CHAPTER 3. BASES IN BANACH SPACES

3.2 Bases of \(C[0,1]\) and \(L_p[0,1]\)

In the previous section we introduced the unit vector bases of \(\ell_p\) and \(c_0\). Less obvious is it to find bases of function spaces like \(C[0,1]\) and \(L_p[0,1]\).

Example 3.2.1. (The Spline Basis of \(C[0,1]\))

Let \((t_n) \subset [0,1]\) be a dense sequence in \([0,1]\), and assume that \(t_1 = 0, t_2 = 1\). It follows that

\[
    \text{mesh}(t_1, t_2, \ldots, t_n) \to 0, \quad \text{if} \quad n \to \infty
\]

where

\[
    \text{mesh}(t_1, t_2, \ldots, t_n) = \max_{i=1,2,\ldots,n} \{|t_i - t_j| : t_j \text{ is neighbor of } t_i\}.
\]

For \(f \in C[0,1]\) we let \(P_1(f)\) to be the constant function taking the value \(f(0)\), and for \(n \geq 2\) we let \(P_n(f)\) be the piecewise linear function which interpolates the \(f\) at the point \(t_1, t_2, \ldots, t_n\). More precisely, let \(0 = s_1 < s_2 < \ldots < s_n = 1\) be the increasing reordering of \(\{t_1, t_2, \ldots, t_n\}\), then define \(P_n(f)\) by

\[
    P_n(f) : [0,1] \to \mathbb{K}, \quad \text{with} \quad P_n(f)(s) = \frac{s_j - s}{s_j - s_{j-1}} f(s_{j-1}) + \frac{s - s_{j-1}}{s_j - s_{j-1}} f(s_j), \quad \text{for} \quad s \in [s_{j-1}, s_j].
\]

We note that \(P_n : C[0,1] \to C[0,1]\) is a linear projection and that \(\|P_n\| = 1\), and that (a), (b), (c) of Proposition 3.1.3 are satisfied. Indeed, the image of \(P_n(C[0,1])\) is generated by the functions \(f_1 \equiv 1, f_2(s) = s, \text{ for } s \in [0,1]\), and for \(n \geq 2, f_n(s)\) is the functions with the property \(f(t_n) = 1, f(t_j) = 0, j \in \{1, 2, \ldots \} \setminus \{t_n\}\), and is linear between any \(t_j\) and the next bigger \(t_i\). Thus \(\dim(P_n(C[0,1])) = n\). Property (b) is clear, and property (c) follows from the fact that elements of \(C[0,1]\) are uniformly continuous, and condition \((3.3)\).

Also note that for \(n > 1\) it follows that \(f_n \in P_n(C[0,1]) \cap N(P_{n-1}) \setminus \{0\}\) and thus it follows from Proposition 3.1.3 that \((f_n)\) is a monotone basis of \(C[0,1]\).

Now we define a basis of \(L_p[0,1]\), the Haar basis of \(L_p[0,1]\). Let

\[
    T = \{(n,j) : n \in \mathbb{N}_0, j = 1, 2, \ldots, 2^n \} \cup \{0\}.
\]

We partially order the elements of \(T\) as follows

\[
    (n_1, j_1) < (n_2, j_2) \iff [(j_2 - 1)2^{-n_2}, j_22^{-n_2}] \subseteq [(j_1 - 1)2^{-n_1}, j_12^{-n_1}]
\]
3.2. BASES OF $C[0, 1]$ AND $L_p[0, 1]$

\[
\iff \quad (j_1 - 1)2^{-n_1} \leq (j_2 - 1)2^{-n_2} < j_22^{-n_2} \leq j_12^{-n_1}, \text{ and } n_1 < n_2
\]
whenever $(n_1, j_1), (n_2, j_2) \in T$

and

\[
0 < (n, j), \quad \text{whenever } (n, j) \in T \setminus \{0\}
\]

Let $1 \leq p < \infty$ be fixed. We define the Haar basis $(h_t)_{t \in T}$ and the in $L_p$
normalized Haar basis $(h^{(p)}_t)_{t \in T}$ as follows.

\[
h_0 = h_0^{(p)} \equiv 1 \text{ on } [0, 1] \text{ and for } n \in \mathbb{N}_0 \text{ and } j = 1, 2 \ldots 2^n \text{ we put}
\]

\[
h_{(n,j)} = 1_{([j-1]2^{-n}, (j-\frac{1}{2})2^{-n})} - 1_{([j-\frac{1}{2}]2^{-n}, j2^{-n})}.
\]

and we let

\[
\Delta_{(n,j)} = \text{supp}(h_{(n,j)}) = ([j-1]2^{-n}, j2^{-n}),
\]

\[
\Delta^+_{(n,j)} = ([j-1]2^{-n}, (j-\frac{1}{2})2^{-n})
\]

\[
\Delta^-_{(n,j)} = ([j-\frac{1}{2}]2^{-n}, j2^{-n}).
\]

We let $h^{(\infty)}_{(n,j)} = h_{(n,j)}$. And for $1 \leq p < \infty$

\[
h^{(p)}_{(n,j)} = \frac{h_{(n,j)}}{\|h_{(n,j)}\|_p} = 2^{n/p} \left(1_{([j-1]2^{-n}, (j-\frac{1}{2})2^{-n})} - 1_{([j-\frac{1}{2}]2^{-n}, j2^{-n})}\right).
\]

Note that $\|h_t\|_p = 1$ for all $t \in T$ and that supp$(h_t) \subset$ supp$(h_s)$ if and only
if $s \leq t$.

**Theorem 3.2.2.** If one orders $(h_t^{(p)})_{t \in T}$ linearly in any order compatible
with the order on $T$ then $(h_t^{(p)})$ is a monotone basis of $L_p[0, 1]$ for all $1 \leq p < \infty$.

**Remark.** a linear order compatible with the order on $T$ is for example the
lexicographical order

\[
h_0, h_{(0,1)}, h_{(1,1)}, h_{(1,2)}, h_{(2,1)}, h_{(2,2)}, \ldots.
\]

Important observation: if $(h_t : t \in T)$ is linearly ordered into $h_0, h_1, \ldots,$
which is compatible with the partial order of $T$, then the following is true:
If $n \in \mathbb{N}$ and and if
\[ h = \sum_{j=1}^{n-1} a_j h_j, \]
is any linear combination of the first $n - 1$ elements, then $h$ is constant on the support of $h_{n-1}$. Moreover $h$ can be written as a step function
\[ h = \sum_{j=1}^{N} b_j 1_{(s_{j-1}, s_j)}, \]
with $0 = s_0 < s_1 < \ldots < s_N$, so that
\[ \int_{s_{j-1}}^{s_j} h_{n}(t) dt = 0. \]

As we will see later, if $1 < p < \infty$, any linear ordering of $(h_t : t \in T)$ is a basis of $L_p[0, 1]$, but not necessarily a monotone one.

**Proof of Theorem 3.2.2.** First note that the indicator functions on all dyadic intervals are in span$(h_t : t \in T)$. Indeed:
\[ 1_{[0,1/2)} = (h_0 + h_{(0,1)})/2, \]
\[ 1_{[1/2,1]} = (h_0 - h_{(0,1)})/2, \]
\[ 1_{[0,1/4)} = 1/2(1_{[0,1/2)} - h_{(1,1)}), \]
and so on.

Since the indicator functions on all dyadic intervals are dense in $L_p[0, 1]$ it follows that $\overline{\text{span}(h_t : t \in T)}$.

Let $(h_n)$ be a linear ordering of $(h_t^{(p)})_{t \in T}$ which is compatible with the ordering of $T$.

Let $n \in \mathbb{N}$ and $(a_i)_{i=1}^{n}$ a scalar sequence. We need to show that
\[ \left\| \sum_{i=1}^{n-1} a_i h_i \right\| \leq \left\| \sum_{i=1}^{n} a_i h_i \right\|. \]

As noted above, on the set $A = \text{supp}(h_n)$ the function $f = \sum_{i=1}^{n-1} a_i h_i$ is constant, say $f(x) = a$, for $x \in A$, therefore we can write
\[ 1_A(f + a_n h_n) = 1_A^+(a + a_n) + 1_A^-(a - a_n), \]
where $A^+$ is the first half of interval $A$ and $A^-$ the second half. From the convexity of $[0, \infty) \ni r \mapsto r^p$, we deduce that
\[ \frac{1}{2} \left[ |a + a_n|^p + |a - a_n|^p \right] \geq |a|^p, \]
and thus
\[
\int |f + a_n h_n|^p dx = \int_{A^c} |f|^p dx + \int_{A} |a + a_n|^p 1_{A^+} + |a - a_n|^p 1_{A^-} dx
\]
\[
= \int_{A^c} |f|^p dx + \frac{1}{2} m(A) [ |a + a_n|^p + |a - a_n|^p ]
\]
\[
\geq \int_{A^c} |f|^p dx + m(A) |a|^p = \int |f|^p dx
\]
which implies our claim. □

**Proposition 3.2.3.** Since for \(1 \leq p < \infty\), and \(1 < q \leq \infty\), with \(\frac{1}{p} + \frac{1}{q} = 1\) it is easy to see that for \(s \hookrightarrow t \in T\) we deduce that \((h^{(q)}_s)_{t \in T}\) are the coordinate functionals of \((h^{(p)}_t)_{t \in T}\).

**Exercises**

1. Decide whether or not the monomial \(1, x, x^n\) are a Schauder basis of \(C[0,1]\).

2. Show that the Haar basis in \(L_1[0,1]\) can be reordered in such a way that it is not a a Schauder basis anymore.
3.3 Shrinking, Boundedly Complete Bases

**Proposition 3.3.1.** Let \((e_n)\) be a Schauder basis of a Banach space \(X\), and let \((e^*_n)\) be the coordinate functionals and \((P_n)\) the canonical projections for \((e_n)\).

Then

(a) \[ P^*_n(x^*) = \sum_{j=1}^{\infty} \langle x^*, e_j \rangle e^*_j = \sum_{j=1}^{\infty} \langle \chi(e_j), x^* \rangle e^*_j, \] for \(n \in \mathbb{N}\) and \(x^* \in X^*\).

(b) \[ x^* = \sigma(X^*, X) - \lim_{n \to \infty} P^*_n(x^*), \] for \(x^* \in X^*\).

(c) \((e^*_n)\) is a Schauder basis of \(\mathop{\text{span}}(e^*_n : n \in \mathbb{N})\) whose coordinate functionals are \((e_n)\).

**Proof.** (a) For \(n \in \mathbb{N}\), \(x^* \in X^*\) and \(x = \sum_{j=1}^{\infty} \langle e^*_j, x \rangle e_j \in X\) it follows that

\[ \langle P^*_n(x^*), x \rangle = \langle x^*, P_n(x) \rangle = \langle x^*, \sum_{j=1}^{\infty} \langle e^*_j, x \rangle e_j \rangle = \sum_{j=1}^{n} \langle x^*, e_j \rangle e^*_j, \] and thus

\[ P^*_n(x^*) = \sum_{j=1}^{n} \langle x^*, e_j \rangle e^*_j. \]

(b) For \(x \in X\) and \(x^* \in X^*\)

\[ \langle x^*, x \rangle = \lim_{n \to \infty} \langle x^*, P_n x \rangle = \lim_{n \to \infty} \langle P^*_n(x^*), x \rangle. \]

(c) It follows for \(m \leq n\) and \((a_i)_{i=1}^{n} \subseteq \mathbb{K}\), that

\[ \left\| \sum_{i=1}^{m} a_i e^*_i \right\| = \sup_{x \in B_X} \left| \sum_{i=1}^{m} a_i \langle e^*_i, x \rangle \right| = \sup_{x \in B_X} \left| \sum_{i=1}^{n} a_i \langle e^*_i, P_m(x) \rangle \right| \leq \left\| \sum_{i=1}^{n} a_i e^*_i \right\| \left\| P_m \right\| \leq \sup_{j \in \mathbb{N}} \left\| P_j \right\| \cdot \left\| \sum_{i=1}^{n} a_i e^*_i \right\|. \]

It follows therefore from Proposition 3.1.9 that \((e^*_n)\) is a basic sequence, thus, a basis of \(\mathop{\text{span}}(e^*_n)\). Since \(\langle \chi(e_j), e^*_i \rangle = \langle e^*_i, e_j \rangle = \delta_{i,j}\), it follows that \((\chi(e_n))\) are the coordinate functionals for \((e^*_n)\). \(\Box\)
3.3. SHRINKING, BOUNDEDLY COMPLETE BASES

Remark. If $X$ is a space with basis $(e_n)$ one can identify $X$ with a vector space of sequences $x = (\xi_n) \subset \mathbb{K}$. If $(e_n^*)$ are coordinate functionals for $(e_n)$ we can also identify the subspace $\text{span}(e_n^* : n \in \mathbb{N})$ with a vector space of sequences $x^* = (\eta_n) \subset \mathbb{K}$. The way such a sequence $x^* = (\eta_n) \in X^*$ acts on elements in $X$ is via the infinite scalar product:

$$\langle x^*, x \rangle = \left\langle \sum_{n \in \mathbb{N}} \eta_n e_n^*, \sum_{n \in \mathbb{N}} \xi_n e_n \right\rangle = \sum_{n \in \mathbb{N}} \eta_n \xi_n.$$

We want to address two questions for a basis $(e_n)$ of a Banach space $X$ and its coordinate functionals $(e_n^*)$:

1. Under which conditions it follows that $X^* = \overline{\text{span}(e_n^*)}$?

2. Under which condition it follows that the map $J : X \to \overline{\text{span}(e_n^*)}$, with

$$J(x)(z^*) = \langle z^*, x \rangle,$$

an isomorphism or even an isometry?

We need first the following definition and some observations.

Definition 3.3.2. [Block Bases]
Assume $(x_n)$ is a basic sequence in Banach space $X$, a block basis of $(x_n)$ is a sequence $(z_n) \subset X \setminus \{0\}$, with

$$z_n = \sum_{j=k_{n-1}+1}^{k_n} a_j x_j, \text{ for } n \in \mathbb{N}, \text{ where } 0 = k_0 < k_1 < k_2 < \ldots \text{ and } (a_j) \subset \mathbb{K}.$$

We call $(z_n)$ a convex block of $(x_n)$ if the $a_j$ are non negative and $\sum_{j=k_{n-1}+1}^{k_n} a_j = 1$.

Proposition 3.3.3. The block basis $(z_n)$ of a basic sequence $(x_n)$ is also a basic sequence, and the basis constant of $(z_n)$ is smaller or equal to the basis constant of $(x_n)$.

Proof. Let $K$ be the basis constant of $(x_n)$, let $m \leq n$ in $\mathbb{N}$, and $(b_i)_{i=1}^n \subset \mathbb{K}$. Then

$$\left\| \sum_{i=1}^m b_i z_i \right\| = \left\| \sum_{i=1}^m \sum_{j=k_{i-1}+1}^{k_i} b_i a_j x_j \right\|.$$
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\[ \sum_{i=1}^{n} \sum_{j=k_{i-1}+1}^{k_i} b_i a_j x_j = K \sum_{i=1}^{n} b_i z_i \]

\[ \leq K \left\| \sum_{i=1}^{n} \sum_{j=k_{i-1}+1}^{k_i} b_i a_j x_j \right\| = K \left\| \sum_{i=1}^{n} b_i z_i \right\| \]

\[ \square \]

Theorem 3.3.4. For a Banach space with a basis \((e_n)\) and its coordinate functionals \((e_n^\ast)\) the following are equivalent.

a) \(X^\ast = \text{span}(e_n^\ast : n \in \mathbb{N})\) (and, thus, by Proposition 3.3.1, \((e_n^\ast)\) is a basis of \(X^\ast\) whose canonical projections are \(P_n^\ast\)).

b) For every \(x^\ast \in X^\ast\),

\[ \lim_{n \to \infty} \| x^\ast |_{\text{span}(e_{j:n})} \| = \lim_{n \to \infty} \sup_{x \in \text{span}(e_{j:n}), \| x \| \leq 1} |\langle x^\ast, x \rangle | = 0. \]

c) Every bounded block basis of \((e_n)\) is weakly convergent to 0.

We call the basis \((e_n)\) shrinking if these conditions hold.

Remark. Recall that by Corollary 2.2.6 the condition (c) is equivalent with

c') Every bounded block basis of \((e_n)\) has a further convex block which converges to 0 in norm.

Proof of Theorem 3.3.4. “(a)⇒(b)” Let \(x^\ast \in X^\ast\) and, using (a), write it as \(x^\ast = \sum_{j=1}^{\infty} a_j e_j^\ast\). Then

\[ \lim_{n \to \infty} \sup_{x \in \text{span}(e_{j:n}), \| x \| \leq 1} |\langle x^\ast, x \rangle | = \lim_{n \to \infty} \sup_{x \in \text{span}(e_{j:n}), \| x \| \leq 1} |\langle x^\ast, (I - P_n)(x) \rangle | \]

\[ = \lim_{n \to \infty} \sup_{x \in \text{span}(e_{j:n}), \| x \| \leq 1} |\langle (I - P_n^\ast)(x^\ast), x \rangle | \]

\[ \leq \lim_{n \to \infty} \| (I - P_n^\ast)(x^\ast) \| = 0. \]

“(b)⇒(c)” Let \((z_n)\) be a bounded block basis of \((x_n)\), say

\[ z_n = \sum_{j=k_{n-1}+1}^{k_n} a_j x_j, \text{ for } n \in \mathbb{N}, \text{ with } 0 = k_0 < k_1 < k_2 < \ldots \text{ and } (a_j) \subset \mathbb{K}. \]

and \(x^\ast \in X^\ast\). Then, letting \(C = \sup_{j \in \mathbb{N}} \| z_j \|\),

\[ |\langle x^\ast, z_n \rangle | \leq \sup_{z \in \text{span}(e_{j:n}), \| z \| \leq C} |\langle x^\ast, z \rangle | \to_{n \to \infty} 0, \text{ by condition (b)}, \]
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thus, \((z_n)\) is weakly null.

“\(\neg (a) \Rightarrow \neg (c)\)” Assume there is an \(x^* \in S_{X^*}\), with \(x^* \notin \text{span}(e^*_j : j \in \mathbb{N})\). It follows for some \(0 < \varepsilon \leq 1\)

\[
(3.5) \quad \varepsilon = \limsup_{n \to \infty} \|x^* - P_n(x^*)\| > 0.
\]

By induction we choose \(z_1, z_2, \ldots \) in \(B_X\) and \(0 = k_0 < k_1 < \ldots\), so that \(z_n = \sum_{j=k_n-1+1}^{k_n} a_j e_j\), for some choice of \((a_j)_{j=k_n-1+1}^{k_n}\) and \(|\langle x^*, z_n \rangle| \geq \varepsilon/(1+K)\), where \(K = \sup_{j \in \mathbb{N}} \|P_j\|\). Indeed, let \(z_1 \in B_X \cap \text{span}(e_j)\), so that \(|\langle x^*, z_1 \rangle| \geq \varepsilon/(1+K)\) and let \(k_1 = \min\{k : z_1 \in \text{span}(e_j : j \leq k)\}\). Assuming \(z_1, z_2, \ldots, z_n\) and \(k_1 < k_2 < \ldots < k_n\) has been chosen. Using (3.5) we can choose \(m > k_n\) so that \(|\langle x^* - P_m(x^*) \rangle| > \varepsilon/2\) and then we let \(z_{n+1} \in B_X \cap \text{span}(e_i : i \in \mathbb{N})\) with

\[
|\langle x^* - P_m(x^*) \rangle, z_{n+1} \rangle| = |\langle x^*, z_{n+1} - P_m(z_{n+1}) \rangle| > \varepsilon/2.
\]

Finally choose

\[
z_{n+1} = \frac{z_{n+1} - P_m(z_{n+1})}{1 + K} \in B_X
\]

and

\[
k_{n+1} = \min \{k : z_{n+1} \in \text{span}(e_j : j \leq k)\}.
\]

It follows that \((z_n)\) is a bounded block basis of \((e_n)\) which is not weakly null. \(\square\)

Examples 3.3.5. Note that the unit vector bases of \(\ell_p\), \(1 < p < \infty\), and \(c_0\) are shrinking. But the unit vector basis of \(\ell_1\) is not shrinking (consider \((1, 1, 1, 1, 1, \ldots) \in \ell_1^* = \ell_\infty\).

Proposition 3.3.6. Let \((e_j)\) be a shrinking basis for a Banach space \(X\) and \((e_j^*)\) its coordinate functionals. Put

\[
Y = \left\{ (a_i) \subset K : \sup_{n} \left\| \sum_{j=1}^{n} a_j e_j \right\| < \infty \right\}.
\]

Then \(Y\) with the norm

\[
\|(a_i)\| = \sup_{n \in \mathbb{N}} \left\| \sum_{j=1}^{n} a_j e_j \right\|
\]

is a Banach space and

\[
T : X^{**} \to Y, \quad x^{**} \mapsto (\langle x^{**}, e_j^* \rangle)_{j \in \mathbb{N}},
\]

is an isomorphism between \(X^{**}\) and \(Y\).

If \((e_n)\) is monotone then \(T\) is an isometry.