A UNIVERSAL REFLEXIVE SPACE FOR THE CLASS OF UNIFORMLY CONVEX BANACH SPACES

E. ODELL AND TH. SCHLUMPRECHT

Dedicated to the memory of V. I. Gurarii

Abstract. We show that there exists a separable reflexive Banach space into which every separable uniformly convex Banach space isomorphically embeds. This solves a problem of J. Bourgain. We also give intrinsic characterizations of separable reflexive Banach spaces which embed into a reflexive space with a block q-Hilbertian and/or a block p-Besselian finite dimensional decomposition.

1. Introduction

J. Bourgain [B] proved that if X is a separable Banach space which contains an isomorph of every separable reflexive space then X contains an isomorph of $C[0,1]$ and hence is universal, i.e., X contains an isomorph of every separable Banach space. He asked if there exists a separable reflexive space X which is universal for the class of all separable uniformly convex (equivalently, all superreflexive [E], [Pi]) Banach spaces. Such an X could not be superreflexive since c_0 and ℓ_1 are finitely representable in any space which contains isomorphs of all ℓ_p's for $1 < p < \infty$.

We shall answer Bourgain's question in the affirmative. S. Prus [P] gave a partial solution by proving that there exists a reflexive Banach space X which is universal for all spaces with a finite dimensional decomposition (FDD) which satisfy (p,q)-estimates for some $1 < q \le p < \infty$.

Definition 1.1. Let (F_n) be an FDD. (x_n) is a *block sequence* of (F_n) if there exist integers $0 = k_0 < k_1 < \cdots$ so that for all $n \in \mathbb{N}$,

$$x_n \in [F_i]_{i\in(k_{n-1},k_n]} \equiv \mathrm{span}\{F_i : k_{n-1} < i \le k_n\} \ .$$

Definition 1.2. Let $1 \le q \le p \le \infty$ and let $C < \infty$. An FDD (F_n) satisfies *C-(p,q)-estimates* if for all block sequences (x_n) of (F_n),

$$C^{-1}\left(\sum \|x_n\|^p\right)^{1/p} \le \|\sum x_n\| \le C\left(\sum \|x_n\|^q\right)^{1/q} \ .$$

We say that (F_n) *satisfies (p,q)-estimates* if it satisfies C-(p,q)-estimates for some $C < \infty$. A *basic sequence (x_n) is said to satisfy (p,q)-estimates* if (E_n) does where $E_n = \mathrm{span}\{x_n\}$ for $n \in \mathbb{N}$.

Terminology. In some of the literature an FDD satisfying $(p,1)$-estimates is called block p-Besselian and one satisfying (∞,q)-estimates is called block q-Hilbertian.

We shall prove that if X is uniformly convex then there exists $1 < q \le p < \infty$ and a space Z with an FDD satisfying (p,q)-estimates such that X embeds into Z. In combination with Prus' result we then obtain the solution to Bourgain's problem.

Research supported by the National Science Foundation.

Theorem 1.3. *There exists a separable reflexive Banach space X which contains an isomorph of every separable superreflexive Banach space.*

To accomplish this we shall characterize when a reflexive space embeds into a reflexive space with an FDD satisfying (p,q)-estimates. Before stating our results in this regard we need some more definitions.

Definition 1.4. If $\mathbf{E}=(E_n)$ is an FDD for a space X, by $P_n^{\mathbf{E}}$ we denote the natural projection of X onto E_n. More generally if I is an interval or finite union of intervals in $\mathbb{N}$, $P_I^{\mathbf{E}}$ shall denote the natural projection on X given by $P_I^{\mathbf{E}}(\sum e_n)=\sum_{n\in I} e_n$ (where $e_n\in E_n$ for all n). The *projection constant* of (E_n) is $\sup\{\|P_I^{\mathbf{E}}\| : I \text{ is an interval in } \mathbb{N}\}$. (E_n) is *bimonotone* if its projection constant is 1. A *blocking* (G_n) of (E_n) is an FDD given by $G_n=[E_i]_{i\in(N_{n-1},N_n]}$ for some sequence of integers $0=N_0<N_1<N_2<\cdots$.

Henceforth all Banach spaces will be assumed to be separable. S_X denotes the unit sphere of X and B_X denotes the unit ball of X.

Definition 1.5. a) $T_\infty\equiv\{(n_1,\dots,n_k) : k\in\mathbb{N} \text{ and } n_1<n_2<\cdots<n_k \text{ are natural numbers}\}$. T_∞ is ordered by $(n_1,\dots,n_k)\le(m_1,\dots,m_\ell)$ iff $k\le\ell$ and $n_i=m_i$ for $i\le k$.

b) A *tree in* a Banach space X is a family in X indexed by T_∞. A *weakly null tree* in X is a tree $(x_\alpha)_{\alpha\in T_\infty}\subseteq X$ with the property that for all $\alpha=(n_1,n_2,\dots n_k)\in T_\infty\cup\{\emptyset\}$, $(x_{(\alpha,n)})_{n>n_k}^\infty$ is weakly null. $(y_i)_{i=1}^\infty$ is a *branch* of $(x_\alpha)_{\alpha\in T_\infty}$ if there exist $n_1<n_2<\cdots$ so that $y_k=x_{(n_1,\dots,n_k)}$ for all $k\in\mathbb{N}$.

c) If $(x_\alpha)_{\alpha\in T_\infty}$ is a tree and if $T'\subset T_\infty$ is such that for each $\alpha\in T'\cup\{\emptyset\}$ there is an infinite $N_\alpha\subset\mathbb{N}$ so that $(\alpha,n)\in T'$ for all $n\in N_\alpha$ we call $(x_\alpha)_{\alpha\in T'}$ a *full subtree*. In this case we can relabel $(x_\alpha)_{\alpha\in T'}$ into $(z_\alpha)_{\alpha\in T_\infty}$. Note that every branch of a full subtree is a branch of the original tree.

Definition 1.6. Let $1\le q\le p\le\infty$ and $C<\infty$. A Banach space X *satisfies C-(p,q)-tree estimates* if for all weakly null trees in S_X there exists branches (y_i) and (z_i) satisfying

$$C^{-1}\Big(\sum|a_i|^p\Big)^{1/p}\le\|\sum a_iy_i\| \ \text{ and } \ \|\sum a_iz_i\|\le C\Big(\sum|a_i|^q\Big)^{1/q}$$

for all $(a_i)\subseteq\mathbb{R}$. X *satisfies (p,q)-tree estimates* if it satisfies C-(p,q)-tree estimates for some $C<\infty$.

Theorem 1.7. *Let X be a reflexive Banach space and let $1\le q\le p\le\infty$. The following are equivalent.*

a) *X satisfies (p,q)-tree estimates.*
b) *X is isomorphic to a subspace of a reflexive space Z having an* FDD *which satisfies (p,q)-estimates.*
c) *X is isomorphic to a quotient of a reflexive space Z having an* FDD *which satisfies (p,q)-estimates.*

Theorem 1.3 is a corollary of this (using Prus' result [P]) since for every uniformly convex space X there exists $K<\infty$ and $1<q\le p<\infty$ such that every normalized 2-basic sequence in X admits K-(p,q)-estimates ([J], [GG]). Indeed it is trivial to extract a 2-basic branch from a normalized weakly null tree. Theorem 1.7 also solves problem IV.3 in [Jo].

2. The Proof

The equivalence of a) and b) in Theorem 1.7 in the case $1 < q = p < \infty$ was established in [OS]. We shall be using some blocking arguments established there and in earlier seminal papers of W.B. Johnson and M. Zippin ([Jo], [JZ1, JZ2]) which we shall recall as needed. A key first step will of course be Zippin's result [Z] that a reflexive space embeds into a reflexive space with an FDD (in fact with a basis).

Before stating Theorem 2.1, which contains the central part of our main Theorem 1.7, we set some more notation. If (E_n) is an FDD then by $c_{00}(\oplus_{n=1}^\infty E_n)$ we mean the subspace of all $x = \sum e_n$ where $e_n \in E_n$ for all n and only finitely many e_n's are nonzero. If Z has an FDD, $\mathbf{F} = (F_n)$, and $1 < p < \infty$ then $Z_p(\mathbf{F})$ denotes the Banach space obtained by completing $c_{00}(\oplus_{n=1}^\infty F_n)$ under $\|\cdot\|_{Z_p}$ given by: for $y = \sum y_n$, $y_n \in F_n$ for all n,

$$\|y\|_{Z_p} = \sup\left\{ \left(\sum_{j=1}^\infty \Big\| \sum_{i=n_{j-1}+1}^{n_j} y_i \Big\|^p \right)^{1/p} : 0 = n_0 < n_1 < \cdots \right\} .$$

Note that (F_n) is a bimonotone FDD for $Z_p(\mathbf{F})$ satisfying 1-$(p,1)$-estimates.

Theorem 2.1. *Let X be a reflexive Banach space and let $1 < p < \infty$. If X satisfies $(p,1)$-tree estimates then*

a) *X can be embedded into a reflexive space Z with an* FDD *satisfying $(p,1)$-estimates.*
   More precisely, if X is a subspace of Z, a reflexive space with an FDD *(E_n) then there exists a blocking $\mathbf{F} = (F_n)$ of (E_n) so that X naturally embeds into the reflexive space $Z_p(\mathbf{F})$.*

b) *X is the quotient of a reflexive space with an* FDD *satisfying $(p,1)$-estimates.*

The proof of a) is much like the proof in [OS]. The proof of b) requires some new ideas. Before starting the proof we need some terminology and preliminary results.

Definition 2.2. Let $\mathbf{E} = (E_i)$ be an FDD for Y and let $\delta = (\delta_i)$ with $\delta_i \downarrow 0$. A sequence $(y_i) \subseteq S_Y$ is called a *δ-skipped block w.r.t.* (E_n) if there exist integers $1 = k_0 < k_1 < \cdots$ so that for all $i \in \mathbb{N}$,

$$\|P^{\mathbf{E}}_{(k_{i-1},k_i)} y_i - y_i\| < \delta_i .$$

Definition 2.3. If $\mathcal{A} \subseteq S_X^\omega$, the set of all normalized sequences in X, and $\varepsilon > 0$ we set

$$\mathcal{A}_\varepsilon = \{(x_n) \in S_X^\omega : \text{ there exists } (y_n) \in \mathcal{A} \text{ with } \|x_n - y_n\| < \frac{\varepsilon}{2^n} \text{ for all } n\} .$$

$\overline{\mathcal{A}_\varepsilon}$ denotes the closure of $\mathcal{A}_\varepsilon$ w.r.t. the product topology of the discrete topology on S_X.

The next result is Theorem 3.3 b) $\Leftrightarrow$ d) in [OS].

Proposition 2.4. *Let X be a Banach space with a separable dual. Then X is (isometrically) a subspace of a Banach space Z having a shrinking FDD (E_n) satisfying the following:*
For $\mathcal{A} \subseteq S_X^\omega$. the following are equivalent.

a) *For all $\varepsilon > 0$ every weakly null tree in S_X has a branch in $\overline{\mathcal{A}_\varepsilon}$.*

b) *For all $\varepsilon > 0$ there exists a blocking (F_i) of (E_i) and $\delta = (\delta_i)$, $\delta_i \downarrow 0$, so that if $(x_n) \subseteq S_X$ is a δ-skipped block w.r.t. (F_i) then $(x_n) \in \overline{\mathcal{A}_\varepsilon}$.*

The following Proposition yields that in the reflexive case the equivalence (a) $\Longleftrightarrow$ (b) in Proposition 2.4 holds for any embedding of X into a reflexive Banach space Z with an FDD.

Proposition 2.5. *Let Z and Y be reflexive spaces with FDDs $\mathbf{E} = (E_n)$ and $\mathbf{F} = (F_n)$, respectively, both containing a space X, and let $\delta = (\delta_n) \subset (0,1)$, with $\delta_n \downarrow 0$, as $n \uparrow \infty$.*

Let C be the maximum of the projection constants of (E_n) and (F_n). Then there is a blocking $\mathbf{G} = (G_n)$ of (F_n), so that every normalized $\frac{\delta}{5C^3}$-skipped block of (G_n) in S_X is a δ-skipped block of (E_n).

Proof. By induction we will choose $0 = M_0 < M_1 < M_2 < \dots$ and $N_1 < N_2 < \dots$ in $\mathbb{N}$ so that for all $k \in \mathbb{N}$

$$(1) \quad \forall x \in S_X \,\forall i \in \{1,2\dots,k,k+1\} \text{ if } \|P^{\mathbf{F}}_{(M_{k-1},\infty)}(x)\| \le \frac{\delta_i}{5C^2} \text{ then } \|P^{\mathbf{E}}_{[N_k,\infty)}(x)\| \le \frac{\delta_i}{2},$$

$$(2) \quad \forall x \in S_X \,\forall i \in \{1,2\dots k,k+1\} \text{ if } \|P^{\mathbf{F}}_{[1,M_k]}(x)\| \le \frac{\delta_i}{5C^2} \text{ then } \|P^{\mathbf{E}}_{[1,N_k]}(x)\| \le \frac{\delta_i}{2}$$

Once accomplished we choose $G_k = \oplus_{i=M_{k-1}+1}^{M_k} F_i$. If (x_n) is a $\delta/5C^3$-skipped block of (G_n) in S_X, there exist $0 = k_0 < k_1 < k_2 < \dots$ such that for all $n \in \mathbb{N}$

$$\|x_n - P^{\mathbf{G}}_{(k_{n-1},k_n)}(x_n)\| = \|x_n - P^{\mathbf{F}}_{(M_{k_{n-1}},M_{k_n})}(x_n)\| \le \frac{\delta_n}{5C^3}.$$

Thus,

$$\|P^{\mathbf{F}}_{[1,M_{k_{n-1}}]}(x_n)\| \le \frac{\delta_n}{5C^2} \text{ and } \|P^{\mathbf{F}}_{(M_{k_n-1},\infty)}(x_n)\| \le \frac{\delta_n}{5C^2}.$$

We deduce from (2) and (1) that

$$\|P^{\mathbf{E}}_{[1,N_{k_{n-1}}]}(x_n)\| \le \frac{\delta_n}{2} \text{ and } \|P^{\mathbf{E}}_{[N_{k_n},\infty)}(x_n)\| \le \frac{\delta_n}{2},$$

which yields that (x_n) is a δ-skipped block of (E_i).

Assume that we have chosen M_{k-1} for some $k \ge 1$. We need to find an N_k which satisfies (1). If such an N_k did not exist, we could find sequences $(x_j) \subset S_X$ and $(i_j) \subset \{1,2\dots k{+}1\}$ so that for any $j > N_{k-1}$,

$$\|P^{\mathbf{F}}_{(M_{k-1},\infty)}(x_j)\| \le \frac{\delta_{i_j}}{5C^2} \text{ and } \|P^{\mathbf{E}}_{[j,\infty)}(x_j)\| > \frac{\delta_{i_j}}{2}.$$

Passing to a subsequence we may assume that $i_j = i$ for all $j \in J$ and some $i \in \{1,2\dots k{+}1\}$, where J is a subsequence of $\mathbb{N}$. Since $\lim_{j\to\infty, j\in J} \|P^{\mathbf{E}}_{[j,\infty)} \circ P^{\mathbf{F}}_{(M_{k-1},\infty)} - P^{\mathbf{E}}_{[j,\infty)}\| = 0$ we deduce that

$$\begin{aligned}\frac{\delta_i}{2} &\le \limsup_{\substack{j\to\infty\\ j\in J}} \|P^{\mathbf{E}}_{[j,\infty)}(x_j)\| \le \limsup_{\substack{j\to\infty\\ j\in J}} \|P^{\mathbf{E}}_{[j,\infty)} \circ P^{\mathbf{F}}_{(M_{k-1},\infty)}(x_j)\| \\ &\le C \limsup_{\substack{j\to\infty\\ j\in J}} \|P^{\mathbf{F}}_{(M_{k-1},\infty)}(x_j)\| \le \frac{\delta_i}{5C^2},\end{aligned}$$

which is a contradiction, and finishes the proof of our claim.

Assume now that we have chosen N_k, but there is no M_k satisfying (2). We could choose a sequence $(x_j) \subset S_X$ and $(i_j) \subset \{1,2\dots k+1\}$ so that for any $j > M_{k-1}$

$$\|P^{\mathbf{F}}_{[1,j]}(x_j)\| \le \frac{\delta_{i_j}}{5C^2} \text{ and } \|P^{\mathbf{E}}_{[1,N_k]}(x_j)\| > \frac{\delta_{i_j}}{2}.$$

After passing to subsequences we can assume that $i_j = i$ for some fixed $i \in \{1, 2, \dots k+1\}$ and $j \in J$, a subsequence of $\mathbb{N}$, and that $(x_j)_{j\in J}$ converges weakly to some $x \in B_X$. Then it follows that

$$\begin{aligned}\|x\| &= \lim_{j_0\to\infty} \|P^{\mathbf{F}}_{[1,j_0]}(x)\| = \lim_{j_0\to\infty} \lim_{\substack{j\to\infty\\ j\in J}} \|P^{\mathbf{F}}_{[1,j_0]}(x_j)\| \\ &\le \limsup_{j_0\to\infty} \limsup_{\substack{j\to\infty\\ j\in J}} \|P^{\mathbf{F}}_{[1,j]}(x_j)\| + \|P^{\mathbf{F}}_{(j_0,j]}(x_j)\| \\ &\le (1+C) \limsup_{\substack{j\to\infty\\ j\in J}} \|P^{\mathbf{F}}_{[1,j]}(x_j)\| \le \frac{2\delta_i}{5C}\end{aligned}$$

and that

$$\|x\| \ge \frac{1}{C}\|P^{\mathbf{E}}_{[1,N_k]}(x)\| = \frac{1}{C} \lim_{\substack{j\to\infty\\ j\in J}} \|P^{\mathbf{E}}_{[1,N_k]}(x_j)\| \ge \frac{\delta_i}{2C},$$

which is a contradiction. □

From Corollary 4.4 in [OS] we have

Proposition 2.6. *Let X be a Banach space which is a subspace of a reflexive space Z with an* FDD $\mathbf{E} = (E_i)$ *having projection constant K. Let $\delta_i \downarrow 0$. Then there is a blocking $\mathbf{F} = (F_i)$ of (E_i) given by $F_n = [E_i]_{i\in(N_{n-1},N_n]}$ for some integers $0 = N_0 < N_1 < \cdots$ with the following property. For all $x \in S_X$ there exists $(x_i) \subseteq X$ and integers (t_i) with $t_i \in (N_{i-1}, N_i]$ for all i such that*

a) $x = \sum_{i=1}^{\infty} x_i$
b) *For $i \in \mathbb{N}$ either $\|x_i\| < \delta_i$ or $\|P^{\mathbf{E}}_{(t_{i-1},t_i)} x_i - x_i\| < \delta_i \|x_i\|$.*
c) $\|P^{\mathbf{E}}_{(t_{i-1},t_i)} x - x_i\| < \delta_i$ *for all $i \in \mathbb{N}$.*
d) $\|x_i\| < K + 1$ *for $i \in \mathbb{N}$.*
e) $\|P^{\mathbf{E}}_{t_i} x\| < \delta_i$ *for $i \in \mathbb{N}$.*

Moreover the above hold for any further blocking of (F_n) (which would redefine the N_i's).

Parts d) and e) were not explicitly stated in [OS] but follow from the proof.

Proof of Theorem 2.1 *a).* Let X be contained in a reflexive space Z with an FDD $\mathbf{E} = (E_i)$ having projection constant K. Assume that X satisfies C-$(p,1)$-tree estimates. Let $\mathcal{A} = \{(x_i) \in S_X^\omega\colon (x_i) \text{ is } \frac{3}{2}\text{-basic and for all scalars } (a_i),\ C\|\sum a_i x_i\| \ge (\sum |a_i|^p)^{1/p}\}$. Choose $\varepsilon > 0$ so that if $(x_i) \in \overline{\mathcal{A}_\varepsilon}$ then (x_i) is 2-basic and satisfies for all $(a_i) \subseteq \mathbb{R}$,

$$2C\|\sum a_i x_i\| \ge \left(\sum |a_i|^p\right)^{1/p} .$$

By Propositions 2.4 and 2.5 there exists $\delta = (\delta_i)$, $\delta_i \downarrow 0$, and a blocking of (E_i), which we still denote by (E_i), so that every δ-skipped block w.r.t. (E_i) is in $\overline{\mathcal{A}}_\varepsilon$. We then use $\bar\delta = (\bar\delta_i)$ where $\bar\delta_i = \delta_i/2K$ to form a new blocking $F_n = [E_i]_{i\in(N_{n-1},N_n]}$ satisfying the conclusion of Proposition 2.6. We assume, as we may, that $\sum_{i=1}^{\infty} \delta_i < 1$. Note that any subsequence of a $\bar\delta$-skipped block w.r.t. (F_i) is then a δ-skipped block w.r.t. (E_i).

Our goal is to prove that X naturally embeds into $Z_p(\mathbf{F})$. To achieve this we prove that if $x \in S_X$ then, for some absolute constant $A = A(K, C)$, $(\sum \|P^{\mathbf{F}}_n x\|^p)^{1/p} \le A$. Since the argument we will give would also work for any blocking of (F_n) (see the "moreover" part of Proposition 2.6) we obtain $\|x\|_{Z_p} \le A$ which finishes the proof of the claim.

Let $x \in S_X$ and write $x = \sum x_i$ with $(x_i) \subseteq X$ and $t_i \in (N_{i-1}, N_i]$ as in Proposition 2.6. Let $y_i = P^{\mathbf{E}}_{(t_{i-1},t_i]} x$ for $i \in \mathbb{N}$. Let $B = \{i \ge 2 : \|x_i\| \ge \bar\delta_i$ and $\|P^{\mathbf{E}}_{(t_{i-1},t_i)} x_i - x_i\| < \bar\delta_i \|x_i\|\}$. Since $(x_i/\|x_i\|)_{i\in B}$ is a δ-skipped block w.r.t. (E_i) we know it is in $\overline{\mathcal{A}_\varepsilon}$ and so $2C\|\sum_{i\in B} x_i\| \ge (\sum_{i\in B}\|x_i\|^p)^{1/p}$. Also for all $i \in \mathbb{N}$,

$$\|y_i\| = \|P^{\mathbf{E}}_{(t_{i-1},t_i)} x - x_i + P^{\mathbf{E}}_{t_i} x + x_i\| \le 2\bar\delta_i + \|x_i\|$$

by c) and e) of Proposition 2.6. Thus $\|y_i\| < K+2$ and

$$\begin{aligned}\sum \|y_i\|^p &= \sum_{i\in B}\|y_i\|^p + \sum_{i\notin B}\|y_i\|^p \le \sum_{i\in B}(2\bar\delta_i + \|x_i\|)^p + \|y_1\|^p + \sum_{i\notin B}(3\bar\delta_i)^p \\ &< \sum_{i\in B} 3^p\|x_i\|^p + (K+2)^p + 1 \le 3^p(2C)^p\|\sum_{i\in B} x_i\|^p + (K+2)^p + 1\ .\end{aligned}$$

Now

$$\|\sum_{i\in B} x_i\| \le 1 + \sum_{i\notin B}\|x_i\| < 1 + \|x_1\| + \sum_i \bar\delta_i < K+3\ .$$

Thus

$$\sum \|y_i\|^p \le (6C)^p(K+3)^p + (K+2)^p + 1 \equiv A'\ .$$

Let $z_i = P^{\mathbf{E}}_{(N_{i-1},N_i]} x = P^{\mathbf{F}}_i x = P^{\mathbf{F}}_i (y_i + y_{i+1})$ for $i \in \mathbb{N}$. Thus $\|z_i\| \le K(\|y_i\| + \|y_{i+1}\|)$ and so

$$\Big(\sum \|z_i\|^p\Big)^{1/p} \le 2K[A']^{1/p} \equiv A\ .$$

To complete the proof of part a) we have the following easy

Lemma 2.7. *Let $\mathbf{F} = (F_i)$ be a shrinking FDD for a Banach space Z. Then for $1 < p < \infty$, $Z_p(\mathbf{F})$ is reflexive.*

Proof. As noted earlier, (F_i) is a bimonotone FDD for $Z_p(\mathbf{F})$ which satisfies 1-$(p,1)$-estimates and hence (F_i) is boundedly complete. Let $\frac{1}{p} + \frac{1}{p'} = 1$ and set

$$\mathcal{F} = \left\{ \sum a_i f_i : \begin{array}{c} (a_i) \in B_{\ell_{p'}} \text{ and } (f_i) \text{ is a (finite or} \\ \text{infinite) block sequence of } (F_n^*) \text{ in } S_{Z^*} \end{array} \right\}.$$

If the above sum $\sum a_i f_i$ is a finite one, say $\sum_{i=1}^n a_i f_i$, then f_n can be supported on $[F_m^*]_{m\in[j,\infty)}$ for some j. It is easy to check that $\mathcal{F}$ is a weak* compact 1-norming subset of $Z_p(\mathbf{F})^*$. Thus X is isometrically a subspace of $C(\mathcal{F})$, the space of continuous function on $\mathcal{F}$. Since every $\|\cdot\|_{Z_p}$-normalized block (z_i) of (F_i) is pointwise null on $\mathcal{F}$, hence weakly null, it follows that (F_i) is a shrinking FDD for $Z_p(\mathbf{F})$. □

To prove part b) we need a blocking result due to Johnson and Zippin.

Proposition 2.8. [JZ1] *Let $T : Z \to W$ be a bounded linear operator from a space Z with a shrinking FDD (G_n) into a space W with an FDD (H_n). Let $\varepsilon_i \downarrow 0$. Then there exist blockings $\mathbf{E} = (E_n)$ of (G_n) and $\mathbf{F} = (F_n)$ of (H_n) so that: for all $i \le j$ and $z \in S_{[E_n]_{n\in(i,j]}}$ we have $\|P^{\mathbf{F}}_{[1,i)} Tz\| < \varepsilon_i$ and $\|P^{\mathbf{F}}_{(j,\infty)} Tz\| < \varepsilon_j$.*

Proof of Theorem 2.1 *b).* By Lemma 3.1 in [OS] we can, by renorming, regard $X^* \subseteq Z^*$ where Z^* is a reflexive space with a bimonotone FDD (E_i^*) such that $c_{00}(\oplus_{i=1}^\infty E_i^*) \cap X^*$ is dense in X^*. Thus we have a quotient map $Q : Z \to X$. By part a) we may regard $X \subseteq W$, a reflexive space with an FDD (F_i) satisfying C-$(p,1)$-estimates for some C. Let K be the projection constant of (F_i).

Choose $\delta = (\delta_i)$, $\delta_i \downarrow 0$, so that if (y_i) is any δ-skipped block of any blocking of (F_i) and (z_i) satisfies $\|z_i - y_i\| < 3K\delta_i$ for all i, then (z_i) is 2-equivalent to (y_i), is $2K$-basic and (y_i) satisfies $2C$-$(p,1)$-estimates.

In addition we require that

$$\sum_{i=1}^{\infty} 3(K+1)K\delta_i < \frac{1}{4}\ , \text{ for } i \in \mathbb{N}, \text{ and } 6/(1-2K\delta_1) < 7 \tag{3}$$

and we choose $\varepsilon_i \downarrow 0$ with $6\varepsilon_i < \delta_i$ for all i.

By Proposition 2.8, blocking and relabeling our FDD's we may assume

$$\text{For all } i \le j \text{ and } z \in S_{[E_n]_{n\in(i,j]}} \text{ we have} \|P^{\mathbf{F}}_{[1,i)} Qz\| < \varepsilon_i \text{ and } \|P^{\mathbf{F}}_{(j,\infty)} Qz\| < \varepsilon_j\ . \tag{4}$$

For $i \in \mathbb{N}$ let $\widetilde{E}_i$ be the quotient space of E_i determined by Q. Thus if $z \in E_i$, the norm on $\tilde{z}$ (the equivalence class of z in E_i) is $|\!|\!|\tilde{z}|\!|\!| = \|Qz\|$. We may assume $\widetilde{E}_i \neq \{0\}$ for all i. More generally for $\tilde{z} = \sum \tilde{z}_i \in c_{00}(\oplus_{i=1}^\infty \widetilde{E}_i)$ with $\tilde{z}_i \in \widetilde{E}_i$ for every i, we set

$$|\!|\!|\tilde{z}|\!|\!| = \sup_{m\le n} \Big\| \sum_{i=m}^{n} Qz_i \| = \sup_{m\le n} \|QP^{\mathbf{E}}_{[m,n]} z\|\ .$$

We let $\widetilde{Z}$ be the completion of $(c_{00}(\oplus_{i=1}^\infty \widetilde{E}_i), |\!|\!|\cdot|\!|\!|)$. Note that if $\tilde{z} = \sum \tilde{z}_i \in c_{00}(\oplus_{i=1}^\infty \widetilde{E}_i)$ then setting $\widetilde{Q}\tilde{z} \equiv \sum \widetilde{Q}\tilde{z}_i \equiv \sum Qz_i$, we have $\|\widetilde{Q}\tilde{z}\| \le |\!|\!|\tilde{z}|\!|\!|$. Thus $\widetilde{Q}$ extends to a norm one map from $\widetilde{Z}$ into X. Before continuing the proof of b) we need

Proposition 2.9.

a) *$(\tilde{E}_i)$ is a bimonotone shrinking* FDD *for $\widetilde{Z}$.*
b) *$\widetilde{Q}$ is a quotient map from $\widetilde{Z}$ onto X. More precisely if $x \in X$ and $z \in Z$ with $Qz = x$, $\|z\| = \|x\|$, and $z = \sum z_i$ with $z_i \in E_i$, then $\tilde{z} = \sum \tilde{z}_i \in \widetilde{Z}$, $|\!|\!|\tilde{z}|\!|\!| = \|z\|$ and $\widetilde{Q}\tilde{z} = x$.*
c) *Let $(\tilde{z}_i)$ be a block sequence of $(\tilde{E}_i)$ in $B_{\widetilde{Z}}$ and assume that $(\widetilde{Q}\tilde{z}_i)$ is a basic sequence with projection constant $\overline{K}$ and $a = \inf_i \|\widetilde{Q}\tilde{z}_i\| > 0$. Then for all scalars (a_i) we have*

$$\|\sum a_i \widetilde{Q}(\tilde{z}_i)\| \le |\!|\!|\sum a_i \tilde{z}_i|\!|\!| \le \frac{3\overline{K}}{a} \|\sum a_i \widetilde{Q}\tilde{z}_i\|\ .$$

Proof. By definition $\tilde{\mathbf{E}} = (\tilde{E}_i)$ is a bimonotone FDD for $\widetilde{Z}$. We will deduce later that it is shrinking. To see b) let $Qz = x$ with $\|z\| = \|x\|$ and write $z = \sum z_i$ with $z_i \in E_i$ for all i. Then for $i \le j$,

$$|\!|\!|\sum_{\ell=i}^{j} \tilde{z}_\ell|\!|\!| = \sup_{i\le n\le m\le j} \|Q(\sum_{\ell=n}^{m} z_\ell)\| \le \sup_{i\le n\le m\le j} \|\sum_{\ell=n}^{m} z_\ell\| = \|\sum_{\ell=i}^{j} z_\ell\|\ .$$

Thus $\sum \tilde{z}_i$ converges in $\widetilde{Z}$ to some $\tilde{z}$ with, $|\!|\!|\tilde{z}|\!|\!| = \|z\|$ and $\widetilde{Q}\tilde{z} = x$.

Next let $(\tilde z_i)$ be as in the statement of c). Since $\|\widetilde Q\| = 1$ we need only prove the right hand inequality in c). Let $(a_i) \in c_{00}$ and choose $k \le m$ so that

$$|\!|\!| \sum a_i \tilde z_i |\!|\!| = \|\widetilde Q P^{\widetilde{\mathbf{E}}}_{[k,m]}(\sum a_i \tilde z_i)\|$$

For all i, $P^{\widetilde{\mathbf{E}}}_{[k,m]}\tilde z_i$ is $\tilde z_i$ or 0 except possibly for 2 values of i, denoted by $i_0 \le i_1$. Thus

$$\|\widetilde Q P^{\widetilde{\mathbf{E}}}_{[k,m]}(\sum a_i \tilde z_i)\| \le |a_{i_0}|\, |\!|\!|\tilde z_{i_0}|\!|\!| + \|\sum_{i\in(i_0,i_1)} a_i \widetilde Q(\tilde z_i)\| + |a_{i_1}|\, |\!|\!|\tilde z_{i_1}|\!|\!| \le \frac{3\overline{K}}{a}\|\sum a_i \widetilde Q(\tilde z_i)\| \ ,$$

using that $(\widetilde Q(\tilde z_i))$ has projection constant $\overline{K}$ and is bounded below in norm by a.

It remains only to prove that $(\widetilde E_i)$ is shrinking. Let $(\tilde z_i)$ be a normalized block sequence of $(\widetilde E_i)$ in $\widetilde Z$. Then $(\widetilde Q \tilde z_i)$ is a bounded sequence in X. Moreover since $c_{00}(\oplus_{i=1}^\infty E_i^*) \cap X^*$ is dense in X^*, $(\widetilde Q\tilde z_i)$ is pointwise null on X^* and hence weakly null in X. We pass to a subsequence which we relabel as $(\widetilde Q\tilde z_i)$ which is either norm null or satisfies $\inf_i \|\widetilde Q z_i\| > 0$ and is basic. In the latter case $(\tilde z_i)$ is weakly null by part c). In the former case, given n we can find a subsequence $(\tilde z_{i_j})_{j=1}^n$ with $\|\widetilde Q \tilde z_{i_j}\| < \frac1n$ if $1 \le j \le n$. Then if $\tilde z = \frac1n \sum_{j=1}^n \tilde z_{i_j}$, for some $k \le m$ and $j_0 \le j_1$

$$|\!|\!|\tilde z|\!|\!| = \|\widetilde Q P^{\widetilde{\mathbf{E}}}_{[k,m]}\tilde z\| \le \Big\|\frac1n \widetilde Q P^{\widetilde{\mathbf{E}}}_{[k,m]}\tilde z_{i_{j_0}}\Big\| + \Big\|\frac1n \sum_{j\in(j_0,j_1)} \widetilde Q\tilde z_{i_j}\Big\| + \Big\|\frac1n \widetilde Q P^{\widetilde{\mathbf{E}}}_{[k,m]}\tilde z_{i_{j_1}}\Big\| < \frac3n \ .$$

Thus in any case every normalized block sequence $(\tilde z_i)$ admits a convex block basis which is norm null and hence $(\tilde z_i)$ is weakly null and so $(\widetilde E_i)$ is shrinking. □

We shall produce $A < \infty$ and a blocking $\widetilde{\mathbf{H}} = (\widetilde H_n)$ of $(\widetilde E_n)$ with the following property. Let $x \in S_X$. Then there exists $\tilde z = \sum \tilde z_n \in \widetilde Z$, with $\tilde z_n \in \widetilde H_n$ for all n, so that if $(\tilde w_n)$ is any blocking of $(\tilde z_n)$ then $(\sum |\!|\!|\tilde w_n|\!|\!|^p)^{1/p} \le A$. Moreover $\|\widetilde Q\tilde z - x\| < \frac12$. Thus if $\widetilde Z_p = \widetilde Z_p(\widetilde{\mathbf{H}})$ then $\widetilde Q : \widetilde Z_p \to X$ remains an onto map. Moreover $\widetilde Z_p$ is reflexive by Proposition 2.9 and Lemma 2.7 and $(\widetilde H_n)$ is an FDD for $\widetilde Z_p$ satisfying 1-$(p,1)$-estimates. To accomplish this we need

Lemma 2.10. *Assume that* (4) *holds for our original map $Q : Z \to X$. Then there exist integers $0 = N_0 < N_1 < \cdots$ so that if we define blockings $C_n = [E_i]_{i\in(N_{n-1},N_n]}$ and $D_n = [F_i]_{i\in(N_{n-1},N_n]}$ we have the following. Set for $n \in \mathbb{N}$,*

$$R_n = \left\{ i \in \mathbb{N} : N_n \ge i > \frac{N_n + N_{n-1}}{2} \right\} \ ,$$
$$L_n = \left\{ i \in \mathbb{N} : N_{n-1} < i \le \frac{N_n + N_{n-1}}{2} \right\} \ ,$$
$$C_{n,R} = [E_i]_{i\in R_n} \quad \textit{and} \quad C_{n,L} = [E_i]_{i\in L_n} \ .$$

Let $x \in S_X$, $i < j$, $\varepsilon > 0$ and assume that $\|P^{\mathbf{D}}_{[1,i]\cup[j,\infty)}x\| < \varepsilon$. Then there exists $z \in B_Z$ with $z \in [C_{i,R} \cup (C_\ell)_{\ell\in(i,j)} \cup C_{j,L}]$ and $\|Qz - x\| < K[2\varepsilon + \delta_i]$.

Proof. By [Jo] (see Lemma 4.3a [OS]) we can choose $0 = N_0 < N_1 < \cdots$ so that if $z \in B_Z$ with $z = \sum z_j$, $z_j \in E_j$ for all j, then for $n \in \mathbb{N}$ there exist $r_n \in R_n$ and $\ell_n \in L_n$ with $\|z_{r_n}\| < \varepsilon_n$ and $\|z_{\ell_n}\| < \varepsilon_n$. Define C_n and D_n as in the statement of the lemma and let $x \in S_X$ and $i < j$ with $\|P^{\mathbf{D}}_{[1,i]\cup[j,\infty)}x\| < \varepsilon$. Let $\|\bar z\| = 1$ with $Q\bar z = x$ and $\bar z = \sum z_j$, $z_j \in E_j$

for all j. Choose $r_i \in R_i$ and $\ell_j \in L_j$ with $\|z_{r_i}\| < \varepsilon_i$ and $\|z_{\ell_j}\| < \varepsilon_j$. Let $z = \sum_{s\in(r_i,\ell_j)} z_s$. Thus $\|z\| \le 1$ and $z \in [C_{i,R} \cup (C_\ell)_{\ell\in(i,j)} \cup C_{j,L}]$.

Now

i) $\|P^{\mathbf{F}}_{[1,r_i)\cup[\ell_j,\infty)} Qz\| < \varepsilon_{r_i} + \varepsilon_{\ell_j-1}$ by (4).

Also if $w = \bar{z} - z = \sum_{s\notin(r_i,\ell_j)} z_j$ then again by (4) and our choice of r_i and ℓ_j,

ii) $\|P^{\mathbf{F}}_{[r_i,\ell_j)} Qw\| = \|P^{\mathbf{F}}_{[r_i,\ell_j)} Q(\sum_{s<r_i} z_s + z_{r_i} + z_{\ell_j} + \sum_{s>\ell_j} z_s)\| < K[\varepsilon_{r_i} + \varepsilon_i + \varepsilon_j + \varepsilon_{\ell_j+1}]$.

From our hypothesis on x,

iii) $\|P^{\mathbf{F}}_{[1,r_i)\cup[\ell_j,\infty)} x\| < 2K\varepsilon$.

Combining i)–iii) we have, since $Qw = x - Qz$,

$$\begin{aligned}\|Qz - x\| &\le \|P^{\mathbf{F}}_{[1,r_i)\cup[\ell_j,\infty)}(Qz - x)\| + \|P^{\mathbf{F}}_{[r_i,\ell_j)}(Qz - x)\|\\ &< \varepsilon_{r_i} + \varepsilon_{\ell_j-1} + 2K\varepsilon + K[\varepsilon_{r_i} + \varepsilon_i + \varepsilon_j + \varepsilon_{\ell_j+1}]\\ &< K[2\varepsilon + 6\varepsilon_i] < K[2\varepsilon + \delta_i]\end{aligned}$$

since by (3) $6\varepsilon_i < \delta_i$. □

We let (C_n) and (D_n) be the blockings given by Lemma 2.10. Finally we block again using Proposition 2.6 for (δ_i) and (D_n) to obtain $G_n = [D_i]_{i\in(k_{n-1},k_n]}$ for some $0 = k_0 < k_1 < \cdots$. We set for $n \in \mathbb{N}$, $H_n = [C_i]_{i\in(k_{n-1},k_n]}$.

Let $x \in S_X$. Then by Proposition 2.6 there exists $(x_i) \subseteq X$ with $x = \sum x_i$ and for all $i \in \mathbb{N}$ there exists $t_i \in (k_{i-1}, k_i]$ so that

a) either $\|x_i\| < \delta_i$ or $\|P^{\mathbf{D}}_{(t_{i-1},t_i)} x_i - x_i\| < \delta_i\|x_i\|$

b) $\|x_i\| < K + 1$.

Let $B = \{i \in \mathbb{N} : \|P^{\mathbf{D}}_{(t_{i-1},t_i)} x_i - x_i\| < \delta_i\|x_i\|\}$ and $y = \sum_{i\in B} x_i$. Then $\|x - y\| \le \sum_{i\notin B}\|x_i\| < \sum \delta_i < 1/4$ by (3). For $i \in B$ set $\bar{x}_i = x_i/\|x_i\|$. From Lemma 2.10 there is a block sequence $(z_i)_{i\in B}$ of (E_n) in B_Z with

$$\|Qz_i - \bar{x}_i\| < K[2\delta_i + \delta_{t_{i-1}}] < 3K\delta_i \ . \tag{5}$$

Indeed the lemma yields that

$$z_i \in [C_{t_{i-1},R} \cup (C_\ell)_{\ell\in(t_{i-1},t_i)} \cup C_{t_i,L}]$$

which ensures that the z_i's are a block sequence. From our choice of (δ_i) (right before (3)) and (5) we have that $(\widetilde{Q}\tilde{z}_i)_{i\in B}$ is $2K$-basic, is 2-equivalent to $(\bar{x}_i)_{i\in B}$, and $(\bar{x}_i)_{i\in B}$ satisfies $2C$-$(p,1)$-estimates.

From Proposition 2.9 c) we have that

$$\begin{aligned}\Big\|\sum_{i\in B} a_i\widetilde{Q}\tilde{z}_i\Big\| &\le \Big\|\sum_{i\in B} a_i\tilde{z}_i\Big\| \le \frac{3(2K)}{\inf\limits_{j\in B}\|\widetilde{Q}\tilde{z}_j\|}\Big\|\sum_{i\in B} a_i\widetilde{Q}\tilde{z}_i\Big\|\\ &\le 7K\Big\|\sum_{i\in B} a_i\widetilde{Q}\tilde{z}_i\Big\| \le 14K\Big\|\sum_{i\in B} a_i\bar{x}_i\Big\| \ .\end{aligned} \tag{6}$$

(We have used that $\inf_{j\in B}\|\widetilde{Q}\tilde{z}_j\| > 1 - 2K\delta_1$ from (5) and $\frac{6}{(1-2K\delta_1)} < 7$ from (3).)

Let $\tilde{z} = \sum_{i\in B}\|x_i\|\tilde{z}_i$. Then from (6), $\tilde{z} \in \widetilde{Z}$ and moreover since $y = \sum_{i\in B}\|x_i\|\bar{x}_i$,

$$\|\widetilde{Q}\tilde{z} - y\| \le \sum_{i\in B}\|x_i\|\,\|\widetilde{Q}\tilde{z}_i - \bar{x}_i\| < \sum_{i\in B}(K+1)3K\delta_i < \frac{1}{4} \ \text{ (by (3))} \ .$$

Thus $\|\widetilde{Q}\tilde z - x\| < 1/2$.

Finally we show that $|\!|\!|\tilde z|\!|\!|_{Z_p} \le A$ where $A = 70CK^2$ and $|\!|\!|\cdot|\!|\!|_{Z_p}$ denotes the norm of $\widetilde{Z}_p(\widetilde{\mathbf{H}})$.

Write $\tilde z = \sum \tilde w_i$ where $(\tilde w_i)$ is any blocking of $(\|x_i\|\tilde z_i)_{i\in B}$. Say $\tilde w_j = \sum_{i\in I_j} \|x_i\|\tilde z_i$ where $I_1 < I_2 < \cdots$ is any partition of B. Then by (6) if $y_j = \sum_{i\in I_j} x_i$,

$$(\sum |\!|\!|\tilde w_j|\!|\!|^p)^{1/p} \le 14K(\sum \|y_j\|^p)^{1/p} < 14K\cdot 2C\|y\| < 35CK \tag{7}$$

since $(\bar x_i)$ satisfies $2C$-$(p,1)$-estimates and $\|y\| < 5/4$.

It remains to show that if we write $\tilde z = \sum \tilde h_n$ where $\tilde h_n \in \widetilde{H}_n$ for all n and $\tilde g_n)$ is any blocking of $(\tilde h_n)$ then

$$(\sum |\!|\!|\tilde g_n|\!|\!|^p)^{1/p} \le 70CK^2 \equiv A\ . \tag{8}$$

As in the proof of a) there exists a blocking $(\tilde w_i)$ of $(\|x_i\|\tilde z_i)_{i\in B}$ with $\tilde g_n = P^{\mathbf{H}}_{(j_{n-1},j_n]}(\tilde w_n + \tilde w_{n+1})$ for some $0 = j_0 < j_1 < \cdots$ and so $|\!|\!|\tilde g_n|\!|\!| \le K(|\!|\!|\tilde w_n|\!|\!| + |\!|\!|\tilde w_{n+1}|\!|\!|)$. Thus (8) follows.

This completes the proof of Theorem 2.1. □

We need some last preliminary results before proving Theorem 1.7.

Lemma 2.11. *Let X be a reflexive Banach space and let $1 < q < \infty$ and $\frac{1}{q} + \frac{1}{q'} = 1$. If X satisfies (∞,q)-tree estimates then X^* satisfies $(q',1)$-tree estimates.*

Proof. By [Z] we may assume that $X \subseteq Z$, a reflexive space with a bimonotone FDD (E_n). Note that if $(f_n) \subseteq S_{X^*}$ is weakly null then there exists a subsequence (f_{n_i}) of (f_n) and a weakly null sequence $(x_i) \subseteq S_X$ with $\lim_i f_{n_i}(x_i) = 1$. Indeed let $(y_n) \subseteq S_X$ with $f_n(y_n) = 1$ for all n. Choose a subsequence (y_{n_i}) which converges weakly to some $y \in X$. Then $f_{n_i}(y_{n_i} - y) \to 1$. Since (E_n) is bimonotone, $\|y_{n_i} - y\| \to 1$ and thus we may take $x_i = (y_{n_i} - y)/\|y_{n_i} - y\|$.

Now let $(f_\alpha)_{\alpha\in T_\infty}$ be a weakly null tree in S_{X^*}. Using the above observation by successively replacing the successors of a given node by a subsequence we obtain a full subtree $(g_\alpha)_{\alpha\in T_\infty}$ of $(f_\alpha)_{\alpha\in T_\infty}$ and a weakly null tree $(x_\alpha)_{\alpha\in T_\infty}$ in S_X so that for all $\alpha \in T_\infty \cup \{\emptyset\}$, $f_{\alpha,i}(x_{\alpha,i}) \to 1$ as $i \to \infty$.

Let $\varepsilon_i \downarrow 0$. By again replacing each successor sequence of nodes by a subsequence we obtain two full subtrees $(f'_\alpha)_{\alpha\in T_\infty}$ and $(x'_\alpha)_{\alpha\in T_\infty}$ satisfying: For all branches $(\alpha_i)_{i=1}^\infty$ of T_∞ and for all $i,j \in \mathbb{N}$ with $i \ne j$,

$$|f'_{\alpha_i}(x'_{\alpha_j})| < \varepsilon_{\max(i,j)} \quad\text{and}\quad f'_{\alpha_i}(x'_{\alpha_i}) > \frac{1}{2}\ .$$

Let $(x'_{\alpha_i})_{i=1}^\infty$ be a branch of $(x_\alpha)_{\alpha\in T_\infty}$ satisfying C-(∞,q)-estimates. Let $(b_i)_{i=1}^\infty \in S_{\ell_{q'}}$ and choose $(a_i)_{i=1}^\infty \in S_{\ell_q}$ with $1 = \sum_{i=1}^\infty a_ib_i$. Then $\|\sum a_jx'_{\alpha_j}\| \le C$ and so

$$\begin{aligned} C\Big\|\sum b_if'_{\alpha_i}\Big\| &\ge (\sum b_if'_{\alpha_i})(\sum a_jx'_{\alpha_j})\\ &\ge \sum_{i=1}^\infty a_ib_if'_{\alpha_i}(x'_{\alpha_i}) - \sum_{i=1}^\infty\sum_{j\ne i}|f'_{\alpha_i}(x'_{\alpha_j})|\\ &> \frac{1}{2} - \sum_{i=1}^\infty[i\varepsilon_i + \sum_{j>i}\varepsilon_j] > \frac{1}{4} \end{aligned}$$

provided that the ε_i's were taken sufficiently small. □

Proposition 2.12. *Let* $\mathbf{F} = (F_i)$ *be a bimonotone* FDD *for a Banach space* Z *and let* $1 < q \le p < \infty$ *and* $C < \infty$. *If* (F_i) *satisfies* C-(∞, q)-*estimates in* Z *then* (F_i) *is a bimonotone* FDD *for* $Z_p(\mathbf{F})$ *satisfying* C-(p,q)-*estimates in* $Z_p(\mathbf{F})$.

Remark. Prus ([P, lemma 3.5] obtained this result with weaker estimates. As written this result is stated with $C = 1$ in [JLPS]. The clever proof we present was shown to us by W.B. Johnson and G. Schechtman.

Proof. Let $z = \sum z_i \in c_{00}(\oplus_{i=1}^\infty F_i)$ with $z_i \in F_i$ for all i. Let $k \in \mathbb{N}$ and $0 = n_0 < n_1 < \cdots < n_k \le \infty$. For some choice of ℓ and $0 = m_0 < m_1 < \cdots < m_\ell$ we have

$$\begin{aligned}
\Big\|\sum z_i\Big\|_{Z_p} &= \Big(\sum_{j=1}^{\ell}\Big\|\sum_{i=m_{j-1}+1}^{m_j} z_i\Big\|^p\Big)^{1/p} \\
&= \Big[\sum_{j=1}^{\ell}\Big\|\sum_{s=1}^{k} P^{\mathbf{F}}_{(n_{s-1},n_s]}\Big(\sum_{i=m_{j-1}+1}^{m_j} z_i\Big)\Big\|^p\Big]^{1/p} \\
&\le C\Big[\sum_{j=1}^{\ell}\Big(\sum_{s=1}^{k}\|P^{\mathbf{F}}_{(n_{s-1},n_s]}\Big(\sum_{i=m_{j-1}+1}^{m_j} z_i\Big)\|^q\Big)^{p/q}\Big]^{1/p} \\
&\le C\Big[\sum_{s=1}^{k}\Big(\sum_{j=1}^{\ell}\|P^{\mathbf{F}}_{(n_{s-1},n_s]}\Big(\sum_{i=m_{j-1}+1}^{m_j} z_i\Big)\|^p\Big)^{q/p}\Big]^{1/q}, \\
&\qquad\text{by the “reverse triangle inequality” in } \ell^k_{p/q} \\
&= C\Big[\sum_{s=1}^{k}\Big(\sum_{j=1}^{\ell}\|P^{\mathbf{F}}_{(m_{j-1},m_j]}(P^{\mathbf{F}}_{(n_{s-1},n_s]}z)\|^p\Big)^{q/p}\Big]^{1/q} \\
&\le C\Big(\sum_{s=1}^{k}\|P^{\mathbf{F}}_{(n_{s-1},n_s]}x\|^q_{Z_p}\Big)^{1/q}.
\end{aligned}$$

□

For the next lemma and the proof of Theorem 1.7 we adopt the convention that for $1 \le p \le \infty$, p' is defined by $1/p + 1/p' = 1$.

Lemma 2.13. [P] *Let* (E_i) *be an* FDD *for a reflexive space* Z. *Let* $1 \le q \le p \le \infty$. *Then* (E_i) *satisfies* (p,q) *estimates iff* (E_i^*) *satisfies* (q',p') *estimates.*

Proof of Theorem 1.7.

a) ⇒ b). Let X be reflexive and satisfy (p,q)-tree estimates. By Lemma 2.11 X^* satisfies $(q',1)$-tree estimates. Thus by Theorem 2.1, X^* is a quotient of a reflexive space Z^* with an FDD $\mathbf{F}^* = (F_n^*)$ satisfying 1-$(q',1)$-estimates. Hence by Theorem 2.1 X embeds into $Z_p(\mathbf{F})$ and, by Lemma 2.13 and Proposition 2.12, (F_n) satisfies (p,q)-estimates.

b) ⇒ c). By Theorem 2.1, if X satisfies b) then X is a quotient of a reflexive space with an FDD satisfying $(p,1)$-estimates. Thus by Lemma 2.13 X^* is a subspace of a reflexive space with an FDD satisfying (∞,p')-estimates. By Lemma 2.11 X^* satisfies $(q',1)$-tree estimates and thus, by Theorem 2.1 and Proposition 2.12, X^* embeds into a reflexive space Z^* with an FDD satisfying (q',p')-estimates. Hence X is a quotient of Z, a reflexive space with an FDD satisfying (p,q)-estimates, again using Lemma 2.13.

c) $\Rightarrow$ a). We assume that X is a quotient of a reflexive space Z having an FDD satisfying (p,q)-estimates. Thus $X^* \subseteq Z^*$ which by Lemma 2.13 has an FDD satisfying (q',p')-estimates. In particular X^* satisfies (q',p')-tree estimates and so by Lemma 2.11, X has $(p,1)$-tree estimates.

Furthermore it is easy to see that if $Q : Z \to X$ denotes the quotient map then given a weakly null sequence $(x_i) \subseteq S_X$ there exists a weakly null $(z_j) \subseteq 2B_Z$ and a subsequence (x_{i_j}) of (x_i) with $Qz_j = x_{i_j}$ for all j. Thus a weakly null tree in S_X can be pruned to obtain a full subtree $(x_\alpha)_{\alpha\in T_\infty}$ and a semi normalized weakly null tree $(z_\alpha)_{\alpha\in T_\infty} \subseteq Z$ with $Qz_\alpha = x_\alpha$ for all α. Since the FDD for Z satisfies (p,q)-estimates it follows that some branch of $(x_\alpha)_{\alpha\in T_\infty}$ admits an upper ℓ_q estimate with an absolute constant. □

Remark 2.14. The following equivalences can be added to Theorem 1.7.

d) X is isomorphic to a subspace of a quotient of a reflexive space Z having an FDD which satisfies (p,q)-estimates.
e) X^* satisfies (q',p')-tree estimates.
f) X is isomorphic to a subspace of a reflexive space Z having an FDD which satisfies 1-(p,q)-estimates.

Indeed f) follows from the proof of a) $\Rightarrow$ b) in Theorem 1.7 since Z has an FDD (F_i) satisfying 1-(∞,q)-estimates and so by Lemma 2.13, $Z_p(\mathbf{F})$ satisfies 1-(p,q)-estimates. Thus we obtain a solution to a question raised in [JLPS] after the statement of Proposition 2.11. We refer the reader to [JLPS] for the relevant definitions.

Corollary 2.15. *Let X be a reflexive Banach space and $1<q\le p<\infty$. The following are equivalent*

a) *X embeds into a reflexive space Z having an* FDD *satisfying* 1-(p,q)*-estimates.*
b) *X can be renormed to be asymptotically uniformly smooth of power type q and X can be renormed to be asymptotically uniformly convex of power type p.*
c) *X can be renormed so as to be both asymptotically uniformly smooth of power type q and asymptotically uniformly convex of power type p.*

The following remains open.

Problem 2.16. Let X be a uniformly convex separable Banach space. Does there exist a uniformly convex space Z with an FDD (or even a basis) so that X embeds into Z?

References

[B] J. Bourgain, *On separable Banach spaces, universal for all separable reflexive spaces*, Proc. Amer. Math. Soc. **79** (1980), no.2, 241–246.

[E] P. Enflo, *Banach spaces which can be given an equivalent uniformly convex norm*, Proceedings of the International Symposium on Partial Differential Equations and the Geometry of Normed Linear Spaces, Jerusalem, 1972, Israel J. Math. **13** (1972), 281–288 (1973).

[GG] V.I. Gurariĭ and N.I. Gurariĭ, *Bases in uniformly convex and uniformly smooth Banach spaces*, Izv. Akad. Nauk SSSR Ser. Mat. **35** (1971), 210–215 (Russian).

[J] Robert C. James, *Super-reflexive spaces with bases*, Pacific J. Math. **41** (1972), 409–419.

[Jo] W.B. Johnson, *On quotients of L_p which are quotients of l_p*, Compositio Math. **34** (1977), no.1, 69–89.

[JZ1] W.B. Johnson and M. Zippin, *On subspaces of quotients of $(\sum G_n)_{l_p}$ and $(\sum G_n)_{c_0}$*, Proceedings of the International Symposium on Partial Differential Equations and the Geometry of Normed Linear Spaces, Jerusalem, 1972, Israel J. Math. **13** (1972), 311–316 (1973).

[JZ2] W.B. Johnson and M. Zippin, *Subspaces and quotient spaces of $(\sum G_n)_{l_p}$ and $(\sum G_n)_{c_0}$*, Israel J. Math. **17** (1974), 50–55.

[JLPS] W.B. Johnson, J. Lindenstrauss, D. Preiss, and G. Schechtman, *Almost Fréchet differentiability of Lipschitz mappings between infinite-dimensional Banach spaces*, Proc. London Math. Soc. (3) **84** (2002), no.3, 711–746.

[OS] E. Odell and Th. Schlumprecht, *Trees and branches in Banach spaces*, Trans. Amer. Math. Soc. **354** (2002), no.10, 4085–4108.

[Pi] G. Pisier, *Martingales with values in uniformly convex spaces*, Israel J. Math. **20** (1975), no.3–4, 326–350.

[P] S. Prus, *Finite-dimensional decompositions with p-estimates and universal Banach spaces* (Russian summary), Bull. Polish Acad. Sci. Math. **31** (1983), no.5–8, 281–288 (1984).

[Z] M. Zippin, *Banach spaces with separable duals*, Trans. Amer. Math. Soc. **310** (1988), no.1, 371–379.

Department of Mathematics, The University of Texas at Austin, 1 University Station C1200, Austin, TX 78712-0257
E-mail address: odell@math.utexas.edu

Department of Mathematics, Texas A&M University, College Station, TX 77843-3368
E-mail address: schlump@math.tamu.edu