{o}_\delta(F) = \alpha$ and denote $\mathcal{D}^{(*,\infty)}_\delta(F) = F^{(o^*_\delta(F))}$.

From the fact that the map $D^*_\delta(\cdot)$ is a Borel measurable map on $\mathcal{K}(B_\infty)$ and from [**Kech1**, Proposition 34.8] it follows that $\mathcal{D}^{(*,\infty)}_\delta(F) = \emptyset$ if and only if $F$ is a countable union of elements of $\mathcal{B}_\delta$. We can apply [**Kech1**, Theorem 34.10] to conclude that

$$\Omega^*_\delta = \{F \in \mathcal{K}(B_\infty) : D^{(*,\infty)}_\delta(F) = \emptyset\} = \bigcup_{\alpha<\omega_1} \{F \in \mathcal{K}(B_\infty) : F^{(\alpha)} = \emptyset\}$$

is co-analyitc and that $\tilde{o}_\delta(\cdot)$ is a co-analytic rank on $\Omega_\delta$. Moreover, by [**KechLouv**, Chapetr VI, Section 1, Theorem 4] The set

$$\Omega^* = \bigcap_{\delta>0} \Omega^*_\delta = \bigcap_{\delta>0} \{F \in \mathcal{K}(B_\infty) : D^{(*,\infty)}_\delta(F) = \emptyset\},$$

is coanalytic and the map

$$\tilde{o}(\cdot) : \mathcal{K}(B_\infty) \to \mathrm{Ord}, \quad F \mapsto \sup_{\delta>0} \tilde{o}_\delta(F) = \sup_{\delta>0, \delta\in\mathbb{Q}} \tilde{o}_\delta(F)$$

defines a coanalyitc rank on $\Omega^*$.

LEMMA 2.4. *If $K \in \mathcal{K}(B_\infty)$, so that $K \subset E(C(\Delta))$, then $o(K) = \tilde{o}(K)$.*

PROOF. Let $\delta > 0$ and $K \in \mathcal{K}(B_\infty)$ with $K \subset E(C(\Delta))$. We first note that for an open $U \subset B_\infty$

$$\begin{aligned}\mathrm{diam}(U\cap K)\le\delta \iff & \mathrm{diam}(\overline{U\cap K})\le\delta\\ &\Longrightarrow \exists f\in C(\Delta)\quad \overline{U\cap K}\subset E(f)+\delta B_\infty\\ &\Longrightarrow \exists n\in \quad \overline{U\cap K}\subset E(f_n)+2\delta B_\infty\\ &\Longrightarrow \overline{U\cap K}\in\mathcal{B}_{2\delta}\Rightarrow \mathrm{diam}\big(\overline{U\cap K}\big)\le 4\delta\end{aligned}$$

and, thus,

$$\begin{aligned}D_{4\delta}(K)&=K\setminus\bigcup\big\{U\subset B_\infty \text{ open, and } \mathrm{diam}(U\cap K)\le 4\delta\big\}\\ &=K\setminus\bigcup\big\{U\subset B_\infty \text{ open, and } \mathrm{diam}(\overline{U\cap K})\le 4\delta\big\}\\ &\subset K\setminus\bigcup\big\{U\subset B_\infty \text{ open, and } U\cap K\in\mathcal{B}_{2\delta}\big\}=D^*_{2\delta}(K)\\ &\subset K\setminus\bigcup\big\{U\subset B_\infty \text{ open, and } \mathrm{diam}(\overline{U\cap K})\le \delta\big\}=D_\delta(K).\end{aligned}$$

This proves that for $K \in \mathcal{K}(B_\infty)$, with $K \subset E(C(\Delta))$, and $\delta > 0$ it follows that $o_\delta(K) \ge \tilde{o}_{2\delta}(K) \ge o_{4\delta}(K)$ and thus $o(K) = \tilde{o}(K)$. □

Combining these results with above Corollary 2.2 we deduce

THEOREM 2.5. *For $\delta > 0$ define:*

$$\mathrm{wco}_\delta : \mathrm{SB}\to\mathrm{SB},\quad \mathrm{wco}_\delta(X) := \begin{cases}\tilde{o}_\delta\big(\overline{E(B_{\Phi(X)})}^{\sigma(\ell_\infty,\ell_1)}\big) & \text{if } \overline{E(B_{\Phi(X)})}^{\sigma(\ell_\infty,\ell_1)}\in\Omega^*_\delta,\\ \omega_1 & \text{otherwise.}\end{cases}$$

*and we call* $\mathrm{wco}(X) = \sup_{\delta>0}\mathrm{wco}_\delta(X)$ *the* index of weak completeness of $X$*.*

a) *$X$ is weakly complete (i.e. every weak Cauchy sequence converges) if and only if for all $\delta > 0$.*

$$\overline{E(B_{\Theta(X)})}^{\sigma(\ell_\infty,\ell_1)}\in\Omega^*_\delta.$$

*The class of weakly complete Banach spaces is co-analytic and* $\mathrm{wco}(\cdot)$ *is a co-analytic rank for that class.*

b) *If $X$ is reflexive then*

$$\mathrm{wco}(X) = \mathrm{Sz}(X^*).$$

PROOF. (a) Lemma 2.1, Corollary 2.2 part (a) and the aforementioned results cited from [**Kech1**] and [**KechLouv**] imply that wco is co-analyitc rank for the set

$$\mathcal{C} := \Big\{X\in\mathrm{SB} : D^*_\infty\big(\overline{E(B_{\Theta(X)})}^{\sigma(\ell_\infty,\ell_1)}\big)=\emptyset\Big\}.$$

We need to show that $\mathcal{C}$ consists of the class of weakly complete separable Banach spaces. If $X$ is weakly complete, then

$$\overline{E(B_{\Theta(X)})}^{\sigma(\ell_\infty,\ell_1)} = E(B_{\Theta(X)}).$$

Indeed, if $y \in \overline{E(B_{\Theta(X)})}^{\sigma(\ell_\infty,\ell_1)}$ it follows from the fact that $B_\infty$ is metrizable that $y = \sigma(\ell_\infty,\ell_1)\lim_{n\to\infty} y_n$, with $y_n = E(x_n) \in E(B_{\Theta(X)})$, for $n \in \mathbb{N}$. By Rosenthal's $\ell_1$ theorem $x_n$ has a subsequence $(z_n)$ which is weak Cauchy, and thus, weakly convergent to some $x \in X \subset C(\Delta)$, or it is equivalent to the unit vector basis in $\ell_1$. The latter can not happen since then $(y_n)$ would not be point wise convergent. We deduce therefore our claim. Since $E(B_{\Theta(X)}) \subset \bigcup_{n\in\mathbb{N}} B_{(\delta,n)}$ for every $\delta > 0$ we deduce that $D_\delta^\infty(E(B_{\Theta(X)})) = \emptyset$ and thus $X \in \mathcal{C}$.

Conversely, if $X$ is not weakly complete there exists an $x^{**} \in B_{X^{**}} \setminus X$ which is the the $\sigma(X^{**}, X^*)$- limit of a sequence $(x_n) \subset B_X$. Consider $(x_n)$ being in $\Theta(X)$. Corollary 2.2 yields that the sequence $(E(x_n))$ converges pointwise to a $y \in \ell_\infty \setminus E(C(\Delta))$. It follows therefore that for some $\delta > 0$ $\text{dist}\big(y, E(C(\Delta))\big) > \delta$ and, thus, that

$$y \in D^*_\infty\Big(\overline{E(B_{\Theta(X)})}^{\sigma(\ell_\infty,\ell_1)}\Big)$$

which implies that $X \notin \mathcal{C}$.

(b) Let $X \in \text{SB}$ be reflexive. Since the map $E : C(\Delta) \to \ell_\infty$ is weakly coninuous, it follows that $K = E(B_X)$ is $\sigma(\ell_\infty,\ell_1)$-compact subset of $B_\infty$. Secondly, we note that $K$ satisfies the conditions of Proposition 1.5 (b), and we deduce that there is subspce $Y$ of $\ell_1$ so that $K = Y^\perp \cap B_{\ell_\infty} = B_{(\ell_1/Y)^*}$ and therefore $X \equiv (\ell_1/Y)^*$ . Thus Lemma 2.4 yields

$$\tilde{o}(K) = o(K) = \text{Sz}(\ell_1/Y) = \text{Sz}(X^*).$$

□

CHAPTER 11

# The result of Argyros and Dodos on strongly boundedness

We present a selfcontained proof of Argyros and Dodos [**AD**] result that the class of reflexive spaces is strongly bounded. The proof is due to Dan Freeman.

DEFINITION 0.6. A class $\mathcal{C} \subset \mathrm{SB}$ is called *strongly bounded* if forall subsets $\mathcal{A} \subset \mathcal{C}$ which are analyitc in SB there is a space $X \in \mathcal{C}$ which is (isomorphically) universal for $\mathcal{A}$.

We will give a direct proof of a result in [**AD**] which states that the class of reflexive spaces with bases, as well as the class of spaces with separable dual with bases, are strongly bounded. We denote by $SB_b$ the subset of $SB$ consisting of Banach spaces with a basis. From Lemma 1.7 and Lemma 1.8 we deduce that $\mathrm{SB}_b$ is an analytic set in SB. As in subsection 2 we denote by $\overline{u} = (u_i)$ the universal bimonotone basis constructed in Theorem 2.1 and define for $t = (n_1, n_2, \ldots, n_\ell) \in \mathbb{N}^{<\mathbb{N}}$ $u_t := u_{n_1+n_2+\ldots+n_\ell}$. Thus $(u_t : t \in \mathbb{N}^{<\mathbb{N}})$ is a bimonotone Schauder tree and for every $\varepsilon > 0$ and any bimonotone basic sequence $(x_i)$ there is a $\sigma \in \mathbb{N}^{\mathbb{N}}$ so that $(x_i) \sim_{1+\varepsilon} (u_{\sigma|_i})$.

For $\mathcal{A} \subset \mathrm{SB}$ we put

$$\tilde{\mathcal{A}} = \big\{ Y \in \mathrm{SB} : \exists X \in \mathcal{A}\ Y \simeq X \big\}$$

and recall that by Corollary 3.6 $\tilde{\mathcal{A}}$ is analytic provided $\mathcal{A}$ is analytic.

THEOREM 0.7. *Let $\mathcal{A} \subset \mathrm{SB}_b$.*
*Then $\tilde{\mathcal{A}}$ is analytic if and only if there is a pruned tree $T$ on $\mathbb{N}$ so that*

(a) $$\forall \sigma \in [T]\ \exists X \in \mathcal{A} \quad \overline{\mathrm{span}(u_{\sigma|_n} : n \in \mathbb{N})} \simeq X$$

(b) $$\forall X \in \mathcal{A}\ \exists \sigma \in [T] \quad \overline{\mathrm{span}(u_{\sigma|_n} : n \in \mathbb{N})} \simeq X.$$

PROOF. "$\Rightarrow$" We first claim that the set

$$C = \big\{ \sigma \in \mathbb{N}^{\mathbb{N}} : \overline{\mathrm{span}(u_{\sigma|_n})} \in \tilde{\mathcal{A}} \big\}$$

is analytic in $\mathbb{N}^{\mathbb{N}}$. Indeed, we first note that

$$\begin{aligned} C_1 &= \Big\{ (\sigma, \overline{x}) \in \mathbb{N}^{\mathbb{N}} \times S^{\mathbb{N}}_{C(\Delta)} : (u_{\sigma|_i}) \sim (x_i) \Big\} \\ &= \bigcup_{N \in \mathbb{N}} \bigcap_{n \in \mathbb{N}} \Big\{ (\sigma, \overline{x}) \in \mathbb{N}^{\mathbb{N}} \times S^{\mathbb{N}}_{C(\Delta)} : (u_{\sigma|_i})_{i=1}^n \sim_N (x_i)_{i=1}^n \Big\} \end{aligned}$$

is Borel in $\mathbb{N}^{\mathbb{N}} \times S^{\mathbb{N}}_{C(\Delta)}$. By Lemma 1.9

$$C_2 = \Big\{ \big( (x_i), \overline{\mathrm{span}(x_i)} \big) : (x_i) \in S^{\mathbb{N}}_{C(\Delta)} \Big\}$$

is Borel in $S_{C(\Delta)}^{\mathbb{N}}\times\mathrm{SB}$ and

$$\mathrm{SpEq} = \{(X,Y)\in\mathrm{SB}^2 : X\simeq\}$$

is analytic in $\mathrm{SB}^2$ by Theorem 3.4. Finally note that:

$$C=\pi_1\big\{\big(\sigma,(x_i),X,Y\big):\big(\sigma,(x_i)\big)\in C_1,\,\big((x_i),X\big)\in C_2 \text{ and } (X,Y)\in\mathrm{SpEq}\big\}.$$

By Proposition 1.2 there is a closed set $F\subset\mathbb{N}^{\mathbb{N}}\times\mathbb{N}^{\mathbb{N}}$ so that $C=\pi_1(F)$. We identify $\mathbb{N}^{\mathbb{N}}\times\mathbb{N}^{\mathbb{N}}$ (canonically) with $(\mathbb{N}\times\mathbb{N})^{\mathbb{N}}$ and define

$$T_0=\big\{t=\mathbb{N}\times\mathbb{N})^{\mathbb{N}}:\exists\alpha\in F, t\prec\alpha\big\}.$$

By Propostion 2.3 $F=[T_0]$. We now let

$$T=\pi_1(T_0)=\left\{t=(m_1,m_2,\dots,m_\ell):\begin{array}{c}\exists(n_1,n_2,\dots,n_\ell)\in N^{<\mathbb{N}}\\ \big((m_1,n_1),(m_2,n_2),\dots(m_\ell,n_\ell)\big)\in T_0\end{array}\right\}.$$

We claim that (a) and (b) are satisfied. Indeed, if $\sigma=(m_i)\in[T]$ then for some $\tau=(n_i)\in\mathbb{N}^{\mathbb{N}}$ it follows that $(m_i,n_i)_{i=1}^n\in T_0$ for all $n\in\mathbb{N}$ and thus $(\sigma,\tau)\in F$ which means that $\sigma\in C$, and thus $\overline{\mathrm{span}(u_{\sigma|_n}:n\in\mathbb{N})}\in\tilde{\mathcal{A}}$. Conversely if $X\in\tilde{\mathcal{A}}$, and $(x_i)$ is a normalized basis of $X$, then there is a $\sigma\in\mathbb{N}^{\mathbb{N}}$ so that $(x_i)\sim(u_{\sigma|_i})$.

Conversely, assume that there is a pruned tree $T$ satisfying (a) and (b). Since the maps

$$\Phi_1:\mathbb{N}^{\mathbb{N}}\to S_{C(\Delta)}^{\mathbb{N}},\quad \sigma\mapsto(u_{\sigma|_n}:n\in\mathbb{N})),\text{ and}$$
$$\Phi_2:S_{C(\Delta)}^{\mathbb{N}}\to\mathrm{SB},\quad (x_n)\mapsto\overline{\mathrm{span}(x_n:n\in\mathbb{N})}$$

are measurable (for $\Phi_2$ see Lemma 1.8) it follows that

$$\Phi=\Phi_2\circ\Phi_1:\mathbb{N}^{\mathbb{N}}\to\mathrm{SB},\quad \sigma\mapsto\overline{\mathrm{span}(u_{\sigma|_n}:n\in\mathbb{N})},$$

is measurable, and thus the set $\mathcal{B}:=\Phi([T])$ is analytic. From (a) and (b) it follows that $\tilde{\mathcal{A}}=\tilde{\mathcal{B}}$ and thus it follows from Corollary 3.6 that $\tilde{\mathcal{A}}$ is analytic. $\square$

Let $T\subset\mathbb{N}^{<\mathbb{N}}$ be a tree and $\overline{x}=(x_t)_{t\in T}$ be a Schauder tree. Recall the following norm on $c_{00}(T)$ introduced in Subsection 1 For $z=(z_t)_{t\in T}\in c_{00}(T)$ we define

$$\|z\|_{\overline{x},T,\infty}=\sup\left\{\max_{i\le\ell}\left\|\sum_{t\in S_i}z_tx_t\right\|:\begin{array}{c}\ell\in\mathbb{N},\ S_i\subset T\text{ disjoint}\\ \text{segments for } i=1,\dots,\ell\end{array}\right\}.$$

We we denote by $X_{\overline{x},T,\infty}$ the completion of $c_{00}(T)$ with respect to $\|\cdot\|_{\overline{x},T,\infty}$. The unit vector basis of $c_{00}(T)$, which is a normalized basis of $X_{\overline{x},T,\infty}$ is, as usual denoted by $(e_t:t\in T)$. From Lemma 5.1 we deduce

PROPOSITION 0.8. *Assume that for all $\sigma\in[T]$ the space $X_\sigma:=\overline{\mathrm{span}(x_{\sigma|_n}:n\in\mathbb{N})}$ is reflexive. Then*

$$V=\bigcup_{\sigma\in[T]}B_{X_\sigma}$$

*is relatively weakly compact in $X_{\overline{x},T,\infty}$.*

PROOF. Let $(x_n) \subset V$. W.l.o.g. we can assume that $x_n \in c_{00}(T)$, for $n \in N$. Thus, for every $n \in \mathbb{N}$ there is an $s(n) = \big(s(n,1), s(n,2), \ldots, s(n,\ell_n)\big) \in T$ so that $x_n \in \text{span}(e_{s(n)|_m : m \le \ell_n})$ for $n \in \mathbb{N}$. Now we apply Lemma 5.1 to the sequence $(s(n) : n \in \mathbb{N})$ and obtain a subsequence $t(n)$ which is either constant, or there is a finite sequence $(a_1, a_2 \ldots a_\ell)$ so that $t(n) = (a_1, a_2, \ldots, a_\ell, \tilde{t}(n))$ where the first elements of the $\tilde{t}_n$'s are pairwise different, or there is an in finite sequence $(a_i)$ in $\mathbb{N}$ and an increasing sequence $(m_n)$ in $\mathbb{N}$, so that $t(n) = (a(1), a(2), \ldots, a(m_n), \tilde{t}_n)$, where either $\tilde{t}(n)$ is empty or the first element of $\tilde{t}_n$ is different form $a(m_n+1)$.

Let $y_i = y_{j_i}$ where $j_i \in \mathbb{N}$ is such that $t(i) = s(j_i)$. In the first case $(y_i)$ has clearly a $w$-convergent subsequence. In the second case we can write $y_i$ as $y_i = u_i + v_i$ with

$$u_i = P_{\text{span}(e_{(a_1,a_2,\ldots a_i)} : i \le \ell)}(x) \in \text{span}(e_{(a_1,a_2,\ldots a_i)} : i \le \ell) \text{ and}$$

$$v_i = P_{\text{span}(e_{(a_1,a_2,\ldots a_\ell,\tilde{t}(i)|j)} j=1,2,\ldots)}(x) \in \text{span}(e_{(a_1,a_2,\ldots a_\ell,\tilde{t}(i)|j)} j = 1, 2, \ldots).$$

Again $(u_i)$ has of course a weakly convergent subsequence and the supports of the $v_i$'s are incomparable and, thus, the $v_i$'s are weakly null.

In the third case we can write $y_i$ as $y_i = u_i + v_i$ with

$$u_i = P_{\text{span}(e_{(a_1,a_2,\ldots a_i)} : i \le m_n)}(x) \in \text{span}(e_{(a_1,a_2,\ldots,a_i)} : i \le m_n), \text{ and}$$

$$v_i = \begin{cases} 0 & \text{if } \tilde{t}_n = \emptyset \\ P_{\text{span}(e_{(a_1,a_2,\ldots a_{m_n},\tilde{t}|_j)} : j=1,2,\ldots)} \in \text{span}(e_{(a_1,a_2,\ldots,a_{m_n},\tilde{t}(i)|_j)} : j = 1, 2, \ldots) & \text{if } \tilde{t}_n \neq \emptyset. \end{cases}$$

The sequence $(u_i)$ is bounded in $X_{(a_i)}$ and has, thus, a convergent subsequence. The supports of the $v_i$ are incomparable, which yields that $v_i$ is weakly null. $\square$

APPENDIX A

# Notes on Axioms of set theory Well ordering and Ordinal Numbers

## 1. Logic and Notation

Any formula in Mathematics can be stated using the symbols

$$\wedge, \neg, \exists, \in, =, (, )$$

and the variables

$$v_j: \text{ where } j \text{ is a natural number.}$$

An *expression* is any finite sequence of symbols and variables. A *formula* is an expression constructed by the following rules.

(F1) $v_i \in v_j$, $v_i = v_j$ are formulae for any $i, j \in \mathbb{N}$.

(F2) if $\Phi$ and $\Psi$ are formulae, then the following expressions are also formulae.

a) $(\Psi) \wedge (\Phi)$.

b) $\neg(\Psi)$.

c) $\exists v_i(\Psi)$ for any $i \in \mathbb{N}$.

In order to simplify mathematical statements and to render them better understandable we are using abbreviations. Here is a list of some of them.

a) $\forall v_i(\Phi) :\iff \neg(\exists v_i(\neg(\Phi)))$.

b) $(\Phi) \vee (\Psi) :\iff \neg((\neg(\Phi)) \wedge (\neg(\Phi)))$.

c) $(\Phi) \to (\Psi) :\iff (\neg(\Phi)) \vee \Psi$.

d) $(\Phi) \leftrightarrow (\Psi) :\iff ((\Phi) \to (\Psi)) \wedge ((\Psi) \leftarrow (\Phi))$

e) $v_i \notin v_j :\iff \neg(v_i \in v_j)$.

f) $v_i \neq v_j :\iff \neg(v_i = v_j)$.

g) $\forall x \in y \ (\Phi) :\iff \forall x \ ((x \in y) \to \Phi)$

h) $\exists x \in y \ (\Phi) :\iff \exists x \ ((x \in y) \wedge \Phi)$

i) $x \subset y :\iff \forall z((z \in x) \to (z \in y))$.

j) If $v_1, v_2, \dots v_n$, $n \in \mathbb{N}$ are variables and $\Phi$ a formula:
$\forall v_1, v_2, \dots v_n \ \Phi :\iff \forall v_1(\forall v_2(\dots(\forall v_n(\Phi))\dots))$
$\exists v_1, v_2, \dots v_n \ \Phi :\iff \exists v_1(\exists v_2(\dots(\forall v_n(\Phi))\dots))$

Also, the variables $v_i$ can be replaced by other letters (Latin, Greek, or Hebrew, capital as well as lower cases) with possibly sub and superscripts. Often, if the context is clear, we suppress parentheses.

From (F1) and (F2) we can deduce that every quantifier (i.e $\forall$ or $\exists$) is followed in the non abbreviated form by exactly one variable, say $v$, we say that *this quantifier acts on* $v$, or that $v$ *lies in the scope of the quantifier*. A variable in a formula is called *bounded variable* if it lies in the scope of a quantifier. Otherwise we call it a *free variable*.

EXAMPLE 1.1. In the formula

$$\exists v_1(v_1 \in v_2)$$

$v_1$ is a bound variable while $v_2$ is free.

NOTATION. If $x_1, x_2, \dots x_n$ are free variable of a formula $\Phi$ we sometimes write $\Phi(x_1, x_2, \dots, x_n)$. In this case $\Phi(y_1, y_2, \dots, y_n)$ is the same formula with $y_i$ replacing $x_i$, for $i = 1, 2, \dots, n$.

With this concept of free variable we can introduce the following abbreviation for unique existence.

k) Assume that $\Phi$ is a formula for which $x$ is a free variable, and let $\Phi'$ be the same formula with $z$ replacing $x$ then:
$\exists! x\,(\Phi) :\iff \big(\exists x\,(\Phi)\big) \wedge \big(\forall z((\Phi') \to (x = z))\big)$.

## 2. Some Axioms of Set Theory

We list the axioms of set theory, for the moment without the the *Axiom of Regularity*, the *Axiom of Choice* and the *Axiom of Infinity*, which will be introduced later.

- *Axiom of Set Existence*

$$\exists\,(x = x) \tag{ASE}$$

- *Axiom of Extensionality*

$$\forall x\,\forall y\big(\forall z\,(z\in x \leftrightarrow z\in y) \leftrightarrow x = y\big) \tag{AE}$$

The Axiom of Extensionality states that, in order to show that two sets are equal it is enough to show that they have the same elements.

- *Axiom of Comprehension Scheme*
  Let $\Phi$ be a formula for which $y$ is not a free variable.

$$\forall z\exists y\forall x(x \in y \leftrightarrow x \in z \wedge \Phi) \tag{ACS}$$

For given $z$ and $\Phi$ the set $y$ (which by (AE) must be unique) is denoted by

$$\{x : x \in z \wedge \Phi\}.$$

Example 2.1. Assume $A, B, \mathcal{F}$ are sets, then the following are also sets:

$$\begin{gathered} A \cap B := \{x : (x \in A) \wedge (x \in B)\} \\ \bigcap \mathcal{F} := \{x : x \in z \wedge \forall y \in \mathcal{F}(x \in y) \\ \text{where } \mathcal{F} \neq \emptyset \text{ and } z \in \mathcal{F} \\ A \setminus B := \{x : (x \in A) \wedge (x \notin B)\} \end{gathered}$$

Exercise 2.2. Using (AE) and (ASE) show formally that the sets $A \cup B$, $\bigcap \mathcal{F}$ and $A \setminus B$ exist and are unique.

By (ASE) there is a set $z$. Using (ACS) we can define the *empty set* by

$$\emptyset = \{x \in z : x \neq x\}.$$

Using (AE) we can show:

Exercise 2.3. There is a unique set, we denote it by $\emptyset$, for which

$$\forall x(x \notin \emptyset).$$

- *Axiom of Unions* (Existence of union)

$$\forall \mathcal{F}\,\exists A \forall Y \forall x\big(x \in Y \wedge Y \in \mathcal{F} \to x \in A\big) \tag{AU}$$

For given $\mathcal{F}$ we can use (AE), (ACS) and (AU) to show that for any set $\mathcal{F}$ there is a unique set, denoted by $\bigcup \mathcal{F}$, for which

$$\forall x\big(x \in \bigcup \mathcal{F} \leftrightarrow \exists B \in \mathcal{F}\,(x \in B)\big).$$

Indeed let $A$ be a set satisfying (AU). Then put

$$\bigcup \mathcal{F} := \{x \in A : \exists B \in \mathcal{F}(x \in B)\}.$$

- *Axiom of Pairing* (Existence of pairs)

$$\text{(PA)} \qquad \forall x \forall y \exists z (x \in z \wedge y \in z).$$

Again for given $x$ and $y$ we can deduce from(AE), (ACS) and (AP) that there is a unique set $A$, we denote it by $\{x, y\}$, for which

$$\forall z (z \in A \leftrightarrow (z = x \vee z = y)).$$

We put $\{x\} = \{x, x\}$ and define the *ordered pair* by

$$(x, y) = \{\{x\}, \{x, y\}\}$$

The union of two sets $A$ and $B$ can be defined by

$$A \cup B = \bigcup \{A, B\}.$$

- *Power Set Axiom*

$$\text{(PSA)} \qquad \forall x \exists y \forall z (z \in y \leftrightarrow z \subset y).$$

- *Axiom of Replacement Schemes.* Assume $\Phi$ is a formula for which $Y$ is not a free variable and $x$ and $y$ are free and let $A$ be a set.

$$\text{(ARS)} \qquad \big(\forall x \in A\, \exists! y\, \Phi(x, y)\big) \to \big(\exists Y\, \forall x \in A\, \exists y \in Y\, \Phi(x, y)\big)$$

Roughly speaking (ARS) means the following: If for each element of a class (see definition at the end of section) we can choose a unique element out of a set $A$, then this class is actually a set.
From (ARS) and (ACS) we deduce the set

$$\{y \in Y : \exists x \in A\ \Phi(x, y)\}$$

exists.

Using (ARS) we can justify the existence *the Cartesian product of two sets $A$ and $B$*

$$A \times B := \{(x, y) : x \in A \wedge y \in B\},\ \textit{i.e.}\ z \in A \times B \leftrightarrow \exists a \in A\, \exists b \in B\, z = (a, b)$$

First we apply the Axiom of Replacement Schemes to the formula $z = (a, b)$ for a fixed $a \in A$ and note that

$$\forall b \in B\, \exists! z\ z = (a, b).$$

Thus using (ARS) together with (AE) we deduce that there is a (and only one) set, which we denote by $\text{prod}(a, B)$, such that

$$\forall z\big(z\in\text{prod}(a,B) \leftrightarrow \exists b\in B\ z=(a,b)\big).$$

Then we consider the formula $Z = \{a\} \times B$ and note that

$$\forall a\in A\,\exists!\,Z = \text{prod}(a,B)$$

and define

$$\text{prod}'(A,B) = \Big\{Z : \exists a\in A\ Z=\text{prod}(a,B)\Big\} = \big\{\text{prod}(a,B) : a\in A\big\}.$$

Finally we use (AU) to define

$$A\times B := \bigcup \text{prod}'(A,B),$$

and note that

$$\forall z\big(z\in A\times B \leftrightarrow \exists a\in A\ z\in\text{prod}(a,B) \leftrightarrow \exists a\in A\,\exists b\in B\,z=(a,b)\big).$$

Definition 2.4. If $\Phi$ is a formula which might have $x$ as a free variable then the *collection* of all $x$ for which $\Phi$ holds might not be a set. If $x$ is not a free variable of $\Phi$ this collection is either empty, or all sets. For example the collection of all sets $x$ for which $x = x$ is not a set. We call these collections *classes* and we call them *proper classes* if they are not sets.

Thus, classes are nothing else but formulae, i.e. to each formula $\Phi$ there corresponds a class

$$\mathbf{C} = \{x : \Phi(x)\},$$

and vice versa. We define the class of all sets by

$$\mathbf{V} = \{x : x = x\}$$

Remark. If $\mathbf{A}$ and $\mathbf{B}$ are classes so is

$$A\times B := \{(a,b)a\in\mathbf{A} \text{ and } b\in\mathbf{B}\}$$

or formally:

$$A\times B := \{z : \exists a\in\mathbf{A}\ \exists b\in\mathbf{B}\quad z=(a,b)\}.$$

## 3. Relations

We now want to abandon the somewhat tedious formal language used in the previous section and return to the "usual language" whenever it seems appropriate. This might be sloppy from a logician's point of view, but better to understand. Nevertheless, we would like to point out that every theorem in Mathematics theoretically could be stated as formula satisfying the rules stated at the beginning of section 2 and can be shown using set theoretic axioms (some of which have yet to be stated).

Definition 3.1. Let $\mathbf{A}$ and $\mathbf{B}$ classes (or sets). A *relation* on $\mathbf{A}$ and $\mathbf{B}$ is a subclass (or subset) $\mathbf{R}$ of $\mathbf{A} \times \mathbf{B}$ . Sometimes we write $a\mathbf{R}b$ if $(a, b) \in \mathbf{R}$ (especially in the case of equivalence relations and orderings).

For a relation $\mathbf{R}$ on $\mathbf{A} \times \mathbf{B}$ we write

$\mathrm{dom}(\mathbf{R}) = \{x : \exists y \ (x, y) \in \mathbf{R}\}$ (*domain of* $R$) and

$\mathrm{ran}(\mathbf{B}) = \{y : \exists x \ (x, y) \in \mathbf{R}\}$ (*range of* $\mathbf{R}$)

$\mathbf{R}^{-1} = \{(y, x) : (x, y) \in \mathbf{R}\}$ (*inverse of* $\mathbf{R}$)

if $\mathbf{A}'$ is a subclass of $\mathbf{A}$: $\mathbf{R}|_{\mathbf{A}'} = (\mathbf{A}' \times \mathbf{B}) \cap \mathbf{R}$ (*restriction of* $\mathbf{R}$ *to* $\mathbf{A}'$.)

A relation $\mathbf{R}$ on $\mathbf{A} \times \mathbf{B}$ is called

- a *function* if

$$\forall x \in \mathrm{dom}(\mathbf{R}) \ \exists! y \in \mathrm{ran}(\mathbf{R}) \ (x, y) \in R,$$

  and we write $\mathbf{R} : \mathbf{A} \to \mathbf{B}$ if $\mathbf{R}$ is a function whose domain is $\mathbf{A}$
- an *injective function* if $\mathbf{R}$ and $\mathbf{R}^{-1}$ are functions.

A relation $\mathbf{R}$ on $\mathbf{A} \times \mathbf{A}$ is called

- *transitive* if

$$\forall x, y, z \in \mathbf{A} \ ((x, y) \in \mathbf{R} \wedge (y, z) \in \mathbf{R} \to (x, z) \in \mathbf{R}).$$

- *reflexive* if

$$\forall x \in \mathbf{A} \ (x, x) \in \mathbf{R}.$$

- *irreflexive* if

$$\forall x \in \mathbf{A} \ (x, x) \notin \mathbf{R}.$$

- *trichotomic* if

$$\forall x, y \in \mathbf{A} \ ((x, y) \in \mathbf{R} \vee (y, x) \in \mathbf{R} \vee x = y).$$

- *symmetric* if

$$\forall x, y \in \mathbf{A} \ ((x, y) \in \mathbf{R} \to (y, x) \in \mathbf{R}).$$

- *anti symmetric* if

$$\forall x, y \in \mathbf{A} \ ((x, y) \in \mathbf{R} \wedge (y, x) \in \mathbf{R} \to x = y).$$

A relation $\mathbf{R}$ on $\mathbf{A} \times \mathbf{A}$ is called

- a *partial order* if it is transitive, reflexive and antisymmetric,
- a *linear order* if it is transitive, antisymmetric and trichotomic,

- a *strict linear order* or *total order* if it is transitive, irreflexive and trichotomic.

Exercise 3.2. Show that a strict linear order is anti symmetric, and thus that a strict linear order is a linear order.

Exercise 3.3. Let $R$ be a linear order. Then there is a reflexive version $R_{ref}$ of $R$,i.e.

$$(a,b) \in R_{ref} \iff (a=b) \vee (a,b) \in R.$$

and a strict version $R_{st}$, i.e.

$$(a,b) \in R_{st} \iff (a \neq b) \wedge (a,b) \in R.$$

Notation. For strict linear orders we will use the symbols $<$,$>$,$\prec$ and $\succ$ and for reflexive linear orders the symbols $\leq$, $\geq$, $\preceq$, and $\succeq$.

Example 3.4. $\subset$ is a transitive, reflexive, symmetric relation on $\mathbf{V}$, thus it is a partial order on $\mathbf{V}$.

Definition 3.5. Two total orders $(S,<)$ and $(S',\prec)$ are called *order isomorphic* if there exists a bijective ($1-1$ and onto) map: $\Phi: S \to S'$ which is *order preserving* i.e.

$$\forall a,b \in S \quad \big(a < b \iff \phi(a) < \phi(b)\big).$$

In that case we call $\phi$ *an order isomorphism between $S$ and $S'$.*

Definition 3.6. A *well order* on a set $S$ is a relation $<$ on $S$ which has the following properties.

(WO1) $(A,<)$ is a total order.
(WO2) Every nonempty subset $A$ of $S$ has a minimum, i.e. there is an $a_0 \in A$ so that for all $a \in A$ either $a_0 < a$ or $a_0 = a$.

In that case the pair $(S,<)$ is called a well ordering of $S$ and we introduce the following notations:

For $a,b \in S$ we write $a \leq b$ if $a < b$ or $a = b$.
For $a,b \in S$, with $a \leq b$ we introduce the following *intervals*:

$$[a,b] = \{x \in S : a \leq x \leq b\}$$
$$[a,b) = \{x \in S : a \leq x < b\}$$
$$(a,b] = \{x \in S : a < x \leq b\}$$
$$(a,b) = \{x \in S : a < x < b\}$$

For a none empty $A \subset S$ we denote the (by (WO2) existing) minimal element of $A$ by $\min(A)$, and write $0_S = \min S$. If $a \in S$ is not a maximal element then the set of *successors of* $a$, namely the set

$$\text{Succ}(a,S,<) = \{x \in S : x > a\},$$

must have a minimal element which we call the *direct successor of* $a$ and denote it by $a^+$. The set of *predecessors* of $a$ is defined by

$$\text{Pred}(a,S,<) = \{x \in S : x < a\} = [0_S, a).$$

PROPOSITION 3.7. *If $(S,<)$ and $(S',\prec)$ are order isomorphic well orderings, then the order isomorphism between them is unique.*

PROOF. W.l.o.g. $S, S' \neq \emptyset$. Let $\Psi, \Phi : S \to S'$ be two order isomorphisms, and assuming that $\Psi \neq \Phi$ we can define:

$$a = \min\{x \in S : \Psi(x) \neq \Phi(x)\}.$$

Since $\Psi(a) \neq \Phi(a)$ we may w.l.o.g. assume that $\Psi(a) < \Phi(a)$. But now it follows that

$$\Phi(S) \subset [0_{S'}, \Psi(a)) \cup \{x' \in S' : x' \succ \Psi(a)\}.$$

Indeed, if $x < a$, then $\Phi(x) = \Psi(x) \prec \Psi(a)$, and, if $x \geq a$ then $\Phi(x) \succeq \Phi(a) \succ \Psi(a)$. This would mean that $\Psi(a) \notin \Phi(S)$, which contradicts the assumption that $\Phi$ is surjective. □

PROPOSITION 3.8. *If $(S,<)$ is a well ordering and $a, b \in S$. Then $[0_S, a]$ and $[0_S, b]$ are order isomorph if and only if $a = b$.*

PROOF. Let $\Phi : [0_S, a] \to [0_S, b]$ be an order isomorphism and consider

$$A = \{x \in [0_S, a] : \Phi(x) \neq x\}.$$

If this set was not empty we could choose $x_0 = \min(A)$, and, using a similar argumentation as in the proof of Proposition 3.7 we would get a contradiction to the assumed surjectivity of $\Phi$. If $A$ is empty, it follows that $\Phi$ is the identity on $[0_S, a]$, and, since $\Phi$ is supposed to be surjective it follows that $a = b$. □

COROLLARY 3.9. *If $(S,<)$ and $(S',\prec)$ are two well orderings and there is an $a' \in S$ so that $(S,<)$ is order isomorph to $[0_{S'}, a')$. Then such an $a' \in S'$ is unique.*

THEOREM 3.10. *Let $(S,<)$ and $(S',\prec)$ be two well orderings. Then one and only one of the following cases occurs.*
*Case 1. $(S,<)$ and $(S',\prec)$ are order isomorphic.*
*Case 2. There is an $a \in S$ so that $[0_S, a)$ and $S'$ are order isomorphic.*
*Case 3. There is an $a' \in S'$ so that $[0_{S'}, a')$ and $S$ are order isomorphic.*

PROOF. Define:

$$A = \{a \in S : \exists a' \in S' \quad [0_S, a] \text{ and } [0_{S'}, a'] \text{ are order isomorphic}\}.$$

Let us first assume that $A = S$. If $S$ has a maximal element $a$ it follows that there exists an order isomorphism between $S = [0_S, a]$ and some interval $[0_{S'}, b']$ (which might or might not be all of $S'$) which is order isomorphic to $S$. This means that we are either in Case 1 or in Case 3 (choose $a' = b'^{+}$). If $S$ has not a maximal element we choose for $a \in S$, an $a' \in S'$ for which an order isomorphism $\Phi_a : [0_S, a] \to [0_{S'}, a']$ exists. Note that by Proposition 3.7 and Corollary 3.9 $a'$ and $\Phi_a$ are unique for every $a \in S$. Then choose

$$\Phi : S \to S', a \mapsto \Phi_a(a).$$

From the uniqueness of the $\Phi_a$'s it follows for $a < b$ in $S$ that $\Phi_a(a) = \Phi_b(a)$. Moreover, if $\Phi(S) = S'$ it follows that $\Phi$ is an order isomorphism between $S$ and $S'$, and if $\Phi(S) \neq S'$ it follows that $\Phi$ is an order isomorphism between $S$ and $(0_{S'}, a')$, where $a' = \min\{x' \in S' : x' \notin \Phi(S)\}$.

If $A \neq S$ we put $a_0 = \min S \setminus A$. For $a < a_0$ we choose (the uniquely existing) $a' \in S'$ and $\Phi_a : [0_S, a] \to [0_{S'}, a']$, where $\Phi_a$ is an order isomorphism and claim that $S' = \{a' : a \in S\}$. Indeed if this where not so, we could put $a'_0 = \min S' \setminus \{a' : a \in S\}$ and define

$$\Phi_{a_0} : [0, a_0] \to [0, a'_0], a \mapsto \begin{cases} \Phi_a(a) \text{ if } a < a_0 \\ a'_0 \text{ if } a = a_0 \end{cases}$$

and deduce from the uniqueness of the $a'$ and $\Phi_a$ for $a \in S$ that $\Phi_{a_0}$ is an order isomorphism between $[0, a_0]$ and $[0, a'_0]$, which contradicts our definition of $a_0$.

Therefore it follows that

$$A' = \{a' \in S' : \exists a \in S \quad [0_S, a] \text{ and } [0_{S'}, a'] \text{ are order isomorphic}\} = S',$$

and using our previous arguments we can show that we are either in Case 1 or Case 2.

Moreover, only one of the three cases can happen. Indeed, assume Cases 1 and 2 hold. This would imply that for some $a \in S$ the sets $S$ and $[0_S, a)$ where order isomorphic. Thus, let $\Phi : S \to [0_S, a)$ be an order isomorphism. But this yields that $\Phi|_{[0_S,a)} : [0_S, a)] \to [0_S, \Phi(a))$ is an order isomorphism, which implies by Proposition 3.8 that $\Phi(a) = a$ which is a contradiction (since $[0_S, a) \subsetneq S$).

Similarly we can show that Cases 1 and 3, as well as Cases 2 and 3 cannot hold at the same time. $\square$

## 4. Definition of ordinal numbers

Our next step will be, roughly speaking, the following: Call two well orderings equivalent if they are order isomorphic. We choose out of each equivalence class of well orderings one representant which we will call *ordinal number.* Since these *classes* are not sets we will not be able to use the Axiom of Choice (which will be introduced later) to do so.

But we can proceed differently and use the fact that "$\in$" can be thought of a relation between sets.

Definition 4.1. An *ordinal number* is a set $\alpha$ which has the following two properties:

(Or1) Every element of $\alpha$ is also a subset of it (such a set is called transitive).

(Or2) $\alpha$ is well ordered by the relation $\in$, i.e. the relation $<$ on $\alpha$ defined by

$$\beta < \gamma \iff \beta \in \gamma \text{ for } \beta, \gamma \in \alpha$$

is a well order on $\alpha$.

If $\alpha$ is an ordinal we write $\alpha \in \mathbf{ON}$. We will show later that $\mathbf{ON}$ is actually a proper class.

Proposition 4.2. *Let* $\alpha \in \mathbf{ON}$.

a) *For all* $\gamma \in \alpha$ *it follows that* $\gamma \in \mathbf{ON}$ *and* $\gamma = [0_\gamma, \gamma)$.

b) *If* $\alpha \neq \emptyset$ *then* $\emptyset \in \alpha$ *and* $\emptyset = 0_\alpha = \min \alpha$. *Therefore we will write instead of* $0_\alpha$ *from now on simply* $0$.

Proof. For $\gamma \in \alpha$ it follows from (Or2) that:

$$[0_\gamma, \gamma) = \{\beta \in \alpha : \beta \in \gamma\} = \{\beta : \beta \in \gamma\} = \gamma.$$

Since intervals of the form $[0_A, a)$, for a well ordering $(A, <)$ and $a \in A$ is also a well ordering it follows that $(\gamma, \in) = \big([0_\alpha, \gamma), \in\big)$ is a well ordering. For $\beta \in \gamma = [0_\alpha, \gamma) \subset \alpha$ it follows that

$$\beta = [0_\alpha, \beta) \subset [0_\alpha, \gamma) = \gamma,$$

thus $\gamma$ also satisfies (Or1), which finishes the proof of (a).

If for some $\alpha \in \mathbf{ON}$ we had $0_\alpha \neq \emptyset$, we would have some $\gamma \in 0_\alpha$ and thus $\gamma < 0_\alpha$ and, by (Or1) $\gamma \in 0_\alpha \subset \alpha$, thus $\gamma \in \alpha$, which contradicts the minimality of $0_\alpha$. ☐

We want to show now that every ordinal $\alpha$ admits a successor, namely a smallest ordinal $\beta$ for which $\alpha < \beta$. The way to do that is to define $\beta = \alpha \cup \{\alpha\}$. It is then easy to see that $\alpha \cup \{\alpha\}$ is also an ordinal, but in order to deduce that $\alpha \neq \alpha \cup \{\alpha\}$ we will need another axiom.

- *Regularity Axiom* or *Wellfoundedness Axiom*

(RA)
$$\forall x\Big(x = \emptyset \vee \exists y\,\big((y \in x) \wedge \forall z(z \in y \to z \notin x)\big)\Big)$$

The Regularity Axiom states that each non empty set $A$ has at least one element $a$ (which is a set itself) all of whose elements are not elements of $A$.

PROPOSITION 4.3. *For every set $A$ it follows that $A \notin A$. In particular it follows for every ordinal $\alpha$ that $\alpha \neq \alpha \cup \{\alpha\}$.*

PROOF. Note that the set $\{A\}$ (this is a set as shown in Section 4) must contain an element (which can only be $A$) which is disjoint from $\{A\}$ thus $A \notin A$ □

PROPOSITION 4.4. *If $\alpha \in \mathbf{ON}$ then*

$$\alpha + 1 := \alpha \cup \{\alpha\} \in \mathbf{ON}.$$

*We call $\alpha + 1$* the successor of $\alpha$
*Assume that $\alpha, \beta \in \mathbf{ON}$, with $\alpha < \beta$, then*

$$\min(\alpha, \beta] = \alpha + 1.$$

PROOF. It is easy to see that $\alpha + 1 = \alpha \cup \{\alpha\}$ satisfies (Or1) and (Or2) and is therefore an ordinal number.

In order to show our second claim we define $\alpha' = \min(\alpha, \beta]$. Since $\alpha' > \alpha$ it follows that $\alpha \in \alpha'$ and, thus, since by Proposition 4.2 $\alpha' \in \mathbf{ON}$, we deduce from (Or1) that $\alpha \subset \alpha'$. Therefore we have $\alpha + 1 \subset \alpha'$.

Assume that $\alpha + 1 \neq \alpha'$ and, thus, $\alpha + 1 \subsetneq \alpha'$. Let $\gamma \in \alpha' \setminus \alpha + 1$, and, thus, $\alpha + 1 \in \gamma$ and (by (Or1)) $\alpha \subset \gamma$, which implies $\alpha \in \gamma \in \alpha'$ which is a contradiction to the assumed minimality of $\alpha'$. □

DEFINITION 4.5. An ordinal $\alpha$ is called a *Successor Ordinal* if it is the successor of some ordinal $\beta$, i.e if $\alpha = \beta + 1$ for some $\beta \in \alpha$. If $\alpha > 0$ is not a successor ordinal it is called a *Limit Ordinal*. The class of all successor ordinal and the class of all limit ordinals are denoted by $\mathrm{Succ}(\mathbf{ON})$ and $\mathrm{Lim}(\mathbf{ON})$, respectively.

THEOREM 4.6. *For any two ordinal numbers $\alpha$,$\beta$ one and only one of the following statements are true.*

a) $\alpha \in \beta$
b) $\beta \in \alpha$
c) $\alpha = \beta$

PROOF. Let $\alpha, \beta \in \mathbf{ON}$. We need to show that either $\alpha \in \beta$ or $\beta \in \alpha$ or $\alpha = \beta$. By Theorem 3.10 we can w.l.o.g assume that there is an injective and order preserving embedding map $\Phi : \alpha \to \beta$ whose image is $[0, \tilde{\beta})$ for some $\tilde{\beta} \in \beta$ or $\tilde{\beta} = \beta$. We need to show that $\alpha = \tilde{\beta}$ and $\Phi(\gamma) = \gamma$ for all $\gamma \in \alpha$. Assuming that this is not true we could choose

$$\gamma_0 = \min\{\gamma \in \alpha : \Phi(\gamma) \neq \gamma\},$$

and put $\gamma_0' = \Phi(\gamma_0)$

We deduce that

$$\beta > \gamma_0' = [0, \gamma_0') \supset \Phi(\{\gamma : \gamma \in \gamma_0\}) = \{\Phi(\gamma) : \gamma \in \gamma_0\} = \gamma_0.$$

(here we think of $\gamma_0$ being the set of all its elements) Thus, by definition of $\gamma_0$ we deduce that $\gamma_0' > \gamma_0$, and, thus,

$$\Phi([0, \alpha)) = \Phi([0, \gamma_0)) \cup \Phi([\gamma_0, \alpha)) = [0, \gamma_0) \cup [\gamma_0', \tilde{\beta}),$$

which contradicts the fact that the image of $\Phi$ is an interval.

Finally the Well Foundedness Principle and Proposition 4.3 yield that only one of the statements $(a)$ $(b)$ or $(c)$ can hold. $\square$

EXERCISE 4.7. For $\alpha \in \mathbf{ON}$, $\alpha > 0$

$$\alpha \in \mathrm{Lim}(\mathbf{ON}) \leftrightarrow \forall \beta < \alpha \quad \beta + 1 < \alpha.$$

THEOREM 4.8. *The class of ordinals itself is well ordered by $\in$, i.e every non empty subclass of* $\mathbf{ON}$ *has a minimal element.*

PROOF. By Theorem 4.6 it is left to show that a non empty sub class $A$ of $\mathbf{ON}$ has a minimum. Let $A \subset \mathbf{ON}$. Since we could pass to a subset of $A$, having the same minimum, if necessary (for example $[0, \alpha + 1) \cap A$ for some $\alpha \in A$) we can assume that $A$ is a set. By the Well Foundedness Principle there is an $\alpha \in A$ which is disjoint from $A$ (as sets). Therefore it must follow from Theorem 4.6 that $\alpha \in \beta$ for all $\beta \in A \setminus \{\alpha\}$. $\square$

COROLLARY 4.9. *If $A$ is a set of ordinals, then*

$$\sup(A) = \bigcup A = \bigcup_{\alpha \in A} \alpha$$

*(Note that this set is* **not** *equal to $A = \{\alpha : \alpha \in A\}$) is also an ordinal and it is the smallest ordinal $\beta$ so that $\alpha \leq \beta$ for all $\alpha \in A$, i.e.*

$$\sup(A) = \min \big\{ \gamma : \forall \alpha \in A \ \gamma \geq \alpha \big\}.$$

PROOF. (Or1) is easy to verify for $\sup(A)$. From Theorem 4.6 we deduce that $\in$ is a total order on $\sup(A)$ and from Theorem 4.8 that it is a well order.

Clearly $\alpha \subset \sup(A)$, for all $\alpha \in A$. If $\beta \in \mathbf{ON}$ has the property that $\alpha \leq \beta$ for all $\alpha \in A$ then $\sup(A) \subset \beta$. $\square$

COROLLARY 4.10. *For an ordinal $\alpha > 0$ it follows that*

$$\alpha \in \mathrm{Lim}(\mathbf{ON}) \iff \alpha = \sup(\alpha) = \sup_{\beta < \alpha} \beta.$$

PROOF. If $\alpha = \beta + 1$, then

$$\sup(\alpha) = \bigcup_{\gamma < \alpha} \gamma = \beta \cup \bigcup_{\gamma < \beta} \gamma = \beta$$

thus $\sup_{\beta < \alpha} \beta \neq \alpha$.

Conversely, assume $\alpha \neq \sup(\alpha)$. (Or1) yields $\sup(\alpha) \subset \alpha$ and from Theorem 4.6 it follows that $\sup(\alpha) \in \alpha$. But this can only mean that $\alpha = \sup(\alpha) \cup \{\sup(\alpha)\}$, because, otherwise, again by Theorem 4.6 $\sup(\alpha) < \sup(\alpha) \cup \{\sup(\alpha)\} < \alpha$ which contradicts the definition of $\sup(\alpha)$ in Corollary 4.9. $\square$

COROLLARY 4.11. $\mathbf{ON}$ *is a proper class.*

PROOF. If $\mathbf{ON}$ were a set it would have a supremum $\sup(\mathbf{ON})$, but this would imply that $\sup(\mathbf{ON}) = \sup(\mathbf{ON}) \cup \{\sup(\mathbf{ON})\}$ which contradicts the Axiom of Well-foundedness. $\square$

THEOREM 4.12. *(Principle of transfinite induction, formal statement) Assume that $\phi(\alpha)$ is a statement and assume that*

$$\forall\alpha\in\mathbf{ON}\ \big((\forall\beta<\alpha\ \phi(\beta))\to\phi(\alpha)\big)$$

*then $\phi(\alpha)$ for all $\alpha\in\mathbf{ON}$.*

PROOF. Assume that there are ordinals $\alpha$ for which $\phi(\alpha)$ is false. By Theorem 4.8 there exists

$$\alpha_0=\min\{\alpha\in\mathbf{ON}:\neg\phi(\alpha)\}.$$

But since for all $\alpha<\alpha_0$ the statement $\phi(\alpha)$ holds, we are led to a contradiction to our assumption. □

THEOREM 4.13. *(Principle of transfinite recursion) Let $\mathbf{F}:\mathbf{V}\to\mathbf{V}$ be a function. Then there is a unique function $\mathbf{G}:\mathbf{ON}\to\mathbf{V}$ so that for all $\alpha\in\mathbf{ON}$ $\mathbf{G}(\alpha)=\mathbf{F}(\mathbf{G}|_\alpha)$.*

REMARK. The statement of Theorem 4.13 means the following:

One would like to define a function $\mathbf{G}:\mathbf{ON}\to\mathbf{ON}$. If one has a *well defined procedure* to define $\mathbf{G}(\alpha)$ assuming one knows $\mathbf{G}(\beta)$ for all $\beta<\alpha$ one can define $\mathbf{G}$ as a function on all of $\mathbf{ON}$.

PROOF OF THEOREM 4.13. The uniqueness of $\mathbf{G}$ follows from Theorem 4.8. Indeed assume that $\mathbf{G}_1$ as well as $\mathbf{G}_2$, both satisfy the conclusion of our claim, but are not equal. Let

$$\alpha_0=\min\{\alpha:\mathbf{G}_1(\alpha)\neq\mathbf{G}_2(\alpha)\}.$$

Then our assumption yields that

$$\mathbf{G}_1(\alpha_0)=\mathbf{F}(\mathbf{G}_1|_{\alpha_0})=\mathbf{F}(\mathbf{G}_2|_{\alpha_0})=\mathbf{G}_2(\alpha_0),$$

which is a contradiction.

Let $\mathcal{A}$ be the class of ordinals $\alpha$, for which there is a function $g_\alpha:\alpha\to\mathbf{V}$ so that for all $\beta<\alpha$ $g_\alpha(\beta)=\mathbf{F}(g_\alpha|_\beta)$.

Using a the same arguments as above, we can observe that if $\alpha\in\mathbf{A}$ there is a unique function $g_\alpha:\alpha\to\mathbf{V}$ so that for all $\beta<\alpha$ $g_\alpha(\beta)=\mathbf{F}(g_\alpha|_\beta)$. This implies in particular that if $\gamma<\beta<\alpha$ it follows that $g_\alpha(\gamma)=g_\beta(\gamma)$. For each $\gamma$ for which there is an $\alpha>\gamma$ so that $\alpha\in\mathbf{A}$ there is a (unique) $g(\gamma)$ so that $g(\gamma)=g_\beta(\gamma)$ whenever $\beta>\gamma$ is in $\mathcal{A}$.

Finally we claim that $\mathcal{A}=\mathbf{ON}$. Indeed, otherwise there would be a minimal $\alpha_0$ for which $g:\alpha_0\to\mathbf{V}$ does not exist. If $\alpha_0\in\mathrm{Succ}(\mathbf{ON})$, say $\alpha_0=\beta_0+1$, then define $g(\beta_0):=F(g|_{\beta_0})$. If $\alpha_0\in\mathrm{Lim}$, define $g:\alpha_0\to\mathbf{V}$ by $g(\beta)=g_{\beta+1}(\beta)$. Finally it is not possible that $\alpha_0=\emptyset$, since we can choose $g_0=\emptyset$. Thus, we derive a contradiction. □

Let us now write down the first ordinals

EXAMPLE 4.14. The following are ordinals

$0 := \emptyset$ (Note that the empty set is always well ordered, no matter how you define $<$)

$1 := 0^+ = \{\emptyset\}$

$2 := 1^+ = \big\{\emptyset, \{\emptyset\}\big\}$

$3 := 2^+ = \Big\{\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}\Big\}$

$\vdots$

We call the the ordinals $0, 1, 2, \ldots$ *natural numbers*. More formally: an ordinal $\alpha$ is called *natural number* if every $\gamma \in \alpha$ is a successor ordinal. We denote the class of natural numbers by $\mathbb{N}$ (we will need to introduce below an additional axiom to deduce that $\mathbb{N}$ is actually a set).

THEOREM 4.15. *(Principle of Induction and Recursion restricted to $\mathbb{N}$)*

a) *For statement $\phi$*

$$\forall n \in \mathbb{N}\, ((\forall m < n\ \phi(m)) \to \phi(n)),$$

*then*

$$\forall n \in \mathbb{N}\ \phi(n)$$

b) *Let $\mathbf{F} : \mathbf{V} \to \mathbf{V}$ be a function. Then there is a unique function $\mathbf{G} : \mathbb{N} \to \mathbf{V}$ so that for all $n \in \mathbb{N}$ $\mathbf{G}(n) = \mathbf{F}(\mathbf{G}|_{\{0,1,\ldots n-1\}})$. (here $n-1$ denotes the predecessor of $n$)*

PROOF. (a) can be obtained by applying Theorem 4.12 to the statement $n \notin \mathbb{N} \vee \phi(n)$ and (b) follows from Theorem 4.13 by restricting the function $\mathbf{G}$ to $\mathbb{N}$. $\square$

REMARK. From now on we will use the principle of induction and recursion on $\mathbb{N}$ in the usual, less formal way: assuming we verified a statement $\phi(m)$ for all $m < n$, $n \in \mathbb{N}$, or defined an, for all $m < n$, $n \in \mathbb{N}$, we can show $\phi(n)$ respectively define $f(n)$. Then the statement $\phi(n)$ is true, or $f(n)$ is defined for all $n \in \mathbb{N}$, respectively.

We would like to continue and write the next ordinal to be $\omega = \sup_{n \text{ natural}} n$, $\omega + 1 = \omega \cup \{\omega\}$ etc. But we need another axiom which postulates the existence of infinite sets

- *Axiom of Infinity*

$$\text{(AI)} \qquad \exists B(\emptyset \in B \wedge (\forall z(z \in B \to z \cup \{z\} \in B).$$

COROLLARY 4.16. *The class of natural numbers is a set. In particular $\omega = \sup_{n \text{ natural}} n$ is an ordinal.*

PROOF. Let $B$ be the set given by (AI). By induction we can prove that for all $n \in \mathbb{N}$ it follows that $n \in B$. Thus, $\mathbb{N}$ is a subclass of a set $B$, thus, a set itself. $\square$

## 5. Arithmetic of ordinals

We will use the principle of transfinite recursion to define addition on ordinals.

THEOREM 5.1. *Let* $\alpha \in \mathbf{ON}$. *Then there is a function*

$$\alpha + \cdot : \mathbf{ON} \to \mathbf{ON}, \quad \beta \to \alpha + \beta,$$

*so that*

$$\alpha + 0 = \alpha \tag{20}$$

$$\alpha + (\beta + 1) = (\alpha + \beta) + 1 \tag{21}$$

*(i.e. the sum of* $\alpha$ *and the successor of* $\beta$ *is the successor of* $\alpha + \beta$*)*

$$\alpha + \sup_{\gamma<\beta} \gamma = \sup_{\gamma<\alpha+\beta} \gamma, \textit{ if } \beta \in \mathit{Lim}(\mathbf{ON}). \tag{22}$$

PROOF. Formally we define $\mathbf{F} : \mathbf{V} \to \mathbf{V}$, by

$\mathbf{F}(\emptyset) = \alpha$

$\mathbf{F}(g) = \min\{\beta : \beta > \gamma \text{ for all } \gamma \in \mathrm{ran}(g)\}$ if $g : \beta \to \mathbf{ON}$ is a function on an ordinal $\beta$

$\mathbf{F}(a) = \emptyset$, in any other case.

Let $\mathbf{G}$ be the function on $\mathbf{ON}$, which uniquely exists by Theorem 4.13 and put $\alpha + \beta = \mathbf{G}(\beta)$.

First we note that if $\beta, \beta' \in \mathbf{ON}$ with $\beta \leq \beta'$ then

$$\begin{aligned}\mathbf{G}(\beta) &= \mathbf{F}(\mathbf{G}|_\beta)\\ &= \min\{\eta : \forall \delta < \beta \quad \mathbf{G}(\delta) < \eta\} \quad \leq \min\{\eta : \forall \delta < \beta' \quad \mathbf{G}(\delta) < \eta\} = \mathbf{G}(\beta').\end{aligned}$$

We note that

$$\mathbf{G}(0) = \mathbf{G}(\emptyset) = \mathbf{F}(\mathbf{G}|_\emptyset) = \mathbf{F}(\emptyset) = \alpha,$$

and if $\beta \in \mathbf{ON}$ then

$$\mathbf{G}(\beta + 1) = \mathbf{F}(\mathbf{G}|_{\beta+1}) = \min\{\delta : \delta > \alpha + \gamma, \gamma \leq \beta\} = \mathbf{G}(\beta) + 1.$$

This verifies (20) and (21), and also proves that $\mathbf{G}$ is strictly increasing. Indeed if $\beta < \beta'$ then

$$\mathbf{G}(\beta) < \mathbf{G}(\beta) + 1 = \mathbf{G}(\beta + 1) \leq \mathbf{G}(\beta').$$

If $\beta \in \mathrm{Lim}(\mathbf{ON})$,

$$\begin{aligned}\mathbf{G}(\sup_{\delta<\beta} \delta) &= \mathbf{G}(\beta)\\ &= \mathbf{F}(\mathbf{G}|_\beta)\\ &= \min\{\delta : \delta > \mathbf{G}(\eta) \text{ for all } \eta < \beta\} = \sup_{\eta<\beta} \mathbf{G}(\eta).\end{aligned}$$

For the last equality we used the fact that $\mathbf{G}$ is strictly increasing, and, thus, that $\mathbf{G}(\beta)$ is a limit ordinal. □

REMARK. From now on we are using the Principle of Transfinite Recursion in a less formal way: We would like to define a function **F** on **ON** or on any subinterval of **ON**.

We do this by prescribing a procedure to define $F(\alpha)$, assuming we defined $F|_{[0,\alpha)}$, i.e. assuming we defined $F(\beta)$ for all $\beta < \alpha$

EXERCISE 5.2. $0 + \alpha = \alpha$

PROPOSITION 5.3. *Assume* $\alpha, \beta, \gamma \in \mathbf{ON}$.

a) $(\alpha + \beta) + \gamma = \alpha + (\beta + \gamma)$. *Thus, the expression* $\alpha + \beta + \gamma$ *is well defined.*
b) *If* $\alpha \leq \beta$ *then* $\alpha + \gamma \leq \beta + \gamma$.
c) *If* $\alpha < \beta$ *then* $\gamma + \alpha < \gamma + \beta$.

EXERCISE 5.4. Show that (b) in Proposition 5.3 does not in general hold for $<$ instead of $\leq$.

EXERCISE 5.5. if $\alpha + \beta = \alpha + \gamma$ then $\beta = \gamma$. but if $\beta + \alpha = \gamma + \alpha$ it follows not necessarily that $\beta = \gamma$.

THEOREM 5.6. *If* $\alpha \in \mathbf{ON}$ *and* $\beta \in Lim(\mathbf{ON})$ *then* $\alpha + \beta \in Lim(\mathbf{ON})$.

PROOF. First note that $\alpha + \beta \geq \beta > 0$ (using Proposition 5.3 and Exercise 5.5. Now assume that our claim is not true and that for some $\gamma \in \mathbf{ON}$ $\alpha + \beta = \gamma + 1$. Since $\beta \in \mathrm{Lim}(\mathbf{ON})$, we deduce that

$$\alpha + \beta = \sup_{\delta<\beta}(\alpha + \delta) = \bigcup_{\delta<\beta}(\alpha + \delta).$$

Thus, there is a $\delta < \beta$ so that $\gamma < \alpha + \delta$, which implies that $\gamma + 1 < \alpha + \delta + 1 = \alpha + (\delta + 1) < \alpha + \beta$, and is a contradiction.

□

EXERCISE 5.7. If $\alpha \leq \beta$ then there is a unique $\gamma$ so that $\alpha + \gamma = \beta$.

DEFINITION 5.8. Using transfinite recursion we define for $\alpha, \beta \in \mathbf{ON}$ the product $\alpha \cdot \beta$ as follows.

$$\alpha \cdot 0 = 0 \tag{23}$$

$$\alpha \cdot \beta = \alpha \cdot \gamma + \alpha, \text{ if } \beta = \gamma + 1 \in \mathrm{Succ}(\mathbf{ON}). \tag{24}$$

$$\alpha \cdot \beta = \sup_{\gamma<\beta} \alpha \cdot \gamma, \text{ if } \beta \in \mathrm{Lim}(\mathbf{ON}). \tag{25}$$

EXERCISE 5.9. Using the formal statement of the Principle of transfinite induction (Theorem 4.13) introduce the multiplication formally.

THEOREM 5.10. *If* $\alpha > 0$ *and* $0 \leq \beta < \gamma$ *then*

$$\alpha \cdot \beta < \alpha \cdot \gamma.$$

PROOF. We prove the claim by transfinite induction for all $\gamma > \beta$.

Assume our claim is true for all $\beta < \gamma' < \gamma$, for some $\gamma > \beta$.

If $\gamma = \gamma' + 1$, for some $\gamma' \geq \beta$ it follows from (24) and Proposition 5.3 (b)

$$\alpha\gamma = \alpha\gamma' + \alpha > \alpha\gamma' + 0 \geq \alpha\beta.$$

If $\gamma = \sup_{\gamma'<\gamma} \gamma'$, then $(\beta, \gamma) \neq \emptyset$ and by (25) and Proposition 5.3 (b)

$$\alpha\gamma = \sup_{\gamma'<\gamma} \alpha\gamma' \geq \alpha(\beta + 1) > \alpha\beta.$$

□

EXERCISE 5.11. a) If $\alpha > 0$ and $\beta > 0$ then $\alpha \cdot \beta > 0$.

b) $(\alpha \cdot \beta) \cdot \gamma = \alpha \cdot (\beta \cdot \gamma)$

c) If $\alpha \cdot \beta = \alpha \cdot \gamma$ then $\beta = \gamma$. But if $\beta \cdot \alpha = \gamma \cdot \alpha$ it follows not necessarily that $\beta = \gamma$.

DEFINITION 5.12. Powers of ordinals

For $\alpha \in \mathbf{ON}$ we define by transfinite induction for each $\beta$, $\alpha^\beta$ so that

$$\alpha^0 = 1 \tag{26}$$

$$\alpha^{\beta+1} = \alpha^\beta \cdot \alpha \text{ and} \tag{27}$$

$$\alpha^\beta = \sup_{\gamma<\beta} \alpha^\gamma \text{if } \beta \in \text{Lim}(\mathbf{ON}) \tag{28}$$

EXERCISE 5.13. Do some of the exercises in Suppes' book, until you feel comfortable

DEFINITION 5.14. On $\mathbf{ON}$ consider the equivalence following relation

$$\alpha \sim \beta :\iff \exists \quad F : \alpha \to \beta, \quad F \text{ is a bijection .}$$

By the property of $\mathbf{ON}$ every equivalence class $\{\beta \in \mathbf{ON} : \beta \sim \alpha\}$ has a minimum We call such an ordinal *cardinal number.*

We denote

$$\omega = \min\{\alpha \in \mathbf{ON} : \alpha \text{ is infinite}\} = \min\{\alpha : \alpha \not\sim n \text{ for all natural numbers} n\}.$$

(first infinite ordinal)

$$\omega_1 = \min\{\alpha : \alpha \notin \omega, \alpha \not\sim \omega\}.$$

(first uncountable ordinal)

EXERCISE 5.15. (Cantor's normal form for countable ordinals)

For every $\gamma < \omega_1$ there is an $k \in \mathbb{N}$, $n_1, n_2, \ldots n_k \in \omega$ $n_i > 0$ for $i = 1, 2 \ldots k$, and $0 \leq alpha_k < \alpha_{k-1} < \ldots \alpha_1$ in $\omega_1$, so that

$$\gamma = n_1\omega^{\alpha_1} + n_2\omega^{\alpha_2} + \ldots + n_k\omega^{\alpha_k}.$$

## 6. The Axiom of Choice, the Lemma of Zorn and the Hausdorff Maximal Principle

DEFINITION 6.1. Let $A$ be a set and $\mathcal{P}(A)$ its power set. A function $f : \mathcal{P}(A) \setminus \{\emptyset\} \to A$ is called a *choice function* if for all $B \subset A$, with $B \neq \emptyset$, $f(B) \in B$.

EXERCISE 6.2. Assume $(A, <)$ is a well ordering. Show that $A$ admits a choice function.

- *Axiom of Choice*

(AC) All sets admit a choice function.

Assuming (AC) we can show the converse of Exercise 6.2.

THEOREM 6.3. *(Numeration Theorem)*
*For any set A there is an ordinal $\alpha$ which admits a bijection $b : \alpha \to A$. In particular, every set can be well ordered.*

PROOF. By (AC) there is a choice function $F : \mathcal{P}(A) \to A$. For $\beta \in \mathbf{ON}$ and a set $B$ we define the following statement $\phi(\beta, B)$.

$$\begin{aligned} \phi(\beta, B) \iff (a) \quad & B \subset A \\ (b) \quad & \exists f : \beta \to B \quad \mathrm{ran}(f) = B \\ & \qquad f(\gamma) = F(A \setminus \mathrm{ran}(f|_\gamma)), \text{ whenever } \gamma < \beta. \end{aligned}$$

If $\phi(\beta, B)$ holds then $f : \beta \to B$ satisfying (b) is uniquely determined and injective. Moreover, if for some $B \subset A$ $\phi(\beta, B)$ and $\phi(\beta', B)$ holds, then $\beta' = \beta$. Indeed, assume $\beta \leq \beta'$, and let $f_\beta : \beta \to B$ and $f_{\beta'} : \beta' \to B$ satisfy (b) for $\beta$ and $\beta'$, respectively. Then we can deduce by transfinite induction that for all $\gamma \in \beta$ $f_\beta(\gamma) = f_{\beta'}(\gamma)$. Thus, $f_{\beta'}[0, \beta) = f_\beta[0, \beta) = B$, and, since $f_{\beta'}$ is injective it follows that $\beta' = \beta$. By the axiom of replacement schemes there is a set $C$ of ordinals so that

$$\beta \in C \leftrightarrow \exists B \subset A \quad \phi(\beta, B).$$

$C$ must be an interval. Secondly, $\sup(C) \in C$. Indeed define $f : [0, \sup(C)) = \bigcup_{\beta \in C} \beta \to A$ by transfinite induction. Assuming $f(\gamma)$ has been chosen for all $\gamma < \alpha$, with $\alpha < \sup(C)$, we can put $f(\alpha) = F(A \setminus \mathrm{ran}(f|_\alpha))$, once we observe that $A \setminus \mathrm{ran}(f|_\alpha) \neq \emptyset$. For our last claim let $\beta \in C$ so that $\alpha \in \beta$, then there is an $f_\beta : \beta \to B$, with $B \subset A$, satisfying (b). By transfinite induction we can show that $f_\beta(\gamma) = f(\gamma)$ for all $\gamma \in \alpha$, and thus $f([0, \alpha)) \subsetneq f_\beta([0, \alpha)) \subset A$.

We finally claim that $\mathrm{ran}(f) = A$. If this was not true we could extend $f$ onto $[0, \sup(C)] = [0, \sup(C) + 1)$ by putting $f(\sup(C)) = F(A \setminus f([0, \sup(C)))$. $\square$

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