ON ASYMPTOTIC STRUCTURE, THE SZLENK INDEX AND UKK PROPERTIES IN BANACH SPACES

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Abstract. Let B be a separable Banach space and let $X = B^*$ be separable. We prove that if B has finite Szlenk index (for all $\varepsilon > 0$) then B can be renormed to have the weak* uniform Kadec-Klee property. Thus if $\varepsilon > 0$ there exists $\delta(\varepsilon) > 0$ so that if (x_n) is a sequence in the ball of X converging ω^* to x so that $\underline{\lim}\, \|x_n - x\| \geq \varepsilon$ then $\|x\| \leq 1 - \delta(\varepsilon)$. In addition we show that the norm can be chosen so that $\delta(\varepsilon) \geq c\varepsilon^p$ for some $p < \infty$ and $c > 0$.

§0. Introduction

The asymptotic structure of a separable infinite dimensional Banach space X as considered in [MT] and [MMT] is a concept which merges the finite dimensional and infinite dimensional structure of X. One obtains for each integer n a class of normalized bases of length n which can each be found arbitrarily far out and arbitrarily separated in X. Usually knowledge of the asymptotic structure does not translate into global information about X. However sometimes it does and one such occurrence is the focal point of this paper.

Our object of attention is the following problem. Suppose X is a space with separable dual having finite Szlenk index for every $\varepsilon > 0$. This means that if one starts with the ball of X^*, B_{X^*}, and then forms the subset of all ω^* limits of ε-separated sequences from B_{X^*} and then forms the new subset of all ω^* limits of ε-separated sequences from this subset and so on then after finitely many such steps one is left with nothing. The question is can X be renormed to have the ω^*-UKK (weak* uniform Kadec-Klee) property? This latter property involves a modulus $\delta(\varepsilon) > 0$. It says that given $\varepsilon > 0$ and $(x_n^*) \subseteq B_{X^*}$, a sequence converging ω^* to x^* with $\underline{\lim}\, \|x_n^* - x^*\| \geq \varepsilon$ then $\|x^*\| \leq 1 - \delta(\varepsilon)$. We show that this problem has an affirmative solution and moreover the modulus $\delta(\varepsilon)$ is of power type.

Our proof leads us into further study of the asymptotic structure of X [MMT]. It turns out that having finite Szlenk index just says that one does not have arbitrarily long bases

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$(e_i^n)_1^n$ amongst the asymptotic structure of X^* with $\sup_n \|\sum_1^n e_i^n\| < \infty$, and this in turn yields uniform lower ℓ_p estimates for some $p < \infty$ on bases in the asymptotic structure. We then carry this lower ℓ_p estimate back to X^* (for a different p) in a certain manner and ultimately obtain the theorem.

A corollary of our result is a solution to a question raised by R. Huff [H] which can be stated as: If X is a reflexive separable space with $X = B^*$ and Szlenk index $S(B, \varepsilon) < \omega$ for all $\varepsilon > 0$, can X be renormed so that X^* has the UKK (uniform Kadec-Klee) property. This is defined like the ω^*-UKK property except that one uses weak convergence rather than weak* convergence. The nonreflexive version of Huff's problem remains open (see below).

A number of authors have written on the ω^*-UKK and UKK properties and the renorming problem [P], [L], [CK], [DGK]. Prus [P] observed that a reflexive X with a basis has an equivalent UKK norm iff some blocking of the basis into an FDD (finite dimensional decomposition) admits for some $p < \infty$ uniform lower ℓ_p estimates on all block bases. The analogous result for the ω^*-UKK property is established for spaces with a shrinking basis in [DGK]. UKK properties in Banach lattices are investigated in [CK]. In [L] Lancien shows that if X has finite Szlenk index then X^* admits an equivalent ω^* lower semicontinuous *ecart* satisfying the ω^*-UKK and solves the renorming problem for the spaces $L_p(X)$ and certain $C(K)$ spaces.

Our result also bears some resemblance with the characterization of superreflexive Banach spaces as those that can be renormed to be uniformly convex [En] and moreover in such a manner as to have modulus of uniform convexity of power type [Pi].

In section 1 we recall the notion of asymptotic structure of a space X with an FDD (E_n) and connect it with certain blockings of the FDD (finite dimensional decomposition). In particular we observe that given $\varepsilon_i \downarrow 0$ there exists a blocking (F_n) of (E_n) so that for all n if $(x_i)_1^n$ is any skipped normalized sequence w.r.t. $(F_i)_n^\infty$ then $(x_i)_1^n$ is (up to ε_n) an element of the n^{th}-asymptotic structure $\{X, (E_i)\}_n$. An infinite version of this result is given in Theorem 2.11. In section 2 we discuss the Szlenk index and introduce some other indices.

Section 3 concerns connections between our indices and lower ℓ_p estimates. In section 4 we solve our problem in the case where X has a shrinking FDD. Section 5 handles the general case. Section 6 concerns some dual results and contains further remarks. In the last section we discuss results concerning the ω^*-UKK modulus $\delta(\varepsilon)$.

Our notation is standard. $X, B, Y, Z, \dots$ will be separable infinite dimensional Banach spaces and $F, G, H, \dots$ will be used for finite dimensional spaces. B_X is the unit ball of X and S_X is the unit sphere of X. $\langle\cdot\rangle$ denotes linear span and $[\cdot]$ is the closed linear span.

§1. Asymptotic Structure

In this section we make a connection between the asymptotic structure of a Banach space X w.r.t. an FDD (E_i) and finite sequences $(x_i)_1^k$, in a certain skipped blocking of (E_i). For example we show that given k and $\varepsilon > 0$ we can find a blocking (F_j) of (E_i) so that, up to ε, every normalized skipped sequence $(x_i)_1^k$ w.r.t. (F_j) belongs to $\{X, (E_i)\}_k$.

We begin by recalling the precise meaning of this last creature [MMT]. Let (E_i) be an FDD for X. We shall assume that (E_n) is monotone. This is not essential but rather is for convenience, here and throughout. Given $k \in \mathbb{N}$ and $C \geq 1$ we let $\mathcal{M}_k(C)$ be the set of all normalized C-basic sequences of length k. $\mathcal{M}_k(C)$ is a compact metric space under the metric $\log d_b(\cdot,\cdot)$ where

$$d_b\big((x_i)_1^k, (y_i)_1^k\big) = \inf\bigg\{AB : \forall\ (a_i)_1^k \subseteq \mathbb{R}\ ,\ A^{-1}\Big\|\sum_1^k a_i y_i\Big\| \leq \Big\|\sum_1^k a_i x_i\Big\| \leq B\Big\|\sum_1^k a_i y_i\Big\|\bigg\}\ .$$

Set $\mathcal{M}_k = \mathcal{M}_k(1)$.

Definition 1.1. Let $(e_i)_1^k \in \mathcal{M}_k$. Then $(e_i)_1^k \in \{X, (E_i)\}_k$ if

$$\begin{aligned}
&\forall\ \varepsilon > 0\ \forall\ n_1\ \exists\ x_1 \in S_{\langle E_i\rangle_{i\geq n_1}}\\
&\forall\ n_2\ \exists\ x_2 \in S_{\langle E_i\rangle_{i\geq n_2}} \cdots\\
&\forall\ n_k\ \exists\ x_k \in S_{\langle E_i\rangle_{i\geq n_k}} \text{ so that } d_b\big((e_i)_1^k, (x_i)_1^k\big) < 1+\varepsilon.
\end{aligned}$$

This notion can also be understood in terms of countably branching trees of length k on S_X. We let T_k be the tree $T_k = \{(n_1, \dots, n_j) : 1 \leq j \leq k \text{ where } \forall\ i,\ n_i \in \mathbb{N}\}$ ordered by $(n_1, \dots, n_j) \leq (m_1, \dots, m_\ell)$ if $j \leq \ell$ and $n_i = m_i$ for $i \leq j$. Then $T_k(X)$ is the set of all trees on S_X indexed by T_k. Thus $\mathcal{T} \in T_k(X)$ if $\mathcal{T} = \{x(n_1, \dots, n_j) : (n_1, \dots, n_j) \in T_k\} \subseteq S_X$ where the order on $\mathcal{T}$ is that induced by T_k. We call such a $\mathcal{T}$ a *block tree* w.r.t.

(E_i) if all initial nodes of $\mathcal{T}$, $(x(n))_{n\in\mathbb{N}}$ are a block basis of (E_i) as indexed by $\mathbb{N}$ and all successors of a node, $\big(x(n_1,\dots,n_j,n)\big)_{n=1}^{\infty}$, also form a block basis of (E_i).

A tree $\mathcal{T}\in T_k(X)$ is *C-basic* if all branches $(x(n_1,\dots,n_j))_{j=1}^k$ are C-basic sequences.

Definition 1.2. Let $\mathcal{T}=(x(n_1,\dots,n_j))_{T_k}\in T_k(X)$. Assume that $\mathcal{T}$ is C-basic and let $(e_i)_1^k\in\mathcal{M}_k(C)$. $\mathcal{T}$ *converges* to $(e_i)_1^k$ if there exists $\varepsilon_n\downarrow 0$ so that for all n and $n_1\ge n$, $(a_i)_1^k\subseteq[-1,1]^k$ and $n_2,\dots,n_k\in\mathbb{N}$,

$$\left|\Big\|\sum_{j=1}^k a_je_j\Big\|-\Big\|\sum_{j=1}^k a_jx(n_1,\dots,n_j)\Big\|\right|<\varepsilon_n\ .$$

The following proposition follows easily from the definitions above.

Proposition 1.3. *Let (E_n) be a monotone FDD for X. $(e_i)_1^k\in\{X,(E_i)\}_k$ iff there exists $\mathcal{T}\in T_k(X)$, a block tree w.r.t. (E_i), which converges to $(e_i)_1^k$.*

Since we shall be concerned as well with subspaces of X we relativize the above definitions. For an interval $I\subseteq\mathbb{N}$, $P_{\langle E_i\rangle_I}$ is the FDD projection of X onto $\langle E_i\rangle_{i\in I}$.

Definitions 1.4. Let (E_n) be a monotone FDD for X and let $Y\subseteq X$ be a subspace. For $k\in\mathbb{N}$ and $(e_i)_1^k\in\mathcal{M}_k$ we say $(e_i)_1^k\in\{Y,(E_i)\}_k$ if $\forall\ \varepsilon>0$

$$\begin{aligned}
&\forall\ n_1\ \forall\ \varepsilon_1>0\ \exists\ y_1\in S_Y \text{ with } \|P_{\langle E_i\rangle_1^{n_1}}y_1\|<\varepsilon_1\\
&\forall\ n_2\ \forall\ \varepsilon_2>0\ \exists\ y_2\in S_Y \text{ with } \|P_{\langle E_i\rangle_1^{n_2}}y_2\|<\varepsilon_2\\
&\dots\\
&\forall\ n_k\ \forall\ \varepsilon_k>0\ \exists\ y_k\in S_Y \text{ with } \|P_{\langle E_i\rangle_1^{n_k}}y_k\|<\varepsilon_k
\end{aligned}$$

such that $d_b((e_i)_1^k,(y_i)_1^k)<1+\varepsilon$.

Definition 1.5. Let (E_i) be a monotone FDD for X and let $Y\subseteq X$. Let $\mathcal{T}=(y(n_1,\dots,n_j))_{T_k}\in T_k(Y)$. $\mathcal{T}$ is an *asymptotic block tree on Y* w.r.t. (E_i), denoted $\mathcal{T}\in a-T_k(Y)$, if for all $s\in\mathbb{N}$, $\lim_{n\to\infty}\|P_{\langle E_i\rangle_1^s}y(n)\|=0$ and $\lim_{n\to\infty}\|P_{\langle E_i\rangle_1^s}(y(n_1,\dots,n_j,n))\|=0$ for all $(n_1,\dots,n_j)\in T_{k-1}$.

Proposition 1.3 becomes

Proposition 1.6. *Let (E_i) be a monotone FDD for X and let $Y \subseteq X$ and $k \in \mathbb{N}$. $(e_i)_1^k \in \{Y, (E_i)\}_k$ iff there exists a 2-basic tree $\mathcal{T} \in a - T_k(Y)$ which converges to $(e_i)_1^k$.*

Of course such an $(e_i)_1^k$ must be 1-basic. Next we relate the asymptotic structure of Y to a certain blocking of the FDD (E_i) for X. Recall that (F_j) is a *blocking* of (E_i) if there exist integers $0 = p_0 < p_1 < \cdots$ so that for all j, $F_j = \langle E_i \rangle_{i=p_{j-1}+1}^{p_j}$. $(H_j)_1^\infty$ is a *skipped blocking* of (F_j) if there exists integers $r_1 \le s_1 < s_1 + 1 < r_2 \le s_2 < s_2 + 1 < r_3 \le s_3 < \cdots$ so that $H_j = \langle F_i \rangle_{i=r_j}^{s_j}$ for all j. $(x_j) \subseteq X$ is a (*skipped*) *block sequence* w.r.t. (F_j) if there exists a (skipped) blocking (H_j) of (F_j) with $x_j \in H_j$ for all j.

Proposition 1.7. *Let (E_n) be a monotone FDD for X. Let $\varepsilon_n \downarrow 0$. Then there exists a blocking (F_j) of (E_n) so that for all k and all skipped normalized sequences $(x_i)_1^k$ w.r.t. $(F_j)_{j=k}^\infty$,*

$$d_b\Big((x_i)_1^k, \{X, (E_i)\}_k\Big) < 1 + \varepsilon_k \ .$$

Rather than prove this we give the proof of the relativized result. This will require that (E_n) be boundedly complete and hence X is naturally a dual space, the dual of $[(E_n^*)] \subseteq X^*$. When we say here and in the sequel that $Y \subseteq X$ is ω^* closed we mean with respect to the ω^* topology thus generated on X. Proposition 1.7 is proved similarly to 1.8 but the boundedly complete hypothesis is never needed. ω^* convergence of a bounded sequence in X is just coordinatewise convergence w.r.t. (E_n).

Proposition 1.8. *Let (E_n) be a monotone boundedly complete FDD for X and let $Y \subseteq X$ be ω^* closed. Let $\varepsilon_n \downarrow 0$. There exist $\delta_k \downarrow 0$ and a blocking (F_j) of (E_n) with the following property. Given $k \in \mathbb{N}$ if $(y_i)_1^k \subseteq S_Y$ satisfies $\exists\ k \le m_0 < m_1 < \cdots < m_k$ so that for $1 \le j \le k$,*

$$\|(I - P_{\langle F_i \rangle_{m_{j-1}+1}^{m_j - 1}})y_j\| < \delta_k$$

then $d_b((y_i)_1^k, \{Y, (E_i)\}_k) < 1 + \varepsilon_k$.

In other words if $(y_i)_1^k$ is almost a normalized skipped block sequence w.r.t. $(F_j)_k^\infty$ then (y_i) is close to being in $\{Y, (E_i)\}_k$. Proposition 1.8 follows by iterating the following fixed k result.

Proposition 1.9. *Let (E_n) be a monotone boundedly complete FDD for X and let $Y \subseteq X$ be ω^* closed. Then for all $\varepsilon > 0$ and $k \in \mathbb{N}$ there exist $N_1 \in \mathbb{N}$, a blocking (F_j) of $(E_i)_{N_1}^\infty$ and $\delta > 0$ so that if $(y_i)_1^k \subseteq S_Y$ satisfies there exists $0 = m_0 < m_1 < \cdots < m_k$ with*

$$\left\|(I - P_{\langle F_j\rangle_{m_{j-1}+1}^{m_{j-1}}})y_j\right\| < \delta \quad \text{for} \quad j \le k$$

then $d_b((y_i)_1^k, \{Y, (E_i)\}_k) < 1 + \varepsilon$.

Proof. We begin by showing how to deduce the proposition from the

Claim. $\exists\ \delta > 0\ \exists\ N_1 \in \mathbb{N}\ \forall\ y_1 \in S_Y$ with $\|P_{\langle E_i\rangle_1^{N_1}} y_1\| < \delta\ \exists\ N_2 \in \mathbb{N}\ \forall\ y_2 \in S_Y$ with $\|P_{\langle E_i\rangle_1^{N_2}} y_2\| < \delta \cdots\ \exists\ N_k \in \mathbb{N}\ \forall\ y_k \in S_Y$ with $\|P_{\langle E_i\rangle_1^{N_k}} y_k\| < \delta$ one has

$$d_b\Big((y_i)_1^k, \{X, (E_i)\}_k\Big) < \sqrt{1+\varepsilon}\ .$$

Indeed assume the claim. Choose $p_1 > N_1$ so that $\{y \in S_Y : \|(I - P_{\langle E_i\rangle_{N_1}^{p_1}})y\| < \delta\} \neq \emptyset$ and let S_1 be a finite δ'-net for this set. Here $\delta' = \delta'(\varepsilon)$ is specified below. Choose p_2 sufficiently large to satisfy the claim ("$\exists\ N_2, \ldots$") for all $y \in S_1$. Define $F_1 = \langle E_i\rangle_{N_1}^{p_1}$ and $F_2 = \langle E_i\rangle_{p_1+1}^{p_2}$. We choose $S_{1,2}$, a finite δ'-net for

$$\{y \in S_Y : \|(I - P_{\langle F_1, F_2\rangle})y\| < \delta\}\ .$$

Choose p_3 sufficiently large to satisfy the claim ("$\exists\ N_2, \ldots$") for all $y \in S_{1,2}$. Set $F_3 = \langle E_i\rangle_{p_2+1}^{p_3}$.

Notation. If F_i have been defined for all $i \in I$, some interval in $\mathbb{N}$, we let S_I be a finite δ'-net for

$$\{y \in S_Y : \|(I - P_{\langle F_j\rangle_I})y\| < \delta\}\ .$$

We shall say intervals $I_1 < \cdots < I_j$ of integers are *skipped* if

$$\max I_i + 1 < \min I_{i+1} \quad \text{for} \quad i < j\ .$$

Suppose that $F_j = \langle E_i\rangle_{p_{j-1}+1}^{p_j}$ has been defined. Choose p_{j+1} large enough to satisfy the claim for all skipped intervals $I_1 < \cdots < I_\ell$ in $\{1, \ldots, j\}$ for any $y_1, \ldots, y_\ell$ with $y_i \in S_{I_i}$ (using "$\exists\ N_{\ell+1} \ldots$"). Define $F_{j+1} = \langle E_i\rangle_{p_j+1}^{p_{j+1}}$.

Let $(y_i)_1^k$ be as in the statement of Proposition 1.9 w.r.t. the blocking (F_j) of $(E_i)_{N_1}^\infty$ just constructed. Thus for some sequence $I_1 < \cdots < I_k$ of skipped intervals we have

$$\|(I - P_{\langle F_j \rangle_{I_\ell}}) y_\ell\| < \delta \;\text{ for }\; \ell \le k \ .$$

For $\ell \le k$ choose $z_\ell \in S_{I_\ell}$ with $\|z_\ell - y_\ell\| < \delta'$. ¿From our construction using the claim we have

$$d_b\Big((z_i)_1^k, \{Y, (E_i)\}_k\Big) < \sqrt{1+\varepsilon} \ .$$

If δ' is sufficiently small (standard perturbation theory depending on k and ε) then $d_b((y_i)_1^k, (z_i)_1^k) < \sqrt{1+\varepsilon}$. The proposition follows: $d_b((y_i)_1^k, \{Y, (E_i)\}_k) < 1+\varepsilon$.

Before proving the claim we require a lemma.

Lemma 1.10. *Let W be a Banach space, $C \ge 1$ and let $\mathcal{T} \in T_k(W)$ be a C-basic tree. Then there exists a subtree $\mathcal{T}'$ of $\mathcal{T}$ order isomorphic to $\mathcal{T}$,*

$$\mathcal{T}' = \{w(n_1, \dots, n_j) : (n_1, \dots, n_j) \in T_k\}$$

which is convergent (to some normalized C-basic sequence).

Proof. Let $\mathcal{T} = \{x(n_1, \dots, n_j) : (n_1, \dots, n_j) \in T_k\}$. Recall that for $M \in [\mathbb{N}]^\omega$, a subsequence of $\mathbb{N}$, $[M]^k = \{(m_1, \dots, m_k) : m_i \in M \text{ and } m_1 < \cdots < m_k\}$. Let $\varepsilon_n \downarrow 0$. Since $\mathcal{M}_k(C)$ is a compact metric space we can cover it by finitely many closed subsets of diameter $< \varepsilon_1$. Thus by Ramsey's theorem there exists $M \in [\mathbb{N}]^\omega$ and one of these closed sets $K_1 \subseteq \mathcal{M}_k(C)$ so that for all $(m_j)_1^k \in [M]^k$, $(x(m_1, \dots, m_j))_{j=1}^k \in K_1$. Thus we have obtained a subtree order isomorphic to $\mathcal{T}$ so that for all $(e_i)_1^k \in K_1$ and any branch $(w_i)_1^k$, $d_b((w_i)_1^k, (e_i)_1^k) < 1 + \varepsilon_1$.

Relabel this tree so it is indexed again by T_k. Let the elements be again called $(x(n_1, \dots, n_j))_{T_k}$. We leave alone the part $(x(1, n_2, \dots, n_j))_{T_{k-1}}$ and apply the above argument to the remaining part of the tree (which still belongs to $T_k(W)$) and a covering of K_1 by closed sets of diameter $< \varepsilon_2$. We continue in this manner ultimately obtaining the tree which we label $(w(n_1, \dots, n_j))_{T_k}$ so that any branch of the tree $\mathcal{T}' = (w(n_1, \dots, n_j))_{j=1}^k$ belongs to K_{n_1}. Since $K_1 \supseteq K_2 \supseteq \cdots$ are closed and $\operatorname{diam} K_n \to 0$, $\bigcap_1^\infty K_n = \{(e_i)_1^k\}$. Clearly $\mathcal{T}'$ converges to $(e_i)_1^k$. $\square$

Proof of the claim. If false then

$$\forall\, \delta > 0\ \forall\, N_1\ \exists\, y_1 \in S_Y\ ,\ \|P_{\langle E_i\rangle_1^{N_1}} y_1\| < \delta$$

$$\forall\, N_2\ \exists\, y_2 \in S_Y\ ,\ \|P_{\langle E_i\rangle_1^{N_2}} y_2\| < \delta$$

$$\dots$$

$$\forall\, N_k\ \exists\, y_k \in S_Y\ ,\ \|P_{\langle E_i\rangle_1^{N_k}} y_k\| < \delta$$

yet $d_b((y_i)_1^k, \{Y,(E_i)\}_k) \geq \sqrt{1+\varepsilon}$.

Fix $\delta > 0$. By the above we can find a tree $\mathcal{T} = \{y(n_1,\dots,n_j) : (n_1,\dots,n_j) \in T_k\} \in T_k(Y)$ so that for all s and all $(n_1,\dots,n_j)$, $j < k$

$$\overline{\lim_{n\to\infty}} \Big\| P_{\langle E_i\rangle_1^s} y(n_1,\dots,n_j,n) \Big\| \leq \delta$$

and for all $(n_1,\dots,n_k)$

$$d_b\Big((y(n_1,\dots,n_j))_{j=1}^k\ ,\ \{Y,(E_i)\}_k \Big) \geq \sqrt{1+\varepsilon}\ .$$

Since for all s, $\overline{\lim}_{n\to\infty} \|P_{\langle E_i\rangle_1^s} y(n)\| \leq \delta$ using that Y is ω^* closed we may choose $y(n_i) \xrightarrow{\omega^*} y_0 \in Y$ with $\|y_0\| \leq \delta$. We then repeat this argument at the next level to the successors of each $y(n_i)$ and so on. Ultimately thus pruning our tree but leaving behind an isomorphic subtree we see that we may assume without loss of generality each $y(n_1,\dots,n_j) = y_0(n_1,\dots,n_{j-1}) + z(n_1,\dots,n_j)$ where $\|y_0(n_1,\dots,n_{j-1})\| \leq \delta$, both $y_0(n_1,\dots,n_{j-1})$ and $z(n_1,\dots,n_{j-1})$ belong to Y and ω^*-$\lim_{n\to\infty} z(n_1,\dots,n_{j-1},n) = 0$. Let

$$w(n_1,\dots,n_j) = \frac{z(n_1,\dots,n_j)}{\|z(n_1,\dots,n_j)\|}\ .$$

Then $(w(n_1,\dots,n_j)_{T_k} \in a - T_k(Y,(E_i))$.

Passing to a subtree we may assume that each branch is 2-basic and hence also (Lemma 1.10) that this tree converges to some $(e_i)_1^k$, which of course is 1-basic. ¿From Proposition 1.6, $(e_i)_1^k \in \{Y,(E_i)\}_k$. ¿From $\|y(n_1,\dots,n_j) - z(n_1,\dots,n_j)\| \leq \delta$ we obtain that

$$\|y(n_1,\dots,n_j) - w(n_1,\dots,n_j)\| \leq \frac{2\delta}{1-\delta}\ .$$

Thus by a perturbation argument choosing $\delta = \delta(\varepsilon)$ sufficiently small we obtain that $d_b((y(n_1,\dots,n_j))_{j=1}^k, (e_j)_1^k) < \sqrt{1+\varepsilon}$ for n_1 large. This is a contradiction. $\square$

Propositions 1.7 and 1.8 have infinite analogues. Let $T_\omega(X)$ be all trees on S_X indexed by $T_\omega = \{(n_1,\dots,n_j) : j\in\mathbb{N},\ n_i\in\mathbb{N} \text{ for } i\le j\}$. Block trees w.r.t. an FDD for X and the relevatized version $\mathcal{T}\in a-T_\omega(Y)$ for $Y\subseteq X$ are defined as above.

We need a definition before stating the infinite version. Let C be a class of normalized infinite 1-basic sequences. If $(x_i^k)_{i=1}^\infty$ is a sequence of normalized 1-basic sequences and (x_i) is a normalized basic sequence we say $(x_i^k)_{i=1}^\infty \to (x_i)$ if for all n, $\lim_k d_b((x_i^k)_{i=1}^n,(x_i)_{i=1}^n)=0$. C is *closed* if whenever $(x_i^k)_{i=1}^\infty\in C$ for all k and $(x_i^k)_{i=1}^\infty\to(x_i)_{i=1}^\infty$, then $(x_i)\in C$. Note that if C is closed then every sequence in C has a convergent subsequence to an element of C. ¿From this it follows that for $\varepsilon>0$, C_ε is also closed where

$$C_\varepsilon \equiv \{(x_i)_1^\infty : (x_i) \text{ is a normalized 1-basic sequence with } d_b((x_i),C)\le 1+\varepsilon\}\ .$$

Theorem 1.11. *Let (E_n) be a monotone FDD for X and let C be a closed class of infinite normalized 1-basic sequences. Suppose that for all block trees w.r.t. (E_n),*

$$\mathcal{T} = (x(n_1,\dots,n_j))_{T_\omega}\in T_\omega(X)\ ,$$

there exists an infinite branch $(x(n_1,\dots,n_j))_{j=1}^\infty\in C$. Then for all $\varepsilon>0$ there exists a blocking (F_n) of (E_n) with the following property. If $(x_i)_1^\infty$ is any skipped normalized sequence w.r.t. $(F_n)_2^\infty$ then there exists $(y_i)_1^\infty\in C$ with $d_b((x_i)_1^\infty,(y_i)_1^\infty)<1+\varepsilon$.

Proof. Every $\bar N=(N_i)_1^\infty\in[\mathbb{N}]^\omega$ determines a blocking $(F_i^{\bar N})_{i=1}^\infty$ of (E_i) via $F_j^{\bar N} = \langle E_i\rangle_{i=N_{j-1}+1}^{N_j}$ (take $N_0=0$). The notation $(x_i^{\bar N})_{i=1}^\infty$ shall mean that $x_i^{\bar N}\in S_{F_{2i}^{\bar N}}$ for $i\in\mathbb{N}$.

Let $\varepsilon>0$. We define

$$\mathcal{A} = \{\bar N\in[\mathbb{N}]^\omega :\ \text{for all } (x_i^{\bar N}), (x_i^{\bar N})\in C_{2\varepsilon})\ .$$

Note that if $\bar N\notin\mathcal{A}$ then there exists $(x_i^{\bar N})\notin C_{2\varepsilon}$. In particular since $C_{2\varepsilon}$ is closed there exists $k\in\mathbb{N}$ so that if $(y_i)_1^\infty$ satisfies $y_i=(x_i^{\bar N})$ for $i\le k$ then $(y_i)\notin C_{2\varepsilon}$. Hence $\mathcal{A}$ is closed in $2^{\mathbb{N}}$ and so Ramsey's theorem ([E] or [O]) applies. We obtain $\bar N\in[\mathbb{N}]^\omega$ so that either $[\bar N]^\omega\subseteq\mathcal{A}$ or $[\bar N]^\omega\cap\mathcal{A}=\emptyset$.

Suppose first that $[\bar N]^\omega\subseteq\mathcal{A}$. Every blocking of $(F_i^{\bar N})$ is equal to $(F_i^{\bar M})$ for some $\bar M\in[\bar N]^\omega$. It follows that if (x_i) is any skipped normalized sequence w.r.t. $(F_i^{\bar N})_2^\infty$ then there exist $\bar{M} \in [\bar{N}]^\omega$ and $(x_i^{\bar{M}})$ so that $(x_i^{\bar{M}}) = (x_i)$. In particular $(x_i) \in C_{2\varepsilon}$ and the theorem follows.

It remains to show that $\mathcal{A} \cap [\bar{N}]^\omega = \emptyset$ is impossible. We shall show that if $\mathcal{A} \cap [\bar{N}]^\omega = \emptyset$ then there exists a block tree $\mathcal{T} \in T_\omega(X)$ w.r.t. (E_i) so that no branch of $\mathcal{T}$ is in C_ε. This will contradict our hypothesis and complete the proof. For each $\bar{M} \in [\bar{N}]^\omega$ there exists $(x_i^{\bar{M}}) \notin C_{2\varepsilon}$. Fix this choice of $(x_i^{\bar{M}})$.

We first consider a simplified situation. Let $\bar{N} = (N_i)$. Suppose that whenever $k \in \mathbb{N}$, $\bar{M} = (M_i)$, $\bar{L} = (L_i)$ belong to $[\bar{N}]^\omega$ and $M_i = L_i$ for $i \le 2k$ then $x_i^{\bar{M}} = x_i^{\bar{L}}$ for $i \le k$. It is then easy to construct $\mathcal{T}$ with the desired properties. Set $x(1) = x_1^{\bar{N}}$. Choose $\bar{M} = (N_2, N_3, \dots)$ and let $x(2) = x_1^{\bar{M}}$. Then let $x(3) = x_1^{\bar{M}}$ for $\bar{M} = (N_3, N_4, \dots)$ and so on. This defines the first level of $\mathcal{T}$. Set $x(1,1) = x_2^{\bar{N}}$. Let $x(1,2) = x_2^{\bar{M}}$ where $\bar{M} = (N_1, N_2, N_4, N_5, \dots)$. Let $x(1,3) = x_2^{\bar{M}}$ for $\bar{M} = (N_1, N_2, N_5, N_6, \dots)$ and so on. The $x(n,m)$'s are defined by copying this pattern. The general elements of $\mathcal{T}$ are obtained likewise. The $\mathcal{T}$ thus constructed is indeed a block tree in $T_\omega(X)$ w.r.t. (E_i). Moreover every branch of $\mathcal{T}$ is a sequence $(x_i^{\bar{M}})$ for some $\bar{M} \in [\bar{N}]^\omega$ and so no branch lies in $C_{2\varepsilon}$.

We copy this idea for the general case. Let $\varepsilon_i \downarrow 0$ rapidly (specified below). By Ramsey's theorem there exist $x(1) \in S_{F_2^{\bar{N}}}$ and $\bar{N}_1 \in [(N_3, N_4, \dots)]^\omega$ so that if $\bar{M} = (N_1, N_2, \bar{\bar{M}})$ where $\bar{\bar{M}} \in [\bar{N}_1]^\omega$ then $\|x(1) - x_1^{\bar{M}}\| \le \varepsilon_1$. This argument is repeated for $\bar{N}$ replaced by $\bar{L} = (N_2, N_3, \dots)$ to get $x(2) \in S_{F_2^{\bar{L}}}$ and $\bar{N}_2 \in [(N_4, N_5, \dots)]^\omega$ so that if $\bar{M} = (N_2, N_3, \bar{\bar{M}})$ where $\bar{\bar{M}} \in [\bar{N}_2]^\omega$ then $\|x(2) - x_1^{\bar{M}}\| \le \varepsilon_1$. In a similar manner we construct all $x(n)$.

To obtain the $x(1,n)$'s we repeat this first level argument. To begin we let $\bar{N}_1 = (N_3^1, N_4^1, \dots)$ and choose (by Ramsey) a subsequence $\bar{N}_{11} \in [(N_5^1, N_6^1, \dots)]^\omega$ and $x(1,1) \in S_{F_2^{(N_1, N_2, \bar{N}_1)}}$ so that if $\bar{M} = (N_1, N_2, N_3^1, N_4^1, \bar{\bar{M}})$ where $\bar{\bar{M}} \in [\bar{N}_{11}]^\omega$ then $\|x(1,1) - x_2^{\bar{M}}\| \le \varepsilon_2$. The tree $\mathcal{T}$ is constructed in this manner.

$\mathcal{T}$ has the property (from the construction) that if $((n_1, \dots, n_j))_{j=1}^\infty$ is any branch of T_ω then there exists $\bar{M} = (M_i) \in [\bar{N}]^\omega$ so that for all k,

$$\|x_k^{\bar{M}} - x(n_1, \dots, n_k)\| \le \varepsilon_k \ .$$

Thus if $(x(n_1, \dots, n_k))_{k=1}^\infty \in C_\varepsilon$ we obtain, provided $\varepsilon_j \downarrow 0$ sufficiently fast, that $(x_k^{\bar{M}}) \in C_{2\varepsilon}$, a contradiction. $\square$

Remark 1.12. One can also give a proof of Theorem 1.11 that imitates the proof of Proposition 1.8. As in the proof of that result it suffices to prove that "$\exists\ N_1\ \forall\ x_1 \in S_{\langle E_i\rangle_{N_1}^\infty}\ \exists\ N_2\ \forall\ x_2 \in S_{\langle E_i\rangle_{N_2}^\infty} \dots$ we have $(x_i)_1^\infty \in C$." This sentence is understood as follows. Consider a two player game in whihc player (I) chooses N_1, then player (II) choose $x_1 \in S_{\langle E_i\rangle_{N_1}^\infty}$, (I) chooses N_2 and so on. Player (I) wins if $(x_i)_1^\infty \in C$. Otherwise (II) wins. The meaning of "$\exists\ N_1 \dots$" above is that (I) has a winning strategy. The negation "$\forall\ N_1\ \exists\ x_1 \in S_{\langle E_i\rangle_{N_1}^\infty}\ \forall\ N_2 \dots (x_i)_1^\infty \notin C$" means that (II) has a winning strategy.

It follows from Martin's theorem that Borel games are determined [M] that either (I) or (II) has a winning strategy. But if (II) has a winning strategy it is easy to construct a block tree $\mathcal{T}$ with no branches in C.

Similarly we obtain

Proposition 1.13. *Let (E_n) be a monotone boundedly complete FDD for X and let $Y \subseteq X$ be a ω^* closed subspace. Let C be a class of normalized basic sequences such that if $\mathcal{T} \in a - T_\omega(Y)$ w.r.t. (E_i) then some branch of $\mathcal{T}$ is in C. Then for all $\varepsilon > 0$ there exists a blocking (F_n) of (E_n) and $\delta_k \downarrow 0$ so that if $(y_i)_1^\infty \subseteq S_Y$ satisfies $\exists\ 1 \leq m_0 < m_1 < \cdots$ with $\|(I - P_{\langle F_i\rangle_{m_{j-1}+1}^{m_j-1}})y_i\| < \delta_j$ for all j then $d_b((y_i)_1^\infty, C) < 1 + \varepsilon$.*

§2. Indices

We define the Szlenk index of a separable Banach space X and another index which we call the H-index and make some connections between them. The latter index is defined in terms of the asymptotic structure in the setting where X has an FDD or is a subspace of a space with an FDD.

Definition 2.1. The Szlenk Index

Let B be a separable Banach space and let $X = B^*$. Thus (B_X, ω^*) is a compact metric space. Let $0 < \varepsilon < 1$. Let $S_0(B,\varepsilon) = B_X$. If $S_\alpha(B,\varepsilon)$ has been defined for $\alpha < \omega_1$ we let

$$S_{\alpha+1}(B,\varepsilon) = \Big\{x : \exists\ (x_n) \subseteq S_\alpha(B,\varepsilon) \text{ with } \omega^*\text{-}\lim_{n\to\infty} x_n = x \text{ and } \varliminf_{n\to\infty} \|x_n - x\| \geq \varepsilon\Big\}\ .$$

If $\alpha < \omega_1$ is a limit ordinal we set

$$S_\alpha(B,\varepsilon) = \bigcap_{\beta<\alpha} S_\beta(B,\varepsilon)\ .$$

Szlenk's original index [S] was defined somewhat differently. However by Rosenthal's ℓ_1 theorem [R] the two indices are equivalent if B contains no isomorph of ℓ_1. Furthermore

$$\sup_{\varepsilon>0}\{\alpha : S_\alpha(B,\varepsilon) \neq \emptyset\} < \omega_1$$

if and only if $X = B^*$ is separable.

We will say that B has *finite Szlenk index* if for all $\varepsilon > 0$ there exists $k \in \mathbb{N}$ with $S_k(B,\varepsilon) = \emptyset$. There is a natural relation between this index and trees on $X = B^*$ (see also [AJO]).

Proposition 2.2. *Let B be a separable Banach space and $X = B^*$. Let $\varepsilon > 0$, $k \in \mathbb{N}$ and $x_0 \in S_{k+1}(B,\varepsilon)$.*

Then there exists a tree $\{x(n_1,\dots,n_j) : (n_1,\dots,n_j) \in T_k\} \subseteq X$ so that

(1) $\omega^*\text{-}\lim x(n) = x_0$

(2) $\omega^*\text{-}\lim_{n\to\infty} x(n_1,\dots,n_j,n) = x(n_1,\dots,n_j)$ *for all* $(n_1,\dots,n_j) \in T_{k-1}$

(3) $\underline{\lim}\, \|x(n) - x_0\| \geq \varepsilon$

(4) $\underline{\lim}_{n\to\infty} \|x(n_1,\dots,n_j,n) - x(n_1,\dots,n_j)\| \geq \varepsilon$ *for all* $(n_1,\dots,n_j) \in T_{k-1}$.

By taking the difference tree of the above tree as we did in the previous section we obtain the following.

Proposition 2.3. *Let $X = B^*$ be a separable dual, $k \in \mathbb{N}$, $\varepsilon > 0$ and assume $S_{k+1}(B,\varepsilon) \neq \emptyset$.*

Then there exists a tree $(z(n_1,\dots,n_j))_{T_k} \subseteq 2B_X$ with for all $(n_1,\dots,n_j)$, $j < k$,

(1) $\underline{\lim}_{n\to\infty} \|z(n_1,\dots,n_j,n)\| \geq \varepsilon$

(2) $\omega^*\text{-}\lim_{n\to\infty} z(n_1,\dots,n_j,n) = 0$

(3) *For all* $(n_1,\dots,n_k)$,

$$\Big\| \sum_{j=1}^{k} z(n_1,\dots,n_j) \Big\| \leq 2\ .$$

This leads us to make the following definitions.

Definition 2.4. Let (x_i) be a basic sequence (of possibly finite length). Let $0 < \varepsilon < 1$. The *strong index* of (x_i) is

$$SI((x_i), \varepsilon) = \sup\{k : \exists\ (a_i)_1^k \text{ with } \varepsilon \leq |a_i| \leq 1 \text{ for } i \leq k \text{ and a normalized block basis } (y_i)_1^k \text{ of } (x_i) \text{ so that } \|\textstyle\sum_1^k a_i y_i\| \leq 1\}.$$

We then use this to define an index based upon the strong index of the asymptotic structure of a space.

Definition 2.5. Let X have a monotone FDD (E_n) and let $Y \subseteq X$ and $\varepsilon > 0$. $H(Y, (E_i), \varepsilon) = \sup\{SI((e_i)_1^k, \varepsilon) : k \in \mathbb{N} \text{ and } (e_i)_1^k \in \{Y, (E_i)\}_k\}$.

As noted in [MMT] it is easy to see that if $(x_i)_1^n$ is a normalized block basis of $(e_i)_1^k \in \{Y, (E_i)\}_k$ then $(x_i)_1^n \in \{Y, (E_i)\}_n$. Thus we have

Proposition 2.6. *Let (E_n) be a monotone FDD for X and let $Y \subseteq X$ and $\varepsilon > 0$. Then $H(Y, (E_i), \varepsilon) = \sup\{k : \exists\ (e_i)_1^k \in \{Y, (E_i)\}_k$ and $(a_i)_1^k \subseteq [\varepsilon, 1]$ with $\|\sum_1^k a_i e_i\| \leq 1\}$.*

Remark 2.7. Our next result yields that the Szlenk index of a space B with separable dual Y is finite iff the H-index of Y w.r.t. a certain FDD is finite as well. Recall that if $B^* = Y$ is separable then B is a quotient of a space with a shrinking basis [DFJP]. It follows that Y is a subspace of a space X with a boundedly complete basis and moreover the ω^* topology on Y induced by B agrees with the relative ω^* topology on Y obtained by regarding the space X as $[(e_n^*)]^*$ where (e_n) is the boundedly complete basis for X. ω^*-convergence in this setting of a bounded sequence is just coordinatewise convergence. For convenience in calculations we take the basis in question or more generally an FDD to be bimonotone.

Proposition 2.8. *Let (E_n) be a bimonotone boundedly complete FDD for X and let $Y = B^*$ be a ω^* closed subspace. Let $\varepsilon > 0$.*

(a) $S_{k+1}(B, \varepsilon) \neq \emptyset \Longrightarrow H(Y, (E_i), \varepsilon/2) \geq k$

(b) $H(Y, (E_i), \varepsilon) \geq k \Longrightarrow S_k(B, \varepsilon/2) \neq \emptyset$.

Proof. Let $\varepsilon > 0$. Suppose that $S_{k+1}(B, \varepsilon) \neq \emptyset$. Let $\overline{\mathcal{T}} = (z(n_1, \ldots, n_j))_{T_k} \subseteq 2B_Y$ be the tree given by Proposition 2.3. Define $w(n_1, \ldots, n_j) = \frac{z(n_1, \ldots, n_j)}{\|z(n_1, \ldots, n_j)\|}$ and let $\mathcal{T} =$

$(w(n_1,\dots,n_j)$ be the corresponding tree. Clearly $\mathcal{T}\in a-T_k(Y)$ and by pruning we may assume all branches are 2-basic. Thus we may assume by Lemma 1.10 that $\mathcal{T}$ converges to some $(e_i)_1^k\in\{Y,(E_i)\}_k$. Note that for all $(n_1,\dots,n_k)$,

$$\|\sum_1^k \|z(n_1,\dots,n_j)\|w(n_1,\dots,n_j)\|\le 2$$

by (3) of Proposition 2.3.

In other words for arbitrarily large n and $\varepsilon'<\varepsilon$ we can find some branch of $\mathcal{T}$ and coefficients all exceeding ε' so that the norm of the ensuing sum does not exceed 2. It follows that $H(Y,(E_i),\varepsilon/2)\ge k$ which proves a).

Next let $H(Y,(E_i),\varepsilon)\ge k$. Thus there exists $(e_i)_1^k\in\{Y,(E_i)\}_k$ and $(a_i)_1^k\subseteq[\varepsilon,1]$ with $\|\sum_1^k a_ie_i\|\le 1$. By Proposition 1.6 there exists $\mathcal{T}=(w(n_1,\dots,n_j))_{T_k}\in a-T_k(Y)$ which converges to $(e_i)_1^k$.

Let $y(n_1,\dots,n_j)=\sum_{i=1}^j a_iw(n_1,\dots,n_i)$. By the convergence of $\mathcal{T}$ to $(e_i)_1^k$ we may assume that $\|y(n_1,\dots,n_j)\|<2$ for all $(n_1,\dots,n_j)\in T_k$. Moreover for $j<k$,

$$\underline{\lim}_{n\to\infty}\|y(n_1,\dots,n_j)-y(n_1,\dots,n_j,n)\|\ge\varepsilon$$

and $\omega^*\text{-}\lim_{n\to\infty}y(n_1,\dots,n_j,n)=y(n_1,\dots,n_j)$. It follows that $S(Y,\varepsilon/2)\ge k$. Indeed $\frac{y(n_1,\dots,n_j)}{2}\in S_{k-j}(B,\varepsilon/2)$ for $1\le j\le k$ and so $0=\omega^*\text{-}\lim\frac{y(n)}{2}$ belongs to $S_k(B,\varepsilon/2)$. $\square$

Proposition 2.9. *Let (E_i) be a bimonotone FDD for X and let $Y\subseteq X$. Let $0<\varepsilon<1$. Then*

(a) $H(Y,(E_i),\varepsilon')\le H(Y,(E_i),\varepsilon)$ *if* $\varepsilon'\ge\varepsilon$.

(b) $H(Y,(E_i),\varepsilon^2)\le[H(Y,(E_i),\varepsilon)+1]^2$

Proof. We need only prove (b). Let $H(Y,(E_i),\varepsilon)=k$. Assume $(e_i)_1^{(k+1)^2}\in\{Y,(E_i)\}_{k+1}$ is such that there exist $(a_i)_1^{(k+1)^2}\subseteq[\varepsilon,1]$ with $\|\sum_1^{(k+1)^2}a_ie_i\|\le 1$. For $1\le j\le k+1$ define $x_j=\frac{1}{b_j}\sum_{i=(j-1)(k+1)+1}^{j(k+1)}a_ie_i$ to be norm 1.

Since (E_i) is bimonotone we see that $b_i\le 1$ for $i\le k+1$. Also $b_i>\varepsilon$ by the definition of $H(Y,(E_i),\varepsilon)=k$. This uses that x_i is formed as a sum of $k+1$ e_j's with coefficients at least ε^2. Note that $\|\sum_1^{k+1}b_jx_j\|\le 1$. But this contradicts our choice of k. $\square$

Definition 2.10. Let X have a bimonotone FDD (E_i) and let $Y \subseteq X$. We say Y has *finite H-index* if $H(Y,(E_i),\varepsilon) < \omega$ for some (and thus by Proposition 2.9 for all) $0 < \varepsilon < 1$.

In the terminology of [HOR] a space B with separable dual Y has finite Szlenk index for all $\varepsilon > 0$ iff $\|\cdot\| : (B_Y, \omega^*) \to \mathbb{R}$ is Baire-1/2. See [HOR] and [KL] for more on the general theory of Baire-1/2 functions.

Remark 2.11. One can define the previous concepts using the weak rather than the ω^* topology. This was done by Huff [H] who attributes the idea to Bourgain. Thus the weak index of X would be given by

$$W_{\alpha+1}(X,\varepsilon) = \{x : \exists\ (x_n) \subseteq W_\alpha(X,\varepsilon)\ ,\ x_n \xrightarrow{\omega} x \text{ and } \underline{\lim}\, \|x_n - x\| \geq \varepsilon\}\ .$$

Of course in the reflexive case, $X = B^*$ we get that for all $\varepsilon > 0$, $W(X,\varepsilon) \equiv \sup\{\alpha : W_\alpha(X,\varepsilon) \neq \emptyset\} < \omega$ iff B has finite Szlenk index.

The notion of *weak asymptotic structure* could also be defined in terms of trees. For $Y \subseteq X$, X having an FDD(E_n), a normalized basic sequence $(e_i)_1^k \in w - \{Y,(E_i)\})k$ if there exists $\mathcal{T} = (y(n_1,\dots,n_j))_{T_k} \in a - T_k(Y)$ w.r.t. (E_i) so that $\mathcal{T}$ converges to $(e_i)_1^k$ and so that for all $(n_1,\dots,n_j) \in T_{k-1}$, $\omega\text{-}\lim_n y(n_1,\dots,n_j,n) = 0$. Of course the weak asymptotic structure could differ from the asymptotic structure but some of the properties of asymptotic structure do still hold in this setting. We state one such result.

Proposition 2.12. *Let X have an FDD (E_i) and let $Y \subseteq X$. Assume that Y does not contain an isomorph of ℓ_1. Let $(e_i)_1^k \in w - \{Y,(E_i)\}$, and let $(y_i)_1^m$ be a normalized block basis of $(e_i)_1^k$. Then $(y_i)_1^m \in w - \{Y,(E_i)\}_m$.*

This follows easily from the following

Lemma 2.13. *Let $\mathcal{T}$ be a tree in B_Y which is order isomorphic to T_k. Assume Y does not contain ℓ_1 and that the initial nodes of $\mathcal{T}$ are weakly null and all successors of a given node in $\mathcal{T}$ are weakly null. Then there exists a subtree $\mathcal{T}' = (y(n_1,\dots,n_j))_{T_k}$ of $\mathcal{T}$ which is order isomorphic to $\mathcal{T}$ and satisfies $\omega\text{-}\lim_n y_m = 0$ whenever $y_m = \sum_{j=1}^k y(n_1^m,\dots,n_j^m)$ for some $(n_1^m,\dots,n_k^m) \in T_k$ with $n_1^m = m$.*

Proof. This can be deduced from the arguments in [K]. $\square$

§3. Lower ℓ_p estimates

Proposition 3.1. *Let (e_i) be a bimonotone basic sequence with $SI((e_i),1/2)\equiv n_0<\infty$. Then there exists $p=p(n_0)\in(1,\infty)$ so that if $(x_i)_1^m$ is any block basis of (e_i) then*

$$\Big\|\sum_1^m x_i\Big\| \geq \frac{1}{2}\Big(\sum_1^m \|x_i\|^p\Big)^{1/p} .$$

Remark 3.1. This lemma is known. It follows from proofs of similar results given in [Ja1] or in [J1]. In the latter the result is presented in an unconditional setting for disjoint blocks but the same proof works in our setting. We choose to present our own proof. The idea of the proof is used for a later result.

Proof. The proof of Proposition 2.6 also yields that $SI((e_i),1/4)\leq [SI((e_i),1/2)+1]^2$. Let $n=4n_0+1$ and choose $p\in(1,\infty)$ with $2^p=n$. We may assume $\|e_i\|=1$ for all i. If (x_i) is a block basis of (e_i) then $SI((e_i),1/2)\geq SI((x_i),1/2)$ so it suffices to prove that for all $(a_i)_1^m\in S_{\ell_p^m}$ that $\|\sum_1^m a_ie_i\|\geq 1/2$.

If this were false choose such an $(a_i)_1^m\in S_{\ell_p^m}$ with $\|\sum_1^m a_ie_i\|<1/2$. Assume m is minimal with this property, (i.e., that such a sequence $(a_i)_1^m$ exists). By the fact that (e_i) is bimonotone, $|a_i|<1/2$ for $i\leq m$. Choose n_1 minimal with $\sum_1^{n_1}|a_i|^p\geq (1/2)^p$. Then choose $n_2>n_1$ minimal so that $\sum_{n_1+1}^{n_2}|a_i|^p\geq(1/2)^p$ and so on until obtaining $n_k<m$ with $\sum_{n_k+1}^m|a_i|^p\leq(1/2)^p$. It follows that

$$\Big(\sum_{n_j+1}^{n_{j+1}}|a_i|^p\Big)^{1/p}\in\Big[\frac{1}{2},2^{1/p}\cdot\frac{1}{2}\Big]\quad\text{for}\quad 0\leq j<k$$

(taking $n_0=0$). Thus $(1-(\frac{1}{2})^p)^{1/p}\leq\frac{1}{2}2^{1/p}k^{1/p}$ which implies that $k>\frac{1}{2}(1-\frac{1}{n})n>\frac{n}{4}$. Set $x_j=\sum_{n_{j-1}+1}^{n_j}a_ie_i$ for $1\leq j\leq k$. By the minimality of m and the fact that $(\sum_{n_{j-1}+1}^{n_j}|a_i|^p)^{1/p}\geq\frac{1}{2}$ we have that $\|x_j\|\geq\frac{1}{4}$. Thus $SI((e_i),\frac{1}{4})\geq k>\frac{n}{4}$. This contradicts $n>4SI((e_i),\frac{1}{4})$. $\square$

Definition 3.2. Let (E_n) be an FDD and $p<\infty$. (E_n) is *block p-Besselian* if there exists $c>0$ so that whenever (x_i) is a block sequence of (E_n),

$$\|\sum x_i\|\geq c\Big(\sum\|x_i\|^p\Big)^{1/p}$$

(E_n) is *skipped block p-Besselian* if the above holds for all skipped sequences of (E_n).

Definition 3.3. Let (E_n) be an FDD and let $p < \infty$. (E_n) is *asymptotically block p-Besselian* if there exists $c > 0$ so that whenever $k \in \mathbb{N}$ and $(x_i)_{i=1}^k$ is a block sequence of $(E_n)_{n=k}^\infty$ then $\|\sum_1^k x_i\| \ge c(\sum_1^k \|x_i\|^p)^{1/p}$.

(E_n) is *asymptotically skipped* block p-Besselian if the above holds for all skipped sequences $(x_i)_{i=1}^k$ of $(E_n)_{n=k}^\infty$.

Proposition 3.4. *Let (E_n) be an FDD which is asymptotically block p-Besselian for some $p < \infty$. Then (E_n) is block q-Besselian for all $q > p$.*

Proof. We may assume that (E_n) is bimonotone. Suppose that $c > 0$ is such that for all k and all block sequences $(x_i)_1^k$ of $(E_n)_k^\infty$,

$$\Big\|\sum_1^k x_i\Big\| \ge c\Big(\sum_1^k \|x_i\|^p\Big)^{1/p} .$$

Let $q > p$. Choose K so large that

$$(*) \qquad cK^{-1}\Big(\frac{K^q}{2} - 1\Big)^{1/p} > 1 .$$

Let $n_0 \in \mathbb{N}$ with $n_0 > K^q + 1$.

Claim. If $(x_i)_1^s$ is a block sequence of $(E_j)_{n_0}^\infty$ then $\|\sum_1^s x_i\| \ge K^{-1}(\sum_1^s \|x_i\|^q)^{1/q}$.

If the claim is true the result follows. Assume the claim is false. Then there exists a normalized block sequence $(e_i)_1^s$ of $(E_j)_{n_0}^\infty$ and scalars $(a_i)_1^s$ with $\sum_1^s |a_i|^q = 1$ and $\|x\| < K^{-1}$ for $x = \sum_1^s a_i e_i$. Furthermore we may assume s is minimal so that such a situation arises. As in the proof of Proposition 3.1 we may write $x = \sum_{i=1}^{N+1} x_i$ where $x_i = \sum_{j=n_{i-1}+1}^{n_i} a_j e_j$ is the shortest vector (after x_{i-1}) with $\|x_i\|_{\ell_q} \ge K^{-1}$ for $i \le N$ and $\|x_{N+1}\|_{\ell_q} < K^{-1}$. Note that $\|x_i\|_{\ell_q} \le K^{-1}2^{1/q}$ for $i \le N$ since $|a_j| < \frac{1}{K}$ by the bimonotone property and the fact that $\|x\| \le K^{-1}$.

Also

$$1 \ge \Big(\sum_{i=1}^N \|x_i\|_{\ell_q}^q\Big)^{1/q} \ge K^{-1}N^{1/q}$$

and so $N \le K^q$. Furthermore

$$\sum_{i=1}^{N+1} (K^{-1}2^{1/q})^q \ge \sum_{i=1}^{N+1} \|x_i\|_{\ell_q}^q = 1$$

and so $2(N+1)K^{-q} \geq 1$ which yields that

$$N^{1/p} > \left(\frac{K^q}{2} - 1\right)^{1/p} .$$

By the minimality of s we have that

$$\|x_i\| \geq K^{-1}\|x_i\|_{\ell_q} \geq K^{-2} \quad \text{for} \quad i \leq N .$$

Combining these with our hypothesis and $(*)$ we have that

$$\begin{aligned} \|x\| &\geq c\Big(\sum_{i=1}^{N} \|x_i\|^p\Big)^{1/p} \geq cK^{-2}N^{1/p} \\ &> K^{-1}\left[cK^{-1}\Big(\frac{K^q}{2} - 1\Big)^{1/p}\right] > K^{-1} \end{aligned}$$

which is a contradiction. □

§4. Blockings in spaces of finite index

In this section we focus on spaces X having an FDD and finite H-index. We prove that the FDD can be blocked to yield certain lower ℓ_p estimates for some $p < \infty$.

Theorem 4.1. *Let (E_n) be an FDD for X.*

a) *If X is of finite H-index w.r.t. (E_n) then there exists $p \in [1,\infty)$ and a blocking (F_j) of (E_n) which is skipped block p-Besselian.*

b) *If X is of finite H-index w.r.t. (E_n) and (E_n) is boundedly complete then there exists a blocking (H_j) of (E_n) and $p \in [1,\infty)$ so that (H_j) is block p-Besselian.*

Proof. a) follows directly from our work thus far. Let (F_n) be the blocking of (E_n) given by Proposition 1.7 for a suitable $\varepsilon_n \downarrow 0$ rapidly. It follows that there exists $n_0 \in \mathbb{N}$ so that if for all k if $(x_i)_1^k$ is a normalized skipped sequence of $(F_n)_k^\infty$ then $SI((x_i)_1^k, 1/2) \leq n_0$. Hence by Propositions 3.1 and 3.4 there exist $p < \infty$ so that (F_n) is skipped block p-Besselian.

To prove part b) we need a trick of W.B. Johnson [J2]. We give the proof because we need a generalization in the next section.

Lemma 4.2. *Let (E_n) be a boundedly complete FDD for X. Let $\varepsilon_n \downarrow 0$. Then there exist integers $0 = n_0 < n_1 < \cdots$ so that if $x = \sum x_j \in S_X$, $x_j \in E_j$ for all j, then for all j there exists $i_j \in (n_{j-1}, n_j]$ so that $\|x_{i_j}\| < \varepsilon_j$.*

Proof. It suffices to show that $\forall\ m\ \forall\ \varepsilon > 0\ \exists\ n > m$ so that if $x = \sum x_i \in S_X$ with $x_i \in E_i$ then there exists $j \in (m,n]$ with $\|x_j\| < \varepsilon$. If not then $\forall\ n\ \exists\ x^n = \sum x_j^n \in S_X$ with $x_j^n \in E_j$ and $\|x_j^n\| \geq \varepsilon$ for all $j \in (m,n]$. Choose a subsequence (x^{n_k}) of (x^n) with $x_j^{n_k} \xrightarrow{k\to\infty} x_j \in E_j$ for all j. Thus $\|x_j\| \geq \varepsilon$ for $j > m$ and $\sup_\ell \|\sum_1^\ell x_i\| < \infty$. This contradicts that (E_j) is boundedly complete. $\square$

Proof of b). Let $\varepsilon_n \downarrow 0$ rapidly. Let (F_j) and p be as a). Let $0 = n_0 < n_1 < \cdots$ be given by Lemma 4.2 and define $H_j = \langle F_i \rangle_{n_{j-1}+1}^{n_j}$. Let $x = \sum x_i = \sum y_i$ with $x \in S_X$, $x_i \in F_i$ and $y_i \in H_j$ for all i,j. For each j choose $i_j \in (n_{j-1}, n_j]$ with $\|x_{i_j}\| < \varepsilon_j$. Set

$$z_j = \sum_{i=i_{j-1}+1}^{i_j} x_i \qquad (i_0 = 0)\ .$$

Then (x_j) is a skipped sequence w.r.t. (F_j) and so $\|\sum z_j\| \geq \frac{1}{2}(\sum \|z_j\|^p)^{1/p}$. Furthermore $\|\sum z_j\| \leq \|x\| + \sum_j \|x_{i_j}\| < 2$ (for suitably small ε_j's). Also for all j, $\|y_j\| \leq \|z_j\| + \|x_{i_j}\| + \|z_{j+1}\|$. Thus

$$\Big(\sum \|y_j\|^p\Big)^{1/p} \leq \Big(\sum_j (\|z_j\| + \|z_{j+1}\| + \varepsilon_j)^p\Big)^{1/p} \leq 9\ ,$$

for suitably small ε_j's. $\square$

Corollary 4.3. *Let (E_n) be a boundedly complete FDD for X and assume that X is of finite H-index w.r.t. (E_n). Then there exist $1 \leq p < \infty$, a blocking (H_j) of (E_n) and an equivalent norm $|\cdot|$ on X so that if $(x_j) \subseteq X$ is any block sequence of (H_j) then $|\sum x_j| \geq (\sum |x_j|^p)^{1/p}$. In particular X can be renormed to have the ω^*-UKK property.*

Proof. Let (H_j) and p be as in b). Define for $x \in X$, $|x| = \sup\{(\sum \|x_i\|^p)^{1/p} : x = \sum x_i$ where (x_i) is a block sequence w.r.t. $(H_j)\}$. $\square$

This result partially solves the problem raised by Huff [H]. If $X = B^*$ is reflexive and B has an FDD and is of finite Szlenk index then X can be renormed to have the UKK. Thus given $\varepsilon > 0$ there exists $\delta(\varepsilon) > 0$ so that if $(x_n) \subseteq B_X$, ω-$\lim_n x_n = x$ and $\|x_n - x_m\| \geq \varepsilon$ for $n \neq m$ then $\|x\| \leq 1 - \delta(\varepsilon)$. In the next section we remove the assumption that X have an FDD.

§5. Blockings and subspaces of finite index

We relativize the results of the previous section to subspaces of X. First we need an extension of Lemma 4.2.

Lemma 5.1. *Let X have a bimonotone boundedly complete FDD (F_n) and let $Y \subseteq X$ be ω^* closed. $\forall\, \varepsilon > 0$ $\forall\, m \in \mathbb{N}$ $\exists\, n > m$ such that if $y = \sum_1^\infty y_i \in B_Y$ with $y_i \in F_i$ for all i then $\exists\, k \in (m, n]$ with*

a) $\|y_k\| < \varepsilon$
b) $\mathrm{dist}(\sum_{i=1}^{k-1} y_i, Y) < \varepsilon$.

Proof. We proved a) in Lemma 4.4. In particular we can find $m = n_0 < n_1 < n_2 < \cdots$ so that if $x = \sum_1^\infty x_i \in B_X$, $x_i \in F_i$ for all i, then for all j there exists $k_j \in (n_{j-1}, n_j]$ with $\|x_j\| < \varepsilon$. Thus if b) fails then for all j there exists $y^j = \sum_i^j \in B_X$, $y_i^j \in F_i$ for all i, so that for all $s < j$ there exists $k(j,s) \in (n_{s-1}, n_s]$ with $\|y^j_{k(j,s)}\| < \varepsilon$ and $\mathrm{dist}(\sum_{i=1}^{k(j,s)-1} y_i^j, Y) \geq \varepsilon$. Passing to a subsequence of (y^j) we may assume that $\lim_{j\to\infty} y_i^j \equiv x_i \in F_i$ exists for all i and that $k(j,s) \equiv k(s)$ for $s \leq j$. By the fact that $\|\sum_1^\ell x_i\| \leq 1$ for all ℓ and the boundedly complete property of (E_n) we have $x = \sum_1^\infty x_i \in B_X$. Also $y^j \xrightarrow{\omega^*} x$ and so $x \in Y$. Thus

$$\mathrm{dist}\left(Y, \sum_{i=1}^{k(s)-1} x_i\right) \xrightarrow{s\to\infty} 0\ ,\ \text{a contradiction.} \qquad \square$$

Theorem 5.2. *Let (E_n) be a bimonotone boundedly complete FDD for X and let Y be a ω^* closed subspace whose predual has finite Szlenk index. There exists a blocking (H_j) of (E_n) and $p = p(H(Y, (E_i), 1/2) \in [1, \infty)$ so that $|\cdot|$ is an equivalent norm on Y where for $x \in X$,*

$$|x| = \sup\left\{ \left(\sum_1^\infty \|x_i\|^p \right)^{1/p} : \exists \text{ a blocking } (G_i) \text{ of } (H_i) \text{ with } x_i \in G_i \text{ for all } i \text{ and } x = \sum_1^\infty x_i \right\}$$

Of course $|x|$ could be infinite for some $x \in X$. We are only claiming it is an equivalent norm on Y. Before proving the theorem we give some corollaries.

Corollary 5.3. *Let Y be a separable dual space whose predual has finite Szlenk index. Then there exist a Banach space Z with a boundedly complete FDD (H_j) and $p \in [1,\infty)$ so that Y embeds isomorphically (norm and ω^*) into Z and $\|\sum z_j\| \geq (\sum \|z_j\|^p)^{1/p}$ for all block bases (z_j) of (H_j).*

Proof. As discussed earlier we may assume that Y is a ω^* closed subspace of a space X having a boundedly complete FDD [DFJP]. We let (H_j) and $|\cdot|$ be as in Theorem 5.2. Define Z to be the completion of $\langle (H_j) \rangle$ under $|\cdot|$. Y embeds into Z by the theorem. □

Corollary 5.4. *Let B be a separable Banach space of finite Szlenk index ($S(B,\varepsilon) < \infty$ for all $\varepsilon > 0$). Then B admits an equivalent ω^*-UKK norm.*

Proof. Let $Y = B^*$. By [DFJP] there exists a space W having a shrinking basis and a quotient map $Q : W \to B$. Thus $Q^* : Y \to W^*$ embeds Y as a ω^* closed subspace of W^*. Moreover Q^* is a ω^* isomorphism as well. Q^*Y has finite index w.r.t. the dual basis of W, a boundedly complete basis for W^*. We then apply Corollary 5.3 obtaining Z, (H_j) and p as in the conclusion of 5.3. Thus we have renormed Y by $\|\cdot\|$ so as to preserve its ω^* topology as the dual space of X in such a manner that Y has a ω^*-UKK norm. The latter comes from the lower ℓ_p estimate for Z. This then defines an equivalent norm on B by regarding Y as the dual of B. Thus for $x \in B$,

$$\|x\| = \sup\{\langle x, y\rangle : y \in B_Y\} \ .$$

So $(Y, \|\cdot\|) = (B, \|\cdot\|)^*$ and $(B, \|\cdot\|)$ has the ω^*-UKK. □

Proof of Theorem 5.2. It will suffice to produce such a p and a blocking (H_j) of (F_j) so that if (G_j) is any further blocking of (H_j) and $y = \sum y_j \in Y$ with $y_j \in G_j$ for all j then

$$\|y\| \geq \frac{1}{13}\Big(\sum \|y_j\|^p\Big)^{1/p} \ .$$

Let $H(Y,(E_i),1/4) \equiv n_0 < \infty$. Let $\varepsilon > 0$ be small (specified below). By Proposition 1.7 there exist $\delta > 0$ and a blocking (F_j) of (E_i) so that if $(y_i)_1^{n_0+1} \subseteq S_Y$ satisfies: there exists a skipped blocking $(G_j)_1^{n_0+1}$ of $(F_j)_2^\infty$ so that

$$\|(I-P_{G_j})y_j\| < \delta \quad \text{for} \quad j \le n_0+1$$

then $d_b((y_i)_1^{n_0+1}, \{Y,(E_i)\}_{n_0+1}) < 1+\varepsilon/2$.

Lemma 5.5. *There exist $p = p(n_0)$ and $\bar\varepsilon_n \downarrow 0$ so that if (G_j) is any skipped blocking of (F_j) and $(y_j) \subseteq Y$ satisfies $\|(I-P_{G_j})y_j\| \le \bar\varepsilon_j \|y_j\|$ for all j then*

$$\Big\|\sum y_j\Big\| \ge \frac{1}{4}\Big(\sum \|y_j\|^p\Big)^{1/p} .$$

Proof. We may assume that each $y_j \ne 0$. By taking $\bar\varepsilon_j$ sufficiently small this will insure that $(y_j/\|y_j\|)$ is $1+\varepsilon$-close to being bimonotone. We claim that

$$SI((y_i),1/2) \le n_0+1 .$$

Indeed if $(z_i)_1^{n_0+2}$ is any normalized block basis of (y_i) then, $d_b((z_i)_2^{n_0+2}, \{Y,(E_i)\}_{n_0+1}) < 1+\varepsilon$ from our initial assumptions on (F_j) and standard perturbation arguments which of course impose restrictions on $(\bar\varepsilon_n)$. Thus $SI((z_i)_2^{n_0+1},1/2) \le n_0$ which gives the claim. If (y_i) were bimonotone we would have the desired estimate by Proposition 3.1, with a lower constant of $1/2$. Since (y_i) is only nearly bimonotone the $1/2$ becomes $1/4$ by taking ε sufficiently small. □

Continuing with the proof of 5.2 we let $\bar{\bar\varepsilon}_n \downarrow 0$ rapidly (specified below) and choose, using Lemma 5.1, integers $0 = m_0 < m_1 < \cdots$ so that for all $y = \sum y_i \in B_Y$ with $y_i \in F_i$ for all i, given $j \in \mathbb{N}$ there exists $i_j \in (m_{j-1}, m_j]$ with $\|y_{i_j}\| < \bar{\bar\varepsilon}_j$ and $d(\sum_1^{i_j-1} y_i, Y) < \bar{\bar\varepsilon}_j$. Define $H_j = \langle F_i\rangle_{(m_{j-1},m_j]}$ for $j \in \mathbb{N}$. Let (G_j) be any further blocking of H_j, say $G_j = \langle H_i\rangle_{(k_{i-1},k_i]}$ for some $0 = k_0 < k_1 < \cdots$. Let $y = \sum y_i \in S_Y$ with $y_i \in F_i$ for all i.

For each j choose $i_j \in (m_{j-1}, m_j]$ with $\|y_{i_j}\| < \bar{\bar\varepsilon}_j$ and $d(\sum_1^{i_j-1} y_i, Y) < \bar{\bar\varepsilon}_j$. Take $i_0 = 0$ and $z_j = \sum_{i=i_{j-1}+1}^{i_j-1} y_i$. Then $d(z_1, Y) < \bar{\bar\varepsilon}_1$ and for $j > 1$

$$d(z_j, Y) < \bar{\bar\varepsilon}_j + \bar{\bar\varepsilon}_{j-1} + \bar{\bar\varepsilon}_{j-1} < 3\bar{\bar\varepsilon}_{j-1} .$$

Choose $w_j \in Y$ with $\|z_j - w_j\| < 3\bar{\bar\varepsilon}_{j-1}$ for $j > 1$ and $\|z_1 - w_1\| < \bar{\bar\varepsilon}_1$.

We claim that $(\sum \|w_j\|^p)^{1/p} \le 5$ and so ($\bar{\bar{\varepsilon}}_j$ sufficiently small) $(\sum \|z_j\|^p)^{1/p} \le 6$. Indeed set $\bar\varepsilon_1 = \bar\varepsilon_0 = \bar\varepsilon_{-1}$ and let $I = \{i : \|z_i\| \ge 2\bar{\bar{\varepsilon}}_{i-2}\}$. If $j \notin I$ then $\|w_j\| \le 3\bar{\bar{\varepsilon}}_{j-1} + 2\bar{\bar{\varepsilon}}_{j-2}$. If $j \in I$ then ($\bar{\bar{\varepsilon}}_j$ suitably small)

$$\|(I - P_{\langle F_i\rangle_{(i_{j-1},i_j)}})w_j\| < \bar\varepsilon_j\|w_j\| \ .$$

Thus by Lemma 5.5

$$\Big(\sum_I \|w_i\|^p\Big)^{1/p} \le 4\Big\|\sum_I w_i\Big\| \le 4\Big(1 + \sum(3\bar{\bar{\varepsilon}}_{j-1} + 2\bar{\bar{\varepsilon}}_{j-2})\Big) < 5$$

if $\bar{\bar{\varepsilon}}_j$ are suitably small. The claim follows.

Finally let $y = \sum b_j$ where $b_j \in G_j$. Then $\|b_j\| \le \|z_{j-1} + y_{i_{j-1}} + z_j + y_{i_j}\|$. This yields

$$\begin{aligned}\Big(\sum \|b_j\|^p\Big)^{1/p} &\le 2\Big(\sum \|z_j\|^p\Big)^{1/p} + 2\Big(\sum \bar\varepsilon_j^p\Big)^{1/p}\\ &\le 13\end{aligned}$$

for suitably small $\bar\varepsilon_j$'s. □

In the case where Y is reflexive we obtain the following.

Theorem 5.6. *Let Y be a reflexive space whose predual has finite index. Then Y can be renormed to have the UKK property. Moreover the UKK modulus is of power type.*

Indeed by a result of Zippin [Z] we can regard $Y \subseteq X$ where X is reflexive and has a basis. The result then follows from our previous results and the following proposition.

Proposition 5.7. *Let Z be the space constructed in Corollary 5.3.*

a) *If X has a basis then Z has a basis.*

b) *If X is reflexive then Z is reflexive.*

Proof. a) is clear. To see b) we first recall that the lower ℓ_p estimate on blocks of (H_j) gave that (H_j) was boundedly complete. It remains to show that (H_j) is shrinking. If not there exists a $|\cdot|$ normalized block basis (x_j) of (H_j) so that for all $(a_i) \subseteq \mathbb{R}^+$ with $\sum a_i = 1$ we have $|\sum a_i x_i| > 1/2$.

Choose $\delta > 0$ so that $\delta^{p-1} < 6^{-p}$. Let $(a_i) \subseteq [0,\delta)$ with $\sum a_i = 1$ and using the definition of the norm $|\cdot|$ choose a blocking (G_j) of (H_j) so that for $x = \sum a_i x_i$, $\frac{1}{2^p} <$

$\sum_j \|P_{G_j}x\|^p$. We assume $P_{G_j}x \neq 0$ for all j. We consider each block G_j and if necessary split it into at most 3 blocks as follows. If $P_{G_j}x_i \neq 0$ for at most one i we do nothing. Otherwise let i be maximal so that $P_{G_j}x_i \neq 0$ and $P_{G_{j+1}}x_i \neq 0$ as well. (If no such x_i exists we do nothing.) We split G_j into two blocks, the first acting on $\langle x_1, \dots, x_{i-1}\rangle$ and the second on x_i. We also make a corresponding split if necessary according to the minimal i so that $P_{G_j}x_i \neq 0$ and $P_{G_{j-1}}x_i \neq 0$.

We let (R_j) be the new blocking. It follows that if $P_{R_j}x_i \neq 0$ for more than one i, then for any such i $P_{R_{j'}}x_i = 0$ for $j \neq j'$. Also for such j, $\|P_{R_j}x\| \leq \sum_{I_j} a_i$ where $I_j = \{i : P_{R_j}x_i \neq 0\}$. Due the the splitting of (G_j) our above estimate becomes

$$\frac{1}{2^p} \leq 3^p \sum_j \|P_{R_j}x\|^p \ .$$

Let $J = \{j : P_{R_j}x_i \neq 0$ for more than one $i\}$ then $\sum_{j\notin J} \|P_{R_j}x\|^p \leq \sum_{i\notin \cup I_j} a_i^p$ since $|x_i| = 1$ for all i. Now we claim that for some $j \in J$, $\|P_{R_j}x\| \geq \delta$. Indeed if not we have

$$\begin{aligned}\frac{1}{6^p} &\leq \sum_{j\in J} \|P_{R_j}x\|^p + \sum_{j\notin J} \|P_{R_j}x\|^p \\ &< \delta^{p-1} \sum_{j\in J} \|P_{R_j}x\| + \sum_{i\notin \cup I_j} a_i^p \\ &< \delta^{p-1} \Bigg[\sum_{i\in \cup I_j} a_i + \sum_{i\notin \cup I_j} a_i\Bigg] = \delta^{p-1} \ .\end{aligned}$$

But this is impossible by our choice of δ.

Hence for such an x, $\|x\| \geq \|P_{R_j}x\| \geq \delta$. But this contradicts that (x_i) is necessarily weakly null for $\|\cdot\|$. Indeed one can always find $(a_i) \subseteq [0,\delta)$ with $\|\sum a_i x_i\| < \delta$ and $\sum a_i = 1$. □

§6. Dual results and further remarks

We next explore dual concepts to those above which will ultimately lead to upper ℓ_q estimates for some $q > 1$.

To say that a basic sequence (x_i) has finite strong index is equivalent to saying that we have uniform lower ℓ_p estimates on all block bases for some $p < \infty$. Thus given K there exists n so that if $(y_i)_1^n$ is a normalized block basis of (x_i) then $\|\sum_1^n y_i\| > K$. In other words (x_i) does not admit (what might be called) ℓ_∞^n+ uniformly as block bases.

The dual notion is an ℓ_1^n+ index.

Definition 6.1. Let (x_i) be a basic sequence and $\varepsilon > 0$.

$$I^+((x_i),\varepsilon) = \sup\Big\{k : \exists \text{ a normalized block basis } (y_i)_1^k \text{ of } (x_i) \text{ satisfying}$$
$$\Big\|\sum_1^k a_i y_i\Big\| \ge \varepsilon \sum_1^k a_i \text{ if } (a_i)_1^k \subseteq \mathbb{R}^+\Big\}\ .$$

It is easy to see that $I^+(x_i) < \infty$ iff there exists $n_0 \in \mathbb{N}$ so that for all normalized block bases $(y_i)_1^{n_0}$ of (x_i) we have $\|\sum_1^{n_0} y_i\| < n_0/2$. Also by James' result that ℓ_1 is not distortable [Ja2] adapted to the ℓ_1^+ situation, $I^+((x_i),\varepsilon) < \infty$ for some $\varepsilon < 1$ iff $I^+((x_i),\varepsilon) < \infty$ for all $\varepsilon < 1$. See [AJO] for more on the I^+ index.

The analog of Proposition 3.1 is

Proposition 6.2. ([Ja1], [J1]) *Let (x_i) be a monotone basis. Suppose that $I^+((x_i),1/2) = n_0 < \infty$. Then there exists $q = q(n_0) > 1$ so that $\|\sum a_i x_i\| \le 6(\sum |a_i|^q)^{1/q}$ for all $(a_i) \subset \mathbb{R}$.*

The same sort of arguments used to prove Theorems 4.1 and 5.2 yield the following. We shall say that if $Y \subseteq X$ where X has an FDD (E_n) then Y is of *finite asymptotic I^+-index* w.r.t. (E_n) if for some $0 < \varepsilon < 1$ (hence all $\varepsilon < 1$)

$$\sup\Big\{I^+((e_i)_1^k,\varepsilon) : (e_i)_1^k \in \{Y,(E_i)\}_k, k \in \mathbb{N}\Big\} < \infty\ .$$

Theorem 6.3. *Let (E_n) be an FDD for X*

a) *If X is of finite asymptotic I^+-index w.r.t. (E_n) then there exist $q > 1$, $K < \infty$ and a blocking (F_j) of (E_n) so that for all block sequences (x_i) w.r.t. (F_j), $\|\sum x_i\| \le K(\sum \|x_i\|^q)^{1/q}$.*

b) *If (E_n) is boundedly complete and $Y \subseteq X$ is ω^* closed of finite asymptotic I^+-index w.r.t. (E_n) then there exist $q > 1$, a blocking (H_j) of (E_n) and $K < \infty$ so that if $y \in Y$ with $y = \sum y_j$ where (y_j) is a block sequence w.r.t. (H_j) then $\|y\| \le K(\sum \|y_i\|^q)^{1/q}$.*

In this theorem we do not need to require skipped sequences in a) because the upper estimate results from the separate estimates applied to $\sum x_{2i}$ and $\sum x_{2i-1}$.

The H-index is a sort of ℓ_∞^+-index. Thus it is natural to ask the following question. Suppose X has infinite H-index w.r.t. (E_n). Is c_0 block finitely representable in (E_n)? The answer is not necessarily.

Example 6.4. There exists a space X with a bimonotone basis (b_i) so that for all n there exists $(e_i)_1^n \in \{X, (b_i)\}_n$ with $\| \sum_1^n e_i \| = 1$ yet c_0 is not block finitely representable in (b_i).

Recall T_ω is the countably branching tree of ω levels, i.e.,

$$T_\omega = \{n_1, \dots, n_j) : j \in \mathbb{N} \ , \ n_1, \dots, n_j \in \mathbb{N}\}$$

ordered by extension. X will be the completion of $c_{00}(T_\omega) \equiv \{f : T_\omega \to \mathbb{R} : f$ has finite support$\}$ under a suitable norm. The node basis $(e_\alpha)_{\alpha \in T_\omega}$ given by $e_\alpha(\beta) = \delta_{\alpha\beta}$ will be a normalized bimonotone basis for X when linearly ordered in any manner that is compatible with the tree order on T_ω. Thus if $\alpha < \beta$ in T_ω then $e_\alpha < e_\beta$ in the basis order.

In addition we will have the following properties.

(1) There exists a basis (e_i) so that if $(\alpha_i)_1^n$ is any initial segment of a branch in T_ω then $(e_{\alpha_i})_1^n$ is 1-equivalent to $(e_i)_1^n$. Moreover $\| \sum_1^n e_i \| = 1$.

(2) If $(x_i)_1^n$ is any normalized block basis of (e_α) then $\| \sum_1^n \varepsilon_i x_i \| \geq n/3$ for some choice of $\varepsilon_i = \pm 1$.

Because of the tree structure (1) yields that $(e_i)_1^n \in \{X, (e_\alpha)\}_n$ for all n. (2) yields that c_0 is not block finitely representable in X.

We shall specify a set $\Gamma \subseteq c_{00}(T_\omega)$ and define for $x \in c_{00}(T_\omega)$,

$$\|x\| = \sup\{\langle f, x \rangle : f \in \Gamma\} \ .$$

$f \in \Gamma$ iff f is finitely supported, $f(\alpha) \in \{0, \pm 1\}$ for all α and on any branch of T_ω, f does not take on successive nonzero values of the same sign. Thus if $\alpha < \beta$ in T_ω and $f(\alpha) = 1$ and $f(\gamma) = 0$ for $\alpha < \gamma < \beta$ then $f(\beta) = -1$ or 0.

All the properties of X are now easily verified except for (2) which requires some effort. Let $(x_i)_1^n$ be a normalized block basis of (e_α). Choose $f_i \in \Gamma$ with $\langle f_i, x_i \rangle = 1$ for $i \leq n$. We may suppose that range $f_i =$ range x_i w.r.t. the linearly ordered basis (e_α); the range of $x \in c_{00}(T_\omega)$ is the smallest interval of α's (in the basis ordering) containing the support of x.

Let I_i be the set of initial nodes w.r.t. the tree order in $\operatorname{supp} f_i$. We shall partition I_i into 3 sets I_i^s, I_i^o and I_i^d and write $f_i = f_i^s + f_i^o + f_i^d$ where f_i^s is f_i restricted to

$\{\beta \in T_\omega : \alpha \le \beta$ for some $\alpha \in I_i^s\}$ and so on. We begin with $i = 2$. Let A_1 be the set of terminal nodes (in the tree order) of $\operatorname{supp} f_1$.

$$\begin{aligned} I_2^s &= \{\beta \in I_2 : \exists\ \alpha \in A_1 \text{ with } \alpha < \beta \text{ and } f_1(\alpha) = f_2(\beta)\} \\ I_2^o &= \{\beta \in I_2 : \exists\ \alpha \in A_1 \text{ with } \alpha < \beta \text{ and } f_1(\alpha) = -f_2(\beta)\} \\ I_2^d &= I_2 \setminus (I_2^s \cup I_2^0)\ . \end{aligned}$$

The letters s, o, d represent same, opposite and disjoint.

Choose $g \in \{f_2^s, f_2^o, f_2^d\}$ so that $\langle g, x_2\rangle \ge 1/3$. If $g = f_2^o$ or f_2^d let $\varepsilon_2 = 1$ and $f(2) = f_1 + g$. If $g = f_2^s$ let $\varepsilon_2 = -1$ and $f(2) = f_1 - g$. It follows that $f(2) \in \Gamma$ and

$$\langle f(2), x_1 + \varepsilon_2 x_2\rangle \ge 1 + \frac{1}{3}\ .$$

We continue in this manner using $f(2)$ to partition I_3 into 3 sets and ultimately determine f_3^s, f_3^o, f_3^d and ε_3 etc. The construction yields (2). $\square$

The analogous question for the I^+-index has a similar answer. If (e_i) is the summing basis for $c_0 = X$ then ℓ_1^{n+} belongs to $\{X, (e_i)\}_n$ for all n but ℓ_1 is not block finitely represented in (e_i).

We do not know how to find reflexive examples with these properties.

Problem 6.5. Does there exist a reflexive space with a basis (e_i) having infinite H-index (respectively, infinite I^+-index) yet c_0 (respectively, ℓ_1) is not block finitely represented in (e_i)?

The H-index was defined for a fixed $\varepsilon > 0$. One can vary the ε at each level and obtain a variable H-index. If (E_n) is an FDD for X, $(x_n) \subseteq X$ is bounded and $x \in X$ we write $x_n \rightharpoonup x$ if (x_n) converges to x coordinatewise w.r.t. (E_n). Let $(\varepsilon_i)_1^n \subseteq (0,1)$. $H_0(X, (E_i), (\varepsilon_i)_1^n) = B_X$. For $k < n$ let

$$\begin{aligned} H_{k+1}(X, (E_i), (\varepsilon_i)_1^n) = \{x : \exists\ (x_j) \subseteq H_k(X, (E_i), (\varepsilon_i)_1^n) \text{ with} \\ x_j \rightharpoonup x \text{ and } \underline{\lim} \|x_j - x\| \ge \varepsilon_{k+1}\}\ . \end{aligned}$$

In this notation having finite H-index just says that for all $\varepsilon > 0$ there exists $n \in \mathbb{N}$ with $H_n(X, (E_i), (\varepsilon)_1^n) = \emptyset$.

Definition 6.6. X has *summable H-index* w.r.t. (E_n) if $\exists\, K<\infty\ \forall\, n\ \forall\, (\varepsilon_i)_1^n \subseteq (0,1)$

$$H_n(X,(E_i),(\varepsilon_i)_1^n) \neq \emptyset \Longrightarrow \sum_1^n \varepsilon_i \le K\ .$$

Again there is a connection with trees and the asymptotic structure of X.

Proposition 6.7. *Let (E_n) be an FDD for X. The following are equivalent.*

(a) *X has summable H-index.*

(b) *There exists $K<\infty$ so that for all n and for all $(e_i)_1^n \in \{X,(E_i)\}_n$,*

$$(e_i)_1^n \textit{ is } K\textit{-equivalent to the unit vector basis of } \ell_1^n$$

(c) *There exists a blocking (H_j) of (E_i) which is skipped asymptotic ℓ_1; i.e., for some $K<\infty$ if $(x_i)_1^n$ is a skipped block sequence of $(H_j)_n^\infty$ then*

$$\Big\|\sum x_i\Big\| \ge K^{-1}\sum \|x_i\|\ .$$

Proof. The equivalence of (b) and (c) follows from Proposition 1.7. The equivalence with (a) comes from the following connection between the variable H-index and trees.

Suppose that $H(X,(E_i),(\varepsilon_i)_0^n)\neq\emptyset$. Then, as in the proof of Proposition 2.8, there exists $\mathcal{T}\in a-T_n(X,(E_i))$ which converges to $(e_i)_1^n\in\{X,(E_i)\}_n$ and satisfies $\|\sum_1^n \varepsilon_i' e_i\|\le 1$ for some $\varepsilon_i/2\le\varepsilon_i'\le 1$. If (b) holds then $\sum_1^n\varepsilon_i'\le K$.

Finally assume (a) and let $(e_i)_1^n\in\{X,(E_i)\}_n$. Assume the variable index of X is $\le K$. Let $(\varepsilon_i)_1^n\subseteq(0,1)$ with $\sum_1^n\varepsilon_i>K$. Suppose $\|\sum_1^n\varepsilon_i e_i\|\le 1$. Choose $\mathcal{T}\in a-T_n(X,(E_i))$ that converges to $(e_i)_1^n$. It follows that $H(X,(E_i),(\varepsilon_i)_1^n)\neq\emptyset$ which is a contradiction. Thus $\|\sum_1^n\varepsilon_i e_i\|\le 1$ implies $\sum_1^n\varepsilon_i\le K$. Since $(\pm e_i)_1^n\in\{X,(E_i)\}_n$ we have $\|\sum_1^n\pm\varepsilon_i e_i\|\le 1$ implies $\sum_1^n\varepsilon_i\le K$. Thus $(e_i)_1^n$ is K-equivalent to the unit vector basis of ℓ_1^n. □

These results can also be generalized to a ω^* closed subspace of a space X with a boundedly complete FDD. By Proposition 6.7 Tsirelson's space T [FJ] has summable H-index. There is a (formally) weaker notion than summable index.

Definition 6.8. Let (E_n) be an FDD for X. We say X has *proportional H-index* w.r.t. (E_n) if there exists $K<\infty$ so that for all $0<\varepsilon<1$, $H(X,(E_n),\varepsilon)\le K/\varepsilon$. It is clear that summable index implies proportional index.

Proposition 6.9. *Let (E_n) be a monotone FDD for X and suppose that X has proportional H-index w.r.t. (E_n). Then X has summable H-index w.r.t. (E_n).*

Proof. If not then for all $\varepsilon > 0$ there exist $k \in \mathbb{N}$ and a block tree w.r.t. (E_n), $\mathcal{T} = (x(n_1, \dots, n_j))_{T_k} \in T_k(X)$, which converges to $(e_i)_1^k \in \{X, (E_i)\}_k$ and such that there exists $(a_i)_1^k \subseteq [0, \infty)$ with $\sum_1^k a_i = 1$ and $\| \sum_1^k a_i e_i \| < \varepsilon$. Without loss of generality we may assume that for some $N \in \mathbb{N}$ each $a_i = \frac{n_i}{N}$ for some $n_i \in \mathbb{N}$ and moreover n_i divides N.

Using Proposition 1.7 and pruning $\mathcal{T}$ we may assume that every collection of $j \le kN$ elements of $\mathcal{T}$, suitably ordered, is essentially in $\{X, (E_n)\}_j$. We form a seminormalized block basis of $\mathcal{T}$ as follows. The order will be $(x_1^1, \dots, x_{n_1}^1, x_1^2, \dots, x_{n_2}^2, \dots x_1^k, \dots, x_{n_k}^k)$. The x_j^1's will be each a sum of $\frac{N}{n_1}$ $x(n)$'s with weight $\frac{n_1}{N}$. This will thus involve N of the $x(n)$'s. The x_j^2's will each be a sum of $\frac{N}{n_2}$ $x(n, m)$'s with weight $\frac{n_2}{N}$ each being a successor to one of the $x(n)$'s in the support of the x_j^1's. And so on.

It follows since X has proportional index that for some fixed $c > 0$,

$$\Big\| \sum_{i=1}^k \sum_{j=1}^{n_i} x_j^i \Big\| \ge c \sum_{i=1}^k n_i = cN \ .$$

However if $(m_1, \dots, m_k) \in T_k$ is such that $x(m_1, \dots, m_j) \in \operatorname{supp} x_{\ell(j)}^j$ for some $\ell(1), \dots, \ell(k)$ then

$$\Big\| \sum_{j=1}^k \frac{n_j}{N} x(\ell(1), \dots, \ell(j)) \Big\| \approx \Big\| \sum_1^k a_i e_i \Big\| < \varepsilon \ .$$

Since there are N such "columns" in the tree we obtain from the triangle inequality that

$$\Big\| \sum_{i=1}^k \sum_{j=1}^{n_i} x_j^i \Big\| < \varepsilon N \ .$$

This yields a contradiction. $\square$

The nonreflexive version of our main theorem remains open when one replaces ω^* convergence by ω convergence.

Problem 6.10. [H] If $W(X, \varepsilon) < \infty$ for all $\varepsilon > 0$ can X be given an equivalent UKK norm?

If the answer is no it still may be true in the case where X does not contain ℓ_1.

We have been largely concerned with conditions that permit us to block a given FDD so as to be block p-Besselian for some $p < \infty$ or to embed a space X into such an FDD. One can ask a more pointed question. Given an FDD (E_n) and $p \in [1, \infty)$ what conditions ensure that this FDD can be blocked to be block p-Besselian or block p-Hilbertian or to be an ℓ_p-FDD? More generally given a space X what conditions ensure that X embeds into such an FDD? Such questions were raised in [J2] where the following is proved. Let $X \subseteq L_p$, $1 < p < 2$ and suppose X satisfies

($*$) There exists $K < \infty$ such that every normalized weakly null sequence in X has a subsequence K-equivalent to the unit vector basis of ℓ_p.

Then X embeds into ℓ_p. In [J2] the question is raised as to whether an FDD for a reflexive space X which satisfies ($*$) can be blocked to be an ℓ_p FDD. The answer is no.

Example 6.11. Let $1 < p < \infty$. There exists a reflexive space X with an unconditional basis so that X satisfies ($*$) for $K = 1 + \varepsilon$, $\varepsilon > 0$ arbitrary, and yet the basis for X cannot be blocked into an ℓ_p FDD.

Fix $1 < q < p$. We define $X = (\sum X_n)_{\ell_p}$ where each X_n is given as follows. X_n will be the completion of $c_{00}(T_n)$ under the norm

$$\|x\|_n = \sup\left\{ \left(\sum_{i=1}^{m} \|x|_{\beta_i}\|_q^p \right)^{1/p} : (\beta_i)_1^m \text{ are disjoint segments in } T_n \right\}.$$

A *segment* β is just an interval in T_n, i.e., for some $\alpha_1 \leq \alpha_2$ in T_n, $\beta = \{\gamma : \alpha_1 \leq \gamma \leq \alpha_2\}$. Clearly the node basis $(e_i)_{\alpha \in T_n}$ given by $e_\alpha(\gamma) = \delta_{\alpha\gamma}$ is a 1-unconditional basis for X_n. Furthermore the unit vector basis of ℓ_q^n is 1-equivalent to $(e_{\alpha_i})_1^n$, if $(\alpha_i)_1^n$ is any branch of T_n and so belongs to $\{X\}_n$ for all n. Thus the basis for X cannot be blocked to be even an asymptotic ℓ_p-FDD. Also each X_n is isomorphic to ℓ_p and thus X is reflexive.

It remains to show that if (x_j) is a normalized weakly null sequence in X and $\varepsilon > 0$ then a subsequence is $1+\varepsilon$-equivalent to the unit vector basis of ℓ_p. By a standard gliding hump argument it suffices to prove this in a fixed X_n. We proceed by induction on n.

For $n = 1$ the result is clear since X_1 is isometric to ℓ_p. Assume the result has been proved for X_{n-1} and let $(x_i)_1^\infty$ be a normalized block basis of the node basis for X_n. If $(k_1, \dots, k_j) \in T_n$ we shall write $x_i(k_1, \dots, k_j)$ rather than $x_i((k_1, \dots, k_j))$. Let $\varepsilon_i \downarrow 0$

rapidly. For $j \in \mathbb{N}$ let P_j be the basis projection of X_n onto $[\{e_{(j,\bar{k})} : \bar{k} = \emptyset \text{ or } \bar{k} \in T_{n-1}\}]$. Passing to a subsequence we may assume that $\lim_{i\to\infty} \|P_j x_i\|_n = a_i$ and from the definition of $\|\cdot\|_n$ we have $(a_i)_1^\infty \in B_{\ell_p}$. Choose $a_0 \geq 0$ so that $(a_0)_1^\infty \in S_{\ell_p}$.

Passing to a subsequence of (x_i) we may assume that there exist integers $1 = N_0 < N_1 < \cdots$ so that

(i) $x_i(j) \neq 0 \Rightarrow j \in [N_i, N_{i+1})$

(ii) $P_j x_i = 0$ for $j \geq N_{i+1}$

(iii) $\Big\| \sum_{j\in[N_i,N_{i+1})} P_j x_i \Big\|_n = \Big(\sum_{j\in[N_i,N_{i+1})} \|P_j x_i\|_n^p \Big)^{1/p}$ is within ε_i of a_0.

(iv) If $j \in [N_i, N_{i+1})$, $i \geq 1$, then if $a_j \neq 0$, $(a_j^{-1} P_j x_\ell)_{\ell > i}$ is $1+\varepsilon_j$-equivalent to the unit vector basis of ℓ_p.

(v) If $j \in [N_0, N_1)$ and $a_j \neq 0$ then $(a_j^{-1} P_j x_\ell)_{\ell=1}^\infty$ is $1+\varepsilon_j$-equivalent to the unit vector basis of ℓ_p.

(vi) $\Big(\sum_{N_1}^\infty a_j^p \Big)^{1/p} < \varepsilon_1$

(vii) If $j \in [N_0, N_1)$ and $a_j = 0$ then $\|P_j x_i\|_n \leq \varepsilon_i$ for all i.

(viii) If $j \in [N_i, N_{i+1})$ and $a_j = 0$ then $\|P_j x_\ell\|_n < \varepsilon_\ell$ for $\ell > i$.

Conditions (iv) and (v) use the induction hypothesis and the fact that for all j, $[\{e_{j,\bar{k})} : \bar{k} \in T_{n-1}\}]$ is isometric to X_{n-1}. Our conditions are sufficient to yield (for suitably small ε_j's) that (x_i) is $1+\varepsilon$-equivalent to the unit vector basis of ℓ_p. We omit the standard yet tedious calculations. □

The example shows that the following question also has a negative answer. Suppose X has a basis so that all spreading models of any normalized block basis are K-equivalent to the unit vector basis of ℓ_p. Can the basis be block to be an asymptotic ℓ_p FDD or even a skipped asymptotic ℓ_p FDD? The following remains open.

Problem 6.12. Let X have a basis (e_i). Let $K < \infty$ and $p \in [1,\infty)$ be such that all spreading models of any normalized block basis of (e_i) are K-equivalent to the unit vector basis of ℓ_p. Is some block basis of X an asymptotic ℓ_p basis?

Using Theorem 1.11 we do get sufficient (and necessary) conditions for an appropriate blocking.

Proposition 6.13. *Let (E_n) be an FDD for X. Let $p \in [1,\infty]$ and let $K < \infty$ so that the following holds.*

For all block trees w.r.t. (E_n), $\mathcal{T} \in T_\omega(X)$ some branch $(x_i)_1^\infty$ satisfies for all $(a_i) \subseteq \mathbb{R}$,

$$\Big\|\sum a_i x_i\Big\| \geq K^{-1}\Big(\sum |a_i|^p\Big)^{1/p}$$

(respectively, $\|\sum a_i x_i\| \leq K(\sum |a_i|^p)^{1/p}$). Then there exists a blocking (F_n) of (E_n) which is skipped block p-Besselian (respectively, block p-Hilbertian). Moreover if (E_n) is boundedly complete one obtains that (F_n) is block p-Besselian.

Similarly one can give a version of Proposition 6.13 for Y a ω^* closed subspace of a space X having a boundedly complete FDD. For example

Proposition 6.14. *Let Y be a ω^* closed subspace of a space with a boundedly complete FDD (E_n). Assume that there exist $K < \infty$ and $p \in [1,\infty)$ so that if $\mathcal{T} \in a - T_\omega(Y)$ w.r.t. (E_n) then some branch $(y_i)_1^\infty$ of $\mathcal{T}$ admits a K-lower ℓ_p estimate. Then Y embeds into a space having a block p-Besselian FDD.*

In regard to the above results we mention the following.

Theorem 6.15. [KW] *Assume X does not contain an isomorph of ℓ_1. Then for all $\varepsilon > 0$ X $1+\varepsilon$-embeds into a space $(\sum E_n)_{\ell_p}$ where the E_n's are finite dimensional iff for all $x \in X$ and all $x_n \stackrel{\omega}{\to} 0$, $\overline{\lim}\,\|x + x_n\| = (\|x\|^p + \overline{\lim}\,\|x_n\|^p)$.*

Theorem 6.16. [GKL] *Assume X does not contain ℓ_1. Then X embeds into c_0 iff for all $\mathcal{T} = (x(n_1,\dots,n_j))_{T_\omega} \subseteq B_X$ with ω-$\lim_n x(n_1,\dots,n_j,n) = 0$ for all $(n_1,\dots,n_j) \in T_j$ there exists a branch $(x_i)_1^\infty$ of $\mathcal{T}$ with $\sup_k \|\sum_1^k x_i\| < \infty$.*

For a space X not containing ℓ_1 the *point of continuity property* (PCP) can be phrased in terms of trees. X fails the PCP iff there exist $\varepsilon > 0$ and a tree $\mathcal{T} = (x(n_1,\dots,n_j))_{T_\omega} \subseteq B_X$ so that $\|x(n_1,\dots,n_j)\| \geq \varepsilon$, ω-$\lim_n x(n_1,\dots,n_j,n) = 0$ and

$$\Big\|\sum_{i=1}^{j} x(n_1,\dots,n_i)\Big\| \leq 1 \quad \text{for all} \quad (n_1,\dots,n_j) \in T_\omega\ .$$

It is shown in [DGK] that if X has an equivalent UKK norm and does not contain ℓ_1 then X has the PCP.

§7. The ω^*-UKK modulus

Let us redefine the modulus for a ω^*-UKK dual space X as follows. Given $\varepsilon > 0$ there exists $\delta = \delta(\varepsilon) > 0$ so that if $(x_n) \subseteq X$, $x \in X$, $\|x + x_n\| \le 1$ and $\|x_n\| \ge \varepsilon$ for all n with $\omega^*\text{-}\lim_{n\to\infty} x_n = 0$ then $\|x\| \le 1 - \delta$.

We have proved that if X (or more properly B where $X = B^*$) has finite Szlenk index then there exists an equivalent norm $\|\cdot\|$ on X (and B) and $p < \infty$ so that for x as above

$$\|x\| \le (1-\varepsilon^p)^{1/p} \sim 1 - \frac{1}{p}\varepsilon^p \text{ for small } \varepsilon\ .$$

So $\delta(\varepsilon) \ge c\varepsilon^p$ for some c.

We examine what can be said about X from knowledge of the ω^*-UKK modulus $\delta(\varepsilon)$. We begin with an easy observation.

Proposition 7.1.

(a) $\ell_1 = c_0^*$ *is* ω^**-UKK with* $\delta(\varepsilon) = \varepsilon$.

(b) *Let* X *be* ω^**-UKK with* $\delta(\varepsilon) \ge c\varepsilon$ *for some* $c > 0$ *and all* $\varepsilon > 0$. *Then every normalized* ω^**-null sequence in* X *admits a subsequence equivalent to the unit vector basis of* ℓ_1.

Proof. (a) is obvious.

(b) The hypothesis yields that if $\omega^*\text{-}\lim x_n = 0$ and $\lambda = \lim \|x + x_n\|$ with $\lim_n \|x_n\| = \varepsilon$ then $\|x\| \le \lambda - c\varepsilon$.

Let (y_n) be normalized ω^*-null in X. Let $\varepsilon_n \downarrow 0$ rapidly. By passing to a subsequence we may assume that for all k and $(a_i)_1^{k+1} \subseteq [-1,1]$, $\ell > k$,

$$\left|\, \Big\|\sum_1^k a_i y_i + a_{k+1} y_\ell\Big\| - \Big\|\sum_1^{k+1} a_i y_i\Big\|\, \right| < \varepsilon_{k+1}\ .$$

Let $\sum_1^{k+1} |a_i| = 1$. Since

$$\lim_{\ell\to\infty} \Big\|\sum_1^k a_i y_i + a_{k+1} y_\ell\Big\| \ge \Big\|\sum_1^k a_i y_i\Big\| + c|a_{k+1}|$$

it follows that

$$\Big\|\sum_1^{k+1} a_i y_i\Big\| \ge \Big\|\sum_1^k a_i y_i\Big\| + c|a_{k+1}| - \varepsilon_{k+1}\ .$$

Iterating the argument we obtain

$$\Big\|\sum_1^{k+1} a_i y_i\Big\| \geq c\sum_1^{k+1} |a_i| - \sum_1^{k+1} \varepsilon_i \geq \frac{c}{2}$$

if $\sum \varepsilon_i < c/2$. $\square$

Actually more can be said.

Remarks 7.2. (1) It follows from [KW] that if $X = B^*$ is as in (b) then B embeds into c_0.

(2) Tsirelson's space T can be renormed for $p > 1$ to have $\delta(\varepsilon) \geq c_p \varepsilon^p$ but of course cannot be renormed to have $\delta(\varepsilon) \geq c\varepsilon$.

(3) Suppose X is as in (b) and X has a boundedly complete FDD, (E_n). Then (E_n) can be blocked into an ℓ_1 FDD for X. This can be deduced either from (1) or from our arguments. More generally if X is a ω^* closed subspace of a space with a boundedly complete FDD (E_n) then there exists a blocking (H_j) of (E_n) so that setting $|x| = \sum \|x_j\|$ for $x = \sum x_j$, $x_j \in H_j$ then X embeds into $\overline{(\langle H_j\rangle, |\cdot|)}$, a space with an ℓ_1-FDD.

Proposition 7.3. *Let Y be a ω^* closed subspace of X, a space with a boundedly complete FDD, (E_n). Assume Y is ω^*-UKK with $\delta(\varepsilon) \geq c\varepsilon^p$ for some $c > 0$, $1 < p < \infty$. Then there exists a blocking (H_j) of (E_n) and a norm $|\cdot|$ on $\langle (H_j)\rangle$ that makes $((H_j), |\cdot|)$ 1-block p-Besselian and so that $|\cdot| \sim \|\cdot\|$ on Y.*

Proof. We may assume (E_n) is bimonotone. ¿From our previous work it suffices to prove that for some $c' > 0$ if $\|y_n\| \geq \varepsilon$, ω^*-$\lim y_n = 0$ and $\lim \|y + y_n\| = \lambda$ for $y, (y_n) \subseteq Y$ then $\|y\|^p \leq \lambda^p - c'\varepsilon^p$.

We present the argument for $p = 2$ where the calculations are simpler.

¿From $\delta(\varepsilon) \geq c\varepsilon^2$ we have $\|\frac{y}{\lambda}\| \leq 1 - c(\frac{\varepsilon}{\lambda})^2$ and so $\|y\| \leq \lambda - \frac{c\varepsilon^2}{\lambda}$. Thus

$$\begin{aligned}\|y\|^2 &\leq \lambda^2 - 2c\varepsilon^2 + c^2\left(\frac{\varepsilon}{\lambda}\right)^2 \varepsilon^2 \\ &\leq \lambda^2 - c\varepsilon^2\end{aligned}$$

since $\varepsilon \leq \lambda$. $\square$

References

[AJO] D. Alspach, R. Judd, E. Odell, *The Szlenk index and local ℓ_1 indices of a Banach space*, in preparation.

[CK] F. Chaatit and M. Khamsi, *Uniform Kadec-Klee property in Banach lattices*, Math. Japonica **43** (1996), 357–364.

[DFJP] W.J. Davis, T. Figiel, W.B. Johnson and A. Pełczyński, *Factoring weakly compact operators*, J. Funct. Anal. **17** (1974), 311–327.

[DGK] S. Dilworth, M. Girardi and D. Kutzarova, *Banach spaces which admit a norm with the uniform Kadec-Klee property.*

[E] E.E. Ellentuck, *A new proof that analytic sets are Ramsey*, J. Symbolic Logic **39** (1974), 163–165.

[En] P. Enflo, *On a problem of Smirnov*, Ark. Mat. **8** (1969), 107–109.

[FJ] T. Figiel and W.B. Johnson, *A uniformly convex Banach space which contains no ℓ_p*, Compositio Math. **29** (1974), 179–190.

[GKL] G. Godefroy, N.J. Kalton and G. Lancien.

[HOR] R. Haydon, E. Odell and H. Rosenthal, *On certain classes of Baire-1 functions with applications to Banach space theory*, Functional Analysis, Springer-Verlag LNM 1470 (1991), 1–35.

[H] R. Huff, *Banach spaces which are nearly uniformly convex*, Rocky Mountain J. Math. **10** (1980), 743–749.

[Ja1] R.C. James, *Super-Reflexive spaces with bases*, Pacific J. Math. **41** (1972), 409–419.

[Ja2] R.C. James, *Uniformly nonsquare Banach spaces*, Ann. of Math. (2) **80** (1964), 542–550.

[J1] W.B. Johnson, *On finite dimensional subspaces of Banach spaces with local unconditional structure*, Studia Math. **51** (1974), 225–240..

[J2] W.B. Johnson, *On quotients of L_p which are quotients of ℓ_p*, Compositio Math. **34** (1977), 69–89.

[KW] N.J. Kalton and D. Werner, *Property (M), M-ideals, and almost isometric structure of Banach spaces.*

[KL] A. Kechris and A. Louveau, *A classification of Baire class 1 functions*, Trans. A.M.S. **318** (1990), 209–236.

[K] H. Knaust, *Array convergence of functions of the first Baire class*, Proc. A.M.S. **112** (1991), 529–532.

[L] G. Lancien, *On uniformly convex and uniformly Kadec-Klee renormings.*

[M] D.A. Martin, *Borel determinacy*, Annals of Math. **102** (1975), 363–371.

[MMT] B. Maurey, V.D. Milman and N. Tomczak-Jaegermann, *Asymptotic infinite-dimensional theory of Banach spaces*, Oper. Theory: Adv. Appl. **77** (1994), 149–175.

[MT] V.D. Milman and N. Tomczak-Jaegermann, *Asymptotic ℓ_p spaces and bounded distortions*, Contemp. Math. **144** (1993), 173–195.

[O] E. Odell, *Applications of Ramsey theorems to Banach space theory*, Notes in Banach spaces (H.E. Lacey, ed.), Univ. of Texas Press, Austin, TX, 1980, pp. 379–404.

[Pi] G. Pisier, *Martingales with values in uniformly convex spaces*, Israel J. Math. **13** (1972), 361–378.

[P] S. Prus, *Nearly uniformly smooth Banach spaces*, Boll. Un. Mat. Ital. **7** (1989), 507–521.

[R] H. Rosenthal, *A characterization of Banach spaces containing ℓ_1*, Proc. Nat. Acad. Sci. U.S.A. **71** (1974), 2411–2413.

[S] W. Szlenk, *The non-existence of a separable reflexive Banach space universal for all separable reflexive Banach spaces*, Studia Math. **30** (1968), 53–61.

[Z] M. Zippin, *Banach spaces with separable duals.*

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