ON SIMPLE SPACES

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For a Banach space $X$ we denote by $\mathcal{L}(X)$ the algebra of all bounded linear operators on $X$. In this paper we are concerned with norm closed two-sided ideals in $\mathcal{L}(X)$. Banach space $X$ is said to be simple if the set $\mathcal{K}(X)$ of all the compact operators is the only non-trivial proper closed ideal in $\mathcal{L}(X)$. It is known that the spaces $\ell_p$ ($1 \leq p < +\infty$) and $c_0$ are simple (see, e.g., [Caradus:74]). No other simple spaces are currently known to the authors.

In this note we consider factorization ideals. In general, the set of all operators in $\mathcal{L}(X)$ that factor through a given Banach space $Z$ need not be an ideal, as it might be not closed under addition. However, the set of all the operators in $\mathcal{L}(X)$ that factor through a power of $Z$ is an ideal. Denote this ideal by $\mathcal{J}_0^Z(X)$, and let $\mathcal{J}^Z(X)$ be the closure of $\mathcal{J}_0^Z(X)$ in $\mathcal{L}(X)$ in the operator norm. Thus, $\mathcal{J}^Z(X)$ is a closed ideal in $\mathcal{L}(X)$.

**Lemma 1.** The following are equivalent

(i) $\mathcal{J}_0^Z(X) = \mathcal{L}(X)$;
(ii) $I \in \mathcal{J}_0^Z(X)$;
(iii) $X$ embeds complementably into a power of $Z$;
(iv) $\mathcal{J}^Z(X) = \mathcal{L}(X)$

**Proof.** The implications (i)$\Rightarrow$(ii)$\Rightarrow$(iii)$\Rightarrow$(iv) are straightforward. The implication (iv)$\Rightarrow$(i) follows from the fact that the closure of a proper ideal in a unital Banach algebra is again a proper ideal (see, e.g., [Conway:90, Corollary VII.2.4]). \qed

Remark 2. In the case when $Z \cong Z \oplus Z$, the ideal $\mathcal{J}_0^Z(X)$ coincides with the set of all the operators that factor through $Z$. Also, in this case the condition (iii) can be replaced with (iii) $X$ embeds complementably into $Z$.

Proposition 3. Suppose that $X$ is a Banach space and $V$ is a complemented subspace of $X$. If $X$ doesn’t embed complementably into a power of $V$, then $X$ is not simple.

Proof. $\mathcal{J}^V(X)$ is a closed ideal in $\mathcal{L}(X)$. It is proper by Lemma \ref{lem:complemented}. To show that $\mathcal{J}^V(X) \neq \mathcal{K}(X)$, let $P : X \to V$ be the canonical projection and let $S : V \to X$ be the inclusion map. Put $T = SP$, then $T \in \mathcal{J}^V(X)$, but $T|_V = \text{id}_V$, so that $T$ is not strictly singular, hence $T \notin \mathcal{K}(X)$. \hfill \Box

Corollary 4. If $X$ is simple then for every complemented subspace $V \subseteq X$, $X$ embeds complementably into a power of $V$.

In view of Remark 2 we also get the following modification of Corollary \ref{cor:complemented}, the proof is similar.

Proposition 5. If $X$ is simple then $X$ embeds complementably into each of its complemented subspace $V$ satisfying $V \cong V \oplus V$.

Corollary 6. Suppose that $X$ is simple and contains a complemented copy of $\ell_p$ $(1 \leq p < \infty)$. Then $X$ is isomorphic to $\ell_p$.

Proof. Indeed, if $V$ is a complemented subspace of $X$ which is isomorphic to $\ell_p$, then, by Proposition 6, $X$ embeds complementably in $\ell_p$. Then $X$ is isomorphic to $\ell_p$. \hfill \Box

Proposition 7. Suppose that there exists a non-strictly singular operator $T : X \to \ell_p$ for some $1 \leq p < \infty$. Then $X$ contains a complemented subspace isomorphic to $\ell_p$.

Proof. Let $W$ be an infinite-dimensional subspace of $X$ such that $T_1 = T|_W$ is an isomorphism. Since every subspace of $\ell_p$ contains a complemented copy of $\ell_p$, then $T(W)$ contains a subspace $V$ isomorphic to
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\[ \ell_p \] and complemented in \( \ell_p \). Let \( P: \ell_p \to V \) be a bounded projection, then \( T^{-1}PT \) is a bounded projection from \( X \) to \( T^{-1}V \). Hence, \( T^{-1}V \) is a complemented subspace of \( X \), isomorphic to \( \ell_p \). \( \square \)

**Corollary 8.** Suppose that a simple Banach space \( X \) has a subspace and a quotient isomorphic to \( \ell_p \) \((1 \leq p < \infty)\). Then \( X \) is isomorphic to \( \ell_p \).

**Proof.** Let \( q \) be the quotient map from \( X \) onto \( \ell_p \), and let \( S \) be an isomorphism from \( \ell_p \) onto a subspace \( V \) of \( X \). Put \( T = Sq: X \to V \). We can view \( T \) as an element of \( L(X) \). Clearly, \( T \) is not compact. Since \( X \) is simple, then every strictly singular operator is compact, hence \( T \) is not strictly singular either. Then it follows from Proposition \( \text{non-ss} \) that \( X \) contains a complemented copy of \( \ell_p \), so that \( X \cong \ell_p \) by Corollary \( \ell \). \( \square \)

**Remark 9.** In the statements of Corollary \( \ell \) and \( \ell \), and of Proposition \( \text{non-ss} \), one can replace \( \ell_p \) with \( c_0 \).

Next, we are going to apply previous results to Orlicz and Tsirelson spaces. Recall that an Orlicz function \( M \) is a continuous non-decreasing convex function defined for \( t \geq 0 \) such that \( M(0) = 0 \) and \( \lim_{t \to \infty} M(t) = \infty \). Given an Orlicz function \( M \), the Orlicz sequence space \( \ell_M \) is the space of all sequences of scalars \( x = (a_1, a_2, \ldots) \) such that \( \sum_{n=1}^{\infty} M(|a_n|/\rho) < \infty \) for some \( \rho > 0 \), equipped with the norm

\[ \|x\| = \inf\{\rho > 0 \mid \sum_{n=1}^{\infty} M(|a_n|/\rho) \leq 1\}. \]

We will be interested only in isomorphic properties of \( \ell_M \), so we can assume, for simplicity, that \( M(1) = 1 \). In this case, the unit vectors \( \{e_n\}_{n=1}^{\infty} \) form a normalized symmetric basic sequence in \( \ell_M \).

Every Orlicz sequence space \( \ell_M \) contains isomorphs of some \( \ell_p \) or \( c_0 \). By \( \text{Lindenstrauss:77} \), Proposition 4.a.4], \( \ell_M \) is separable \( \iff \ell_M \) does not contain \( \ell_\infty \) \( \iff \{e_n\}_{n=1}^{\infty} \) is a boundedly complete basis for \( \ell_M \).

**Corollary 10.** Let \( \ell_M \) be a simple Orlicz sequence space. Then it is isomorphic to \( \ell_p \) for some \( 1 \leq p < \infty \).
Proof. First we observe that $\ell_M$ must be separable. Suppose not, then $\ell_M$ contains $\ell_\infty$, necessarily complemented because $\ell_\infty$ is injective. Notice that $L_1(0,1)$ has a subspace isomorphic to $\ell_2$ and thus $\ell_2$ is a quotient of subspace isomorphic to $\ell_2$, it follows from Corollary 8 that $\ell_\infty$ is simple. Since $\ell_M$ contains $\ell_\infty$ complementably, it follows from Proposition 5 that $\ell_M$ embeds complementably into $\ell_\infty$, but this contradicts $\ell_\infty$ being prime.

Thus, we can assume that $\ell_M$ is separable. Then $\{e_n\}_{n=1}^\infty$ is a symmetric boundedly complete basis for $\ell_M$. This implies, in particular, that $c_0$ doesn’t embed in $\ell_M$. It follows that $\ell_M$ contains $\ell_p$ isomorphically, for some $1 \leq p < \infty$. If $\ell_M$ contains $\ell_1$ then, since $\ell_1$ is injective in Banach spaces with an unconditional basis (see Johnson:74[J74]), we are done by Corollary 8. Finally, if $\ell_M$ doesn’t contain $\ell_1$, but contains $\ell_p$ for some $1 < p < \infty$, then $\ell_M$ is reflexive by Lindenstrauss:77[LT77, Theorem 1.c.12(a)]. It follows now from Lindenstrauss:77[LT77, Theorem 4.b.3(iv)] that $\ell_M$ has a quotient isomorphic to $\ell_p$. Thus by Corollary 8, $\ell_M$ must be isomorphic to $\ell_p$. \hfill $\square$

Note that not every Orlicz sequence space contains a complemented copy of $\ell_p$. For the examples of such spaces, see Lindenstrauss:77[LT77, 4.c]. $L_\infty(0,1) \cong \ell_\infty$. Since $\ell_\infty$ also has a

**Proposition 11.** If $T$ is the Tsirelson space, then neither $T$ nor $T^*$ is simple.

**Proof.** It follows from Casazza:89[CS89, Theorem VI.b.4] that there exists a fast increasing subsequence $n_k$ of integers such that the subspace $Z = \text{span}\{e_{n_k}\}_k$ spanned by the corresponding subsequence of the basis does not contain a complemented subspace isomorphic to $T$. On the other hand, for every subspace $Z$ spanned by a subsequence we have $Z \cong Z \oplus Z$. Indeed, taking any subsequence $m_k$ such that $n_k < m_k < n_{k+1}$ we get another subspace isomorphic to $Z$, and the subspace corresponding to a subsequence $(n_k, m_k)_k$ is also isomorphic to $Z$. Formally, letting
$Z_1 = \text{span}\{e_{m_k}\}_k$ we have $Z_1 \cong Z$ and $Z \oplus Z_1 = \text{span}\{e_{m_k}, e_{m_k}\}_k \cong Z$.

Now Proposition 5 implies that $T$ is not simple.

Furthermore, notice that $Z^*$ is a complemented subspace of $T^*$, and $Z^* \cong Z^* \oplus Z^*$. Since $T$ is reflexive, we conclude that $T$ does not embed complementably into $Z^*$. Hence, $T^*$ is also not simple by Proposition 5.

If $Z$ is a Banach space, we use the following notation:

$$\ell_p(Z) = \left( \bigoplus_{i=1}^{\infty} Z \right)_p \quad \text{and} \quad \ell_p^n(Z) = \left( \bigoplus_{i=1}^{n} Z \right)_p.$$ 

Now consider $\mathcal{J}_0^{\ell_p^n(Z)}(X)$ and $\mathcal{J}^{\ell_p^n(Z)}(X)$. Since $\ell_p(Z) \cong \ell_p(Z) \oplus \ell_p(Z)$, Remark 2 implies that $\mathcal{J}_0^{\ell_p^n(Z)}(X)$ is the set of all operators that factor through $\ell_p(Z)$. Again, $\mathcal{J}^{\ell_p^n(Z)}(X)$ is proper if and only if $\mathcal{J}_0^{\ell_p^n(Z)}(X)$ is proper.

**Proposition 12.** If $X$ is simple then $\ell_p(V) \cong \ell_p(X)$ for every complemented subspace $V$ in $X$ and every $1 \leq p < \infty$.

**Proof.** Suppose that $X$ is simple and $V$ is a complemented subspace of $X$. Then $\ell_p(V)$ is a complemented subspace of $\ell_p(X)$, so that $\ell_p(X) = \ell_p(V) \oplus Z_1$ for some $Z_1$. On the other hand, it follows from Corollary 4 that $X$ embeds complementably into $\ell_p(V)$, so that $\ell_p(X)$ embeds complementably into $\ell_p(\ell_p(V)) \cong \ell_p(V)$. Thus, $\ell_p(V) \cong \ell_p(X) \oplus Z_2$. Now, using Pelczyński’s Decomposition method, one can prove a Schröder-Bernstein-type statement that $\ell_p(V) \cong \ell_p(X)$. Indeed,

$$\ell_p(X) \oplus \ell_p(V) \cong \ell_p(X) \oplus \ell_p(X) \oplus Z_2 \cong \ell_p(X) \oplus Z_2 \cong \ell_p(V),$$

and

$$\ell_p(V) \oplus \ell_p(X) = \ell_p(V) \oplus \ell_p(V) \oplus Z_1 \cong \ell_p(V) \oplus Z_1 = \ell_p(X).$$

Suppose that $T \in \mathcal{J}_0^Z(X)$, then $T$ factors through $Z^n$ for some $n$. Since $Z^n \cong \ell_p^n(Z)$, then $T$ factors through $\ell_p(Z)$. It follows that $\mathcal{J}_0^Z(X) \subseteq \mathcal{J}_0^{\ell_p^n(Z)}(X)$ and, therefore, $\mathcal{J}^Z(X) \subseteq \mathcal{J}^{\ell_p^n(Z)}(X)$.
Proposition 13. Suppose that $\ell_p$ doesn’t embed complementably into $X$ for some $1 \leq p < \infty$. Then for every Banach space $Z$, $J^Z(X)$ is proper if and only if $J^{\ell_p(Z)}(X)$ is proper. The same statement is valid with $\ell_p$ replaced with $c_0$.

In the proof of Proposition 13 we will need the following result.

Theorem 14. Suppose that $1 \leq p \leq \infty$ and $\{X_i\}$ is a sequence of infinite-dimensional Banach spaces. Put $X = \bigoplus_{i=1}^{\infty} X_i$ if $p < \infty$, or $X = \bigoplus_{i=1}^{\infty} X_i$ if $p = \infty$. If $Y$ is a complemented subspace of $X$ then either $\ell_p$ (respectively, $c_0$) embeds complementably into $Y$, or $Y$ actually embeds complementably into a finite sum $\bigoplus_{i=1}^{n} X_i$ for some $n$.

Proof of Proposition 13. Since $J^Z(X) \subseteq J^{\ell_p(Z)}(X)$ is always true, we only need to prove that if $J^{\ell_p(Z)}(X) = \mathcal{L}(X)$ then $J^Z(X) = \mathcal{L}(X)$. In view of Lemma 1, it suffices to show that if $J^{\ell_p(Z)}_0(X) = \mathcal{L}(X)$ then $J^Z_0(X) = \mathcal{L}(X)$. Suppose $J^{\ell_p(Z)}_0(X) = \mathcal{L}(X)$, then by Lemma 1 $X$ embeds complementably into $\ell_p(Z)$. By Theorem 14 we conclude that $X$ embeds complementably into $\ell_p^n(Z)$ for some $n$, so that $J^Z_0(X) = \mathcal{L}(X)$ by Lemma 1.

Proof of Theorem 14. For an integer $n \geq 1$, let $P_n : X \to X$ be the natural projection onto $\bigoplus_{i=1}^{n} X_i$. By $Q : X \to X$ denote a bounded projection onto the subspace $Y$. It is clear that at least one of the following two cases holds.

(I): $\exists \delta \exists m_0 \parallel QP_m y \parallel \geq \delta \parallel y \parallel$ for all $y \in Y$.

(II): $\forall \varepsilon > 0 \forall m \exists y \in Y \parallel QP_m y \parallel \leq \varepsilon \parallel y \parallel$.

Case (I): First note that for all $y \in Y$ one has $\delta \parallel y \parallel \leq \parallel QP_m y \parallel \leq \parallel P_m y \parallel \parallel Q \parallel$, so that

$$\text{eq-1} \quad \parallel P_m y \parallel \geq \delta_1 \parallel y \parallel \quad \text{for all } y \in Y,$$

where $\delta_1 = \delta/\parallel Q \parallel$. Set $F = P_{m_0}(Y)$ and $T = P_{m_0} | Y$. Then (1) immediately implies that $T : Y \to F$ is an isomorphism from $Y$ onto $F$.
and $\|T^{-1}\| \leq \delta_1^{-1}$. Let $S = P_{m_0}Q|F$. Then $S : F \to F$ and we have

\begin{equation}
\tag{2}
\|Sf\| \geq \delta_1\|f\| \quad \text{for all } f \in F.
\end{equation}

Indeed, let $f \in F$ and let $f = P_{m_0}y$ for some $y \in Y$. Then, by \eqref{eq:1},

\[\|Sf\| = \delta_1\|QP_{m_0}y\| \geq \delta_1\delta\|y\| = \delta_1\delta\|f\|.
\]

Thus \eqref{eq:2} implies $\|S^{-1}\| \leq (\delta\delta_1)^{-1}$. Consider the operator $\tilde{Q} = S^{-1}P_{m_0}Q : X \to F$. Clearly, $\tilde{Q}$ is a projection from $X$ onto $F$ and $\|\tilde{Q}\| \leq (\delta\delta_1)^{-1}\|Q\|$. Thus $F$ is isomorphic to $Y$ and complemented in $X$, in this case.

**Case (II):** Let $R_n = I - P_n$. First observe that if a vector $y \in Y$ satisfies condition (II) then we also have

\begin{equation}
\tag{3}
\|R_{m}y\| \geq (1 - \varepsilon)\|Q\|^{-1}\|y\| =: c\|y\|.
\end{equation}

Indeed, $\|QR_{m}y\| \geq \|Qy\| - \|QP_{m}y\| \geq (1 - \varepsilon)\|y\|$; and on the other hand, $\|QR_{m}y\| \leq \|Q\|\|R_{m}y\|$.

Assume for simplicity that $\varepsilon = 0$ in condition (II) and that all vectors and functionals involved in the remaining part of the argument have always finite support. The general case follows from this by a standard approximation. Under this additional assumption, condition (II) says

\begin{equation}
\tag{II'}
\forall m \exists y \in Y \text{ } QP_{m}y = 0.
\end{equation}

Construct by induction a sequence of integers $0 = n_0 < n_1 < n_2 < \ldots$ and sequences of vectors $\{u_j\}$ in $Y$ and of functionals $\{u^*_j\}$ in $X^*$ such that for every $j = 1, 2, \ldots$ we have

\begin{enumerate}
  \item $\|u_j\| = 1$, and $QP_{n_{j-1}}u_j = 0$, $R_{n_j}u_j = 0$, and $\|R_{n_{j-1}}u_j\| \geq c$;
  \item $\|u^*_j\| = 1$, $u^*_j(u_j) \geq c$, and $P^*_{n_{j-1}}u^*_j = 0$, $R^*_{n_j}u^*_j = 0$, and

\end{enumerate}

\begin{equation}
\tag{u-star}
R^*_nQ^*u^*_j = 0 \quad \text{whenever } n \geq n_j.
\end{equation}

Indeed, given $n_{j-1}$, let $u_j$ be any norm 1 vector satisfying condition (II’) for $m = n_{j-1}$, and let $n'$ be such that $R_{n'}u_j = 0$. Then by \eqref{eq:3}, conditions \eqref{u-star} are satisfied whenever $n_j \geq n'$.

Let $u^*_j$ be a norming functional for $R_{n_{j-1}}u_j$, so that $P^*_{n_{j-1}}u^*_j = 0$, $R^*_{n'}u^*_j = 0$ and $\|u^*_j\| = 1$, and $u^*_j(u_j) = u^*_j(R_{n_{j-1}}u_j) = \|R_{n_{j-1}}u_j\| \geq c$.

Finally, let $n_j \geq n'$ be an integer such that $R^*_nQ^*u^*_j = 0$. It follows that $R^*_nQ^*u^*_j = 0$ still if $n \geq n_j$. It is clear that (II) is satisfied.
We shall show that \( \{u_j\} \) is equivalent to the unit vector basis in \( \ell_p \), and its span is complemented in \( X \).

Set \( w_j = R_{n_{j-1}}u_j \), for \( j = 1, 2, \ldots \). By (4) we have \( Qu_j = Qw_j \) and \( w_j \) is supported on the (open) interval \((n_{j-1}, n_j)\), for \( j = 1, 2, \ldots \). Let \( \{a_j\} \) be a finite sequence of scalars. Then

\[
\| \sum_j a_j u_j \| = \| \sum_j a_j Q u_j \| = \| \sum_j a_j Q w_j \| \leq \| Q \| \| \sum_j a_j w_j \| \leq \| Q \| \left( \sum_j |a_j|^p \right)^{1/p}.
\]

To get the lower \( \ell_p \)-estimate, fix a finite sequence of scalars \( \{a_j\} \). For an arbitrary finite scalar sequence \( \{b_j\} \) set \( x^* = \sum_j b_j u_j^* \), and consider \( x^*(\sum_k a_k u_k) \).

Fix \( k = 1, 2, \ldots \). Since \( u_j^* = R_{n_k} u_j^* \) and \( R_{n_k} u_k = 0 \) for \( j > k \), we infer that \( u_j^*(u_k) = 0 \) for all \( j > k \). If \( j < k \), then (4-star) yields \( Q^* u_j^* = P_{n_{k-1}}^* Q^* u_j^* \), while (4) implies \( Q P_{n_{k-1}} u_k = 0 \). Therefore,

\[
u_j^*(u_k) = u_j^*(Q u_k) = Q^* u_j^*(u_k) = P_{n_{k-1}}^* Q^* u_j^*(u_k) = u_j^*(Q P_{n_{k-1}} u_k) = 0.
\]

Thus, \( x^* (\sum_k a_k u_k) = \sum_k a_k b_k u_k^* (u_k) \). Since \( u_k^*(u_k) \geq c \) for all \( k \), choosing an appropriate sequence \( \{b_k\} \) with \( \sum_k |b_k|^{p'} = 1 \) we get

\[
c \left( \sum_k |a_k|^p \right)^{1/p} \leq \sum_k a_k b_k u_k^* (u_k) \leq \| x^* \| \| \sum_k a_k u_k \|.
\]

(Here \( 1/p + 1/p' = 1 \), for \( p > 1 \) and \( p' = \infty \), for \( p = 1 \), with an appropriate interpretation of the \( \ell_p' \)-norm.)

Finally note that since \( u_j^* \) is supported on the (open) interval \((n_{j-1}, n_j)\), for \( j = 1, 2, \ldots \), for any finite scalar sequence \( \{b_j\} \) we have \( \| \sum_j b_j u_j^* \| \leq \left( \sum_j |b_j|^{p'} \right)^{1/p'} \); hence \( \| x^* \| \leq 1 \). Thus

\[
c \left( \sum_k |a_k|^p \right)^{1/p} \leq \| \sum_k a_k u_k \|.
\]

To show that \( \text{span}\{u_j\} \) is complemented, set \( w_j^* = u_j^*/u_j^*(w_j) \), for \( j = 1, 2, \ldots \). Then \( \| w_j^* \| \leq 1/c \) for all \( j \), and for any finite scalar
sequence \( \{b_j\} \) we have

\[ \left\| \sum_j b_j w_j^* \right\| \leq (1/c) \left( \sum_j |b_j|^p \right)^{1/p}. \]  

(5)

Define an operator \( \widetilde{Q} : X \to \text{span}\{u_j\} \) by \( \widetilde{Q}x = \sum w_j^*(x)u_j \), for \( x \in X \). For all \( j \) we have \( u_j^* = R_{n_j-1} u_j \), and hence \( w_j^*(u_j) = 1 \).

As before, \( w_j^*(u_k) = 0 \) whenever \( k \neq j \), \( j, k = 1, 2, \ldots \). Thus \( \widetilde{Q} \) is a projection. The following estimate for the norm \( \|\widetilde{Q}\| \) is standard.

Let \( x \in X \). For an appropriate scalar sequence \( \{b_j\} \) satisfying \( \sum_j |b_j|^p = 1 \) we have, using inequalities (4) and (5),

\[ \|\widetilde{Q}x\| \leq \|Q\| \left( \sum_j |w_j^*(x)|^{p'} \right)^{1/p'} = \|Q\| \left| \sum_j b_j w_j^*(x) \right| \]

\[ \leq \|Q\| \left\| \sum_j b_j w_j^* \right\| \|x\| \leq (1/c) \|Q\| \|x\|. \]

So \( \|\widetilde{Q}\| \leq (1/c)\|Q\| \), and this completes the proof. \( \Box \)

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