The lattice of closed ideals in the Banach algebra of operators on a certain dual Banach space

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Abstract

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1 Introduction

The aim of this note is to classify the closed ideals in the Banach algebra $\mathcal{B}(F)$ of (bounded, linear) operators on the Banach space

$$F := \left( \bigoplus_{n \in \mathbb{N}} \ell_n^2 \right)_{\ell_1}.$$  

More precisely, we shall show that there are exactly four closed ideals in $\mathcal{B}(F)$, namely $\{0\}$, the compact operators $\mathcal{K}(F)$, the closure $\mathcal{F}_{\ell_1}(F)$ of the set of operators factoring through $\ell_1$, and $\mathcal{B}(F)$ itself.

The collection of Banach spaces $E$ for which a classification of the closed ideals in $\mathcal{B}(E)$ exists is very sparse. Indeed, the following list appears to be the complete list of such spaces.

(i) For a finite-dimensional Banach space $E$, $\mathcal{B}(E) \cong M_n$, where $n$ is the dimension of $E$, and so it is ancient folklore that $\mathcal{B}(E)$ is simple in this case.

(ii) In 1941 Calkin [2] classified all the ideals in $\mathcal{B}(\ell_2)$. In particular he proved that there are only three closed ideals in $\mathcal{B}(\ell_2)$, namely $\{0\}$, $\mathcal{K}(\ell_2)$, and $\mathcal{B}(\ell_2)$.

(iii) In 1960 Golberg, Markus, and Feldman [5] extended Calkin’s theorem to the other classical sequence spaces. More precisely, they showed that $\{0\}$, $\mathcal{K}(E)$, and $\mathcal{B}(E)$ are the only closed ideals in $\mathcal{B}(E)$ for each of the spaces $E = c_0$ and $E = \ell_p$, where $1 \leq p < \infty$.

(iv) Later in the 1960’ies Gramsch [6] and Luft [9] independently extended Calkin’s theorem in a different direction by classifying all the closed ideals in $\mathcal{B}(H)$ for each Hilbert space $H$ (not necessarily separable). In particular, they showed that these ideals are well-ordered by inclusion.
(v) In 2003 Laustsen, Loy, and Read [7] proved that, for the Banach space

$$E := \left( \bigoplus_{n \in \mathbb{N}} \ell_2^n \right)_{\ell_1},$$

there are exactly four closed ideals in $\mathcal{B}(E)$, namely $\{0\}$, the compact operators $\mathcal{K}(E)$, the closure $\mathcal{F}_{\ell_1}(E)$ of the set of operators factoring through $\ell_1$, and $\mathcal{B}(E)$ itself.

Note that the Banach space $F$ given by (1.1) is the dual of the Banach space $E$ given by (1.2), and so the result of this note can be seen as a ‘dualization’ of [7]. In fact, our strategy draws heavily on the methods introduced in [7].

2 The classification theorem

We begin this section by recalling various definitions and results from [7]. For simplicity we state the results only in the generality that is required for our present purposes, but emphasize that a number of them hold true in greater generality.

2.1 $\ell_1$-direct sums. Let $(E_n)$ be a sequence of Banach spaces. We denote by $\left( \bigoplus E_n \right)_{\ell_1}$ the $\ell_1$-direct sum of $E_1, E_2, \ldots$, that is, the collection of sequences $(x_n)$ such that $x_n \in E_n$ for each $n \in \mathbb{N}$ and

$$\left\| (x_n) \right\| := \sum_{n=1}^{\infty} \left\| x_n \right\| < \infty.$$

This is a Banach space for coordinatewise defined addition and scalar multiplication and norm given by (2.1).

Set $E := \left( \bigoplus E_n \right)_{\ell_1}$. For each $m \in \mathbb{N}$, we write $J_m^E$ for the canonical embedding of $E_m$ into $E$ and $Q_m^E$ for the canonical projection of $E$ onto $E_m$. Both $J_m^E$ and $Q_m^E$ are operators of norm one; in fact, the former is an isometry, and the latter is a quotient map. When no ambiguity may arise, we omit the superscript $E$ from the operators $J_m^E$ and $Q_m^E$.

We use similar notation and conventions for finite collections of Banach spaces and operators.

2.2 Definition. Let $(E_n)$ and $(F_n)$ be sequences of Banach spaces, and let $T: \left( \bigoplus E_n \right)_{\ell_1} \to \left( \bigoplus F_n \right)_{\ell_1}$ be an operator. We associate with $T$ the infinite matrix $(T_{m,n})$, where

$$T_{m,n} := Q_m^F T J_n^E: E_n \to F_m \quad (m, n \in \mathbb{N}).$$

The support of the $n^{th}$ column of $T$ is

$$\text{colsupp}_n(T) := \{ m \in \mathbb{N} \mid T_{m,n} \neq 0 \} \quad (n \in \mathbb{N}).$$

We say that $T$ has finite columns if each column has finite support, and in this case we set $\mu_n(T) := \max(\text{colsupp}_n(T))$. 

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The significance of operators with finite columns lies in the fact that, in the case where each of the spaces \( E_n \) \((n \in \mathbb{N})\) is finite-dimensional, for each operator \( T: (\bigoplus E_n)_{\ell_1} \to (\bigoplus F_n)_{\ell_1} \) we can find a perturbation \( \tilde{T}: (\bigoplus E_n)_{\ell_1} \to (\bigoplus F_n)_{\ell_1} \) with finite columns such that the difference \( T - \tilde{T} \) is compact and has arbitrarily small norm (see [7, Lemma 2.7(i)]).

### 2.3 Diagonal operators

Let \((E_n)\) and \((F_n)\) be sequences of Banach spaces, and, for each \( n \in \mathbb{N} \), let \( T_n : E_n \to F_n \) be an operator. Suppose that \( \sup \|T_n\| < \infty \). Then we can define the diagonal operator

\[
\text{diag}(T_n) : (x_n) \mapsto (T_n x_n), \quad \left( \bigoplus E_n \right)_{\ell_1} \to \left( \bigoplus F_n \right)_{\ell_1}.
\]

Clearly, we have \( \|\text{diag}(T_n)\| = \sup \|T_n\| \).

The following construction is a dual version of [7, Construction 4.2].

### 2.4 Construction

Let \((E_n)\) and \((F_n)\) be sequences of Banach spaces, and set \( E := (\bigoplus E_n)_{\ell_1} \) and \( F := (\bigoplus F_n)_{\ell_1} \). Further, set \( \tilde{F} := (\bigoplus \tilde{F}_n)_{\ell_1} \), where \( \tilde{F}_n := F \) for each \( n \in \mathbb{N} \). Let \( T : E \to F \) be an operator. Since \( \|TJ_n^E\| \leq \|T\| \) for each \( n \in \mathbb{N} \), we have a diagonal operator \( \text{diag}(TJ_n^E) : E \to \tilde{F} \). We claim that there is an operator \( W : \tilde{F} \to F \) such that

\[
T = W \text{diag}(TJ_n^E).
\]  

Indeed, suppose that \( y = (y_n) \in \tilde{F} \), so that \( y_n \in F \) for each \( n \in \mathbb{N} \) and \( \sum_{n=1}^{\infty} \|y_n\| < \infty \). Then, for each \( m \in \mathbb{N} \), the series \( \sum_{n=1}^{\infty} Q_m y_n \) is absolutely convergent in \( F_m \), and \( \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} Q_m y_n \| \leq \|y\| \). It follows that we can define an operator

\[
W : y \mapsto \left( \sum_{n=1}^{\infty} Q_m y_n \right)_{m \in \mathbb{N}}, \quad \tilde{F} \to F.
\]

We note that \( \|W\| < 1 \), and (2.2) is satisfied because \( Q_m^F W \text{diag}(TJ_n^E)J_k^E = Q_m^F T J_k^E \) for each \( k, m \in \mathbb{N} \).

A linear subspace \( G \) of a vector space \( E \) is termed cofinite if the quotient \( E/G \) is finite-dimensional. It is a standard elementary fact that the intersection of finitely many cofinite subspaces is again cofinite. More precisely, we have the following upper bound on the codimension.

### 2.5 Lemma

Let \( n \in \mathbb{N} \), and let \( G_1, \ldots, G_n \) be cofinite linear subspaces of a vector space \( E \). Then

\[
\dim \frac{E}{G_1 \cap G_2 \cap \cdots \cap G_n} \leq \sum_{j=1}^{n} \dim E/G_j.
\]
2.6 Definition. (i) Let $G$ be a closed subspace of a Hilbert space $H$. We denote by $G^\perp$ the orthogonal complement of $G$ in $H$, and write $\text{proj}_G$ for the orthogonal projection of $H$ onto $G$ (so that $\text{proj}_G$ is the idempotent operator on $H$ with image $G$ and kernel $G^\perp$).

(ii) Let $m \in \mathbb{N}$, let $E$ be a Banach space, and let $K_1, \ldots, K_m$ be Hilbert spaces. For each operator $T: E \to (K_1 \oplus \cdots \oplus K_m)_{\ell_1}$ and each $\varepsilon > 0$, set

$$n_\varepsilon(T) := \sup \left\{ n \in \mathbb{N}_0 \left| \| (\text{proj}_{G_1^+} \oplus \cdots \oplus \text{proj}_{G_m^+}) T \| > \varepsilon \right. \text{ whenever } G_j \text{ is a subspace of } K_j \text{ with } \dim G_j \leq n \right\} \in \mathbb{N}_0 \cup \{\pm \infty\}.$$

2.7 Lemma. Let $m \in \mathbb{N}$, let $H$ and $K_1, \ldots, K_m$ be Hilbert spaces, let $T: H \to (K_1 \oplus \cdots \oplus K_m)_{\ell_1}$ be an operator, and let $0 < \varepsilon < \|T\|$.

(i) Suppose that $n_\varepsilon(T)$ is finite. Then there are operators $R: H \to \ell_1$ and $S: \ell_1 \to (K_1 \oplus \cdots \oplus K_m)_{\ell_1}$ such that $\|T - SR\| \leq \varepsilon$, $\|R\| \leq \|T\| \sqrt{n_\varepsilon(T) + 1}$, and $\|S\| \leq 1$.

(ii) For each natural number $n \leq n_\varepsilon(T)/2 + 1$, there are operators $U: \ell_2^m \to H$ and $V: (K_1 \oplus \cdots \oplus K_m)_{\ell_1} \to \ell_2^n$ such that $I_{\ell_2^m} = VTU$, $\|U\| \leq 1/\varepsilon$, and $\|V\| \leq 1$.

(iii) Let $k \in \mathbb{N}$, let $H_0$ be a closed cofinite subspace of $H$, and suppose that $n_\varepsilon(T) \geq \dim H_0^\perp + k$. Then $n_\varepsilon(T|_{H_0}) \geq k$.

Proof. Parts (i) and (ii) are the dual versions of [7, Lemma 5.3(i)-(ii)]. Indeed, they follow by using [7, Lemma 5.3(i)-(ii)] together with the fact that, for any operators $T_1: (K_1 \oplus \cdots \oplus K_m)_{\ell_\infty} \to H$ and $T_2: H \to (K_1 \oplus \cdots \oplus K_m)_{\ell_1}$, we have $m_\varepsilon(T_1) = n_\varepsilon(T_2)$ and $n_\varepsilon(T_2) = m_\varepsilon(T_2^\perp)$, where $m_\varepsilon(\cdot)$ is defined as in [7, Definition 5.2(ii)], and where $T_1^\perp$ and $T_2^\perp$ are the mixed dual/adjoint operators given by

$$T_1^\perp := \sum_{j=1}^m J_j(T_j)^*: \quad (K_1 \oplus \cdots \oplus K_m)_{\ell_\infty} \to H$$

and

$$T_2^\perp := \sum_{j=1}^m (Q_jT_j^*)^*: \quad (K_1 \oplus \cdots \oplus K_m)_{\ell_\infty} \to H$$

(Here we write $(K_1 \oplus \cdots \oplus K_m)_{\ell_\infty}$ for the direct sum of $K_1, \ldots, K_m$ equipped with the $\ell_\infty$-norm $\|(x_1, \ldots, x_m)\| := \max\{\|x_1\|, \ldots, \|x_m\|\}$, and we use $^*$ to denote the adjoint of an operator between Hilbert spaces.)

(iii). For each $j = 1, \ldots, m$, let $G_j$ be a subspace of $K_j$ with $\dim G_j \leq k$. Set $F_j := G_j + Q_j T(H_0^\perp)$. Then $F_j$ is finite-dimensional with $\dim F_j \leq n_\varepsilon(T)$, and so we can find $x \in H$ such that $\|x\| \leq 1$ and $\| (\text{proj}_{G_1^+} \oplus \cdots \oplus \text{proj}_{G_m^+}) T x \| > \varepsilon$. It follows that

$$\| (\text{proj}_{G_1^+} \oplus \cdots \oplus \text{proj}_{G_m^+}) T|_{H_0} \| \geq \| (\text{proj}_{G_1^+} \oplus \cdots \oplus \text{proj}_{G_m^+}) T(\text{proj}_{H_0} x) \| \geq \| (\text{proj}_{F_1^+} \oplus \cdots \oplus \text{proj}_{F_m^+}) T(\text{proj}_{H_0} x) \| = \| (\text{proj}_{F_1^+} \oplus \cdots \oplus \text{proj}_{F_m^+}) T x \| > \varepsilon.$$
and so \( n_\varepsilon(T|_{H_0}) \geq k \).

2.8 Remark. Let \((K_n)\) be a sequence of Hilbert spaces, and let \(T\) be an operator on \((\bigoplus K_n)_{\ell_1}\) with finite columns. As in [7, Remark 5.4], there is a natural way to define \(n_\varepsilon(T|_{J_m})\) for each \(\varepsilon > 0\) and each \(m \in \mathbb{N}\), namely by ignoring the cofinite number of Hilbert spaces \(K_k\) such that \(Q_k T J_m = 0\).

For each pair \((E, F)\) of Banach spaces, set
\[
\mathcal{G}_{\ell_1}(E, F) := \{TS \mid S \in \mathcal{B}(E, \ell_1), T \in \mathcal{B}(\ell_1, F)\}.
\]
The fact that \(\ell_1 \cong \ell_1 \oplus \ell_1\) implies that \(\mathcal{G}_{\ell_1}\) is an operator ideal, and so its closure \(\overline{\mathcal{G}}_{\ell_1}\) is a closed operator ideal. As usual, we write \(\mathcal{G}_{\ell_1}(E)\) instead of \(\mathcal{G}_{\ell_1}(E, E)\).

2.9 Lemma. Let \(E\) be a Banach space and \(\mathcal{I}\) be an ideal in \(\mathcal{B}(E)\). If \(P\) is an idempotent operator on \(E\) and \(P \in \mathcal{I}\), then in fact \(P \in \mathcal{I}\).

Proof. Let \((T_n)\) be a sequence in \(\mathcal{I}\) converging to \(P\). Replacing \(T_n\) with \(PT_nP\) we may assume that \(T_n \in P\mathcal{B}(E)P\) for all \(n \in \mathbb{N}\). Note that \(P\mathcal{B}(E)P\) is a Banach algebra with unit \(P\), and so there exists \(n\) such that \(T_n\) is invertible. Thus there is an operator \(U \in \mathcal{B}(E)\) with \(P = (PUP)T_n\), which implies that \(P \in \mathcal{I}\). 

We can now state and prove our main theorem.

2.10 Theorem. Set \(F := \bigoplus \ell_2\). The lattice of closed ideals in \(\mathcal{B}(F)\) is given by
\[
\{0\} \subseteq \mathcal{K}(F) \subseteq \overline{\mathcal{G}}_{\ell_1}(F) \subseteq \mathcal{B}(F).
\]
Further, the following dichotomy holds for each operator \(T\) on \(F\) with finite columns:

(i) \(T \in \overline{\mathcal{G}}_{\ell_1}(F)\) if and only if \(\sup \{n_\varepsilon(T|_{J_k}) \mid k \in \mathbb{N}\} < \infty\) for each \(\varepsilon > 0\);

(ii) there are operators \(U\) and \(V\) on \(F\) such that \(VTU = I_F\) if and only if \(\sup \{n_\varepsilon(T|_{J_k}) \mid k \in \mathbb{N}\} = \infty\) for some \(\varepsilon > 0\).

Proof. We begin by proving the implications ‘\(\Leftarrow\)’ in (i) and (ii) for each operator \(T\) with finite columns.

(i), \(\Leftarrow\). Let \(0 < \varepsilon < \|T\|\), and suppose that \(c := \sup \{n_\varepsilon(T|_{J_k}) \mid k \in \mathbb{N}\} < \infty\). Then, for each \(k \in \mathbb{N}\), there are operators \(R_k: \ell_2^c \to \ell_1\) and \(S_k: \ell_1 \to F\) such that \(\|T J_k - S_k R_k\| \leq \varepsilon\), \(\|R_k\| \leq \|T\| \sqrt{c + 1}\), and \(\|S_k\| \leq 1\) by Lemma 2.7(i). In the notation of Construction 2.4 (with \(E_n = F_n = \ell_2^c\)), we see that the diagonal operators \(\text{diag}(R_k): F \to (\bigoplus \ell_1)_{\ell_1}\) and \(\text{diag}(S_k): (\bigoplus \ell_1)_{\ell_1} \to \tilde{F}\) satisfy
\[
\|\text{diag}(T|_{J_k}) - \text{diag}(S_r)\text{diag}(R_k)\| = \sup \|T J_k - S_k R_k\| \leq \varepsilon.
\]
It follows that \(\text{diag}(T|_{J_k}) \in \overline{\mathcal{G}}_{\ell_1}(F, \tilde{F})\) because \((\bigoplus \ell_1)_{\ell_1}\) is isomorphic to \(\ell_1\) and \(\varepsilon\) is arbitrary, and so by (2.2) we conclude that \(T \in \overline{\mathcal{G}}_{\ell_1}(F)\), as desired.
(ii), $\Leftrightarrow$. Suppose that $\sup \{u_\varepsilon(T^j k) \mid k \in \mathbb{N}\} = \infty$ for some $\varepsilon > 0$. We construct inductively a strictly increasing sequence $(k_j)$ of natural numbers such that, for each $j \in \mathbb{N}$, the following assertions hold:

(a) $\text{closupp}_{k_j}(T) \neq \emptyset$ and $\mu_{k_{j+1}}(T) \geq \mu_{k_j}(T)$.

(b) Set $m_0 := 0$, $m_j := \mu_{k_j}(T)$, and $E_j := (\bigoplus_{i=m_{j-1}+1}^{m_j} \ell^2_i)^1$, and let $P_j : F \to E_j$ be the canonical projection. Then there are operators $U_j : \ell^2_j \to \ell^2_j$ and $V_j : E_j \to \ell^2_j$ such that the diagram

\[
\begin{array}{ccc}
\ell^2_j & \xrightarrow{I} & \ell^2_j \\
\downarrow U_j & & \downarrow V_j \\
\ell^2_j & \xrightarrow{J_{k_j}} & F \\
& \xrightarrow{T} & F \\
& \xrightarrow{P_j} & E_j
\end{array}
\]

is commutative, $\|U_j\| \leq 1/\varepsilon$, $\|V_j\| \leq 1$, and $\text{im} U_j \subseteq \bigcap_{i=1}^{m_j-1} \ker T_{i,k_j}$. (The latter condition is ignored for $j = 1$.)

We start the induction by choosing $k_1 \in \mathbb{N}$ such that $n_\varepsilon(T^j k_1) \geq 1$. Then $\text{closupp}_{k_1}(T) \neq \emptyset$ and $\|T^j k_1\| > \varepsilon$. Take a unit vector $x \in \ell^2_1$ such that $\|T^j k_1 x\| > \varepsilon$, and define

\[U_1 : \alpha \mapsto \frac{\alpha x}{\|T^j k_1 x\|}, \quad \ell^2_1 \to \ell^2_1.\]

Further, take a functional $V_1 : E_1 \to \ell^2_1$ of norm 1 such that $V_1(P_1 T^j k_1 x) = \|P_1 T^j k_1 x\|$. Then the diagram in (b) is commutative because $\|P_1 T^j k_1 x\| = \|T^j k_1 x\|$.

Now let $j \geq 2$, and suppose that $k_1 < k_2 < \cdots < k_{j-1}$ have been chosen in accordance with (a)-(b). Set $h := \sum_{i=1}^{m_{j-1}} i$, take $k_j > k_{j-1}$ such that $n_\varepsilon(T^j k_j) \geq h + 2(j - 1)$, and set $H := \bigcap_{i=1}^{m_{j-1}} \ker T_{i,k_j}$. Lemma 2.5 shows that

\[\dim H^\perp = \dim \ell^2_2 / H \leq \sum_{i=1}^{m_{j-1}} \dim \ell^2_2 / \ker T_{i,k_j} = \sum_{i=1}^{m_{j-1}} \dim \text{im} T_{i,k_j} \leq h,\]

and hence $n_\varepsilon(T^j k_j |_H) \geq 2(j - 1)$ by Lemma 2.7(iii). In particular $T^j k_j |_H \neq 0$, and (a) is satisfied. Further, we note that $n_\varepsilon(P_1 T^j k_j |_H) = n_\varepsilon(T^j k_j |_H)$ because $Q_1 T^j k_j |_H = 0$ whenever $i \leq m_{j-1}$ or $i > m_j$. Lemma 2.7(ii) then implies that there are operators $U_j : \ell^2_2 \to H \subseteq \ell^2_k$ and $V_j : E_j \to \ell^2_2$ such that (b) is satisfied. This completes the inductive construction.

We ‘glue’ the operators $U_j$ ($j \in \mathbb{N}$) together in the following way to obtain an operator $U$ on $F$. Given $x \in F$, define $y_i \in \ell^2_i$ by

\[y_i := \begin{cases} U_j Q_j x & \text{if } i = k_j \text{ for some } j \in \mathbb{N} \\ 0 & \text{otherwise} \end{cases} \quad (i \in \mathbb{N}).\]
Then we have
\[ \sum_{i=1}^{\infty} \| y_i \| = \sum_{j=1}^{\infty} \| U_j Q_j x \| \leq \frac{\| x \|}{\varepsilon} < \infty, \]
and so \( Ux := (y_i) \) defines an operator \( U \) on \( F \). Similarly, since
\[ \sum_{j=1}^{\infty} \| V_j P_j x \| \leq \sum_{j=1}^{\infty} \| P_j x \| = \| x \|, \]
the assignment \( Vx := (V_j P_j x) \) defines an operator \( V \) on \( F \).

We \textit{claim} that \( VTU = I_F \). To this end, it suffices to check that
\[
Q_i VTU J_j x = \begin{cases} x & \text{if } i = j \\ 0 & \text{otherwise} \end{cases} \quad (i, j \in \mathbb{N}, x \in \ell_1^J). \tag{2.4}
\]

By definition, we have \( Q_i VTU J_j x = V_i P_i T J_{k_i} U_j x \). For \( i = j \), the diagram in (b), above, shows that this is \( x \). For \( i < j \), we have \( U_j x \in \ker T_{h, k_j} (1 \leq h \leq m_{j-1}) \), and so
\[ P_i T J_{k_i} U_j = \sum_{h=m_{i-1}+1}^{m_i} J_h Q_h T J_{k_i} U_j x = \sum_{h=m_{i-1}+1}^{m_i} J_h T_{h, k_j} U_j x = 0. \]

For \( i > j \), \( P_i T J_{k_i} = \sum_{h=m_{i-1}+1}^{m_i} J_h T_{h, k_j} = 0 \) because \( T_{h, k_j} = 0 \) whenever \( h \geq m_j \). This completes the proof of (2.4).

Next we establish (2.3). It is clear that \( \{0\} \subset \mathcal{H}(F) \subset \mathcal{G}_{\ell_1}(F) \) (\( F \) contains \( \ell_1 \) as a complemented subspace, the projection onto which is an example of a non-compact operator in \( \mathcal{G}_{\ell_1}(F) \)). To see that \( \mathcal{G}_{\ell_1}(F) \) is a proper ideal in \( \mathcal{B}(F) \), first note that by Lemma 2.9, if \( I_F \in \mathcal{G}_{\ell_1}(F) \), then \( I_F \in \mathcal{G}_{\ell_1}(F) \), and hence \( F \) would be isomorphic to \( \ell_1 \). It is known, however, that \( F \) is not isomorphic to \( \ell_1 \), although this is by no means obvious. One may for example use that \( \ell_1 \) has a unique unconditional basis up to equivalence (a fact that essentially relies on Khintchine’s inequality), whereas it is easy to see that \( F \) does not have this property.

We now show that the ideals in (2.3) are the only closed ideals of \( \mathcal{B}(F) \). Standard basis arguments show that the identity on \( \ell_1 \) factors through any non-compact operator in \( \mathcal{B}(F) \) (see for example [7, §3]). It follows that for each non-zero, closed ideal \( \mathcal{J} \) in \( \mathcal{B}(F) \), either \( \mathcal{J} = \mathcal{H}(F) \) or \( \mathcal{G}_{\ell_1}(F) \subset \mathcal{J} \).

Suppose that \( \mathcal{J} \) is a closed ideal in \( \mathcal{B}(F) \) properly containing \( \mathcal{G}_{\ell_1}(F) \). Take \( T \in \mathcal{J} \setminus \mathcal{G}_{\ell_1}(F) \), and take \( \tilde{T} \in \mathcal{B}(F) \) with finite columns such that \( T - \tilde{T} \) is compact (cf. [7, Lemma 2.7(i)])). Then \( \tilde{T} \) is also in \( \mathcal{J} \setminus \mathcal{G}_{\ell_1}(F) \). By (the contrapositive of) (i), \( \Rightarrow \), we conclude that \( \sup \{ n_\varepsilon (\tilde{T} J_k) \mid k \in \mathbb{N} \} = \infty \) for some \( \varepsilon > 0 \), and hence (ii), \( \Leftarrow \), implies that \( I_F = VTU \) for some operators \( U \) and \( V \) on \( F \). It follows that \( \mathcal{J} = \mathcal{B}(F) \), as required.

It remains to prove the implications ‘\( \Rightarrow \)’ in (i) and (ii) for each operator \( T \) with finite columns. This is done by contraposition.
(i), ⇒. Suppose that \( \sup \{ n_\vee(TJ_k) \mid k \in \mathbb{N} \} = \infty \) for some \( \vee > 0 \). Then, by (ii), \( \Leftarrow \), there are operators \( U \) and \( V \) on \( F \) such that \( I_F = VTV \), and so \( T \not\in \mathcal{F}_\ell_1(F) \) because \( \mathcal{F}_\ell_1(F) \) is a proper ideal in \( \mathcal{B}(F) \).

(ii), ⇒. This is similar. \( \square \)

In [1, §8] Bourgain, Casazza, Lindenstrauss, and Tzafriri prove that every infinite-dimensional, complemented subspace of the Banach space \( F := (\bigoplus \ell_2^n)_{\ell_1} \) is isomorphic to either \( F \) or \( \ell_1 \). Here we present a new proof of this fact using only the ideal structure of \( \mathcal{B}(F) \). More precisely, we shall deduce it from the dichotomy in Theorem 2.10 for operators in \( \mathcal{B}(F) \) with finite columns.

2.11 Remark. In [7, §6] a new proof is presented for the corresponding result of Bourgain, Casazza, Lindenstrauss, and Tzafriri for the space \( E := (\bigoplus \ell_2^n)_{c_0} \), which says that every infinite-dimensional, complemented subspace of \( E \) is isomorphic to either \( E \) or \( c_0 \). This new proof in [7] relies on a result of Casazza, Kottman and Lin [3] that implies that \( E \) is primary. The result of [3], however, does not show that \( F \) is primary, and so the argument in [7] cannot be used here. The proof we present below uses only the classification result, Theorem 2.10, and it also works for the space \( E \).

We start with an easy strengthening of part (ii) of Theorem 2.10.

2.12 Proposition. Let \( T \) be an operator on \( F \). If \( T \not\in \mathcal{F}_\ell_1(F) \) then there exist operators \( A \) and \( B \) on \( F \) such that \( I_F = ATB \).

Proof. Let \( K \) be a compact operator on \( F \) such that \( T - K \) has finite columns. Note that by the ideal property we have \( T - K \not\in \mathcal{F}_\ell_1(F) \). By Theorem 2.10 there are operators \( U \) and \( V \) on \( F \) such that \( I_F = U(T - K)V \). Thus \( UTV \) is a compact perturbation of the identity, and hence it is a Fredholm operator. It follows that for some \( W \in \mathcal{B}(F) \) the operator \( WUTV \) is a cofinite-rank projection. Since \( E \) is isomorphic to its finite-codimensional subspaces the result follows. \( \square \)

2.13 Theorem. (Bourgain, Casazza, Lindenstrauss, Tzafriri [1]) Every infinite-dimensional, complemented subspace of \( F = (\bigoplus \ell_2^n)_{\ell_1} \) is isomorphic to either \( F \) or \( \ell_1 \).

Proof. Let \( Y \) be an infinite-dimensional, complemented subspace of \( F \), and let \( P \in \mathcal{B}(F) \) be an idempotent operator with image \( Y \). If \( P \in \mathcal{F}_\ell_1(F) \), then by Lemma 2.9 we have \( P \in \mathcal{F}_\ell_1(F) \), and hence \( Y \) is isomorphic to \( \ell_1 \). If \( P \not\in \mathcal{F}_\ell_1(F) \), then by Proposition 2.12 the identity on \( F \) factors through \( P \), i.e., \( F \) is isomorphic to a complemented subspace of \( Y \). We can thus write \( F \sim Y \oplus V \) and \( Y \sim F \oplus W \) for suitable Banach spaces \( V \) and \( W \). We
now use Pełczynski’s decomposition method to show that $Y$ is isomorphic to $F$.

\[
F \sim Y \oplus V \\
\sim F \oplus W \oplus V \\
\sim (F \oplus F \oplus \ldots)_{\ell_1} \oplus W \oplus V \\
\sim (Y \oplus V \oplus Y \oplus V \oplus \ldots)_{\ell_1} \oplus W \oplus V \\
\sim (Y \oplus V \oplus Y \oplus V \oplus \ldots)_{\ell_1} \oplus W \\
\sim F \oplus W \sim Y,
\]

where we also used the fact that $F$ is isomorphic to $(F \oplus F \oplus \ldots)_{\ell_1}$. \qed

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## References


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