September 2011

Research Statement

Sheng-Chi Liu

1. Overview

My research interest is analytic number theory, especially on modular forms and L-functions. The best well-known example of an L-function is the Riemann zeta function which is defined by

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_{p \text{ prime}} (1 - \frac{1}{p^s})^{-1} \quad \text{for } \mathrm{Re}(s) > 1.$$

The Riemann zeta function encodes a lot of arithmetic information. For example, the divergence of $\zeta(s)$ at $s = 1$ gives that there are infinitely many primes. One of the most famous problems in number theory, *the Riemann hypothesis*, is to understand the distribution of zeros of $\zeta(s)$.

A modular form f on $SL_2(\mathbb{Z})$ is a holomorphic function on the complex upper half plane $\mathbb{H}$ satisfying a transformation law

$$f\left(\frac{az+b}{cz+d}\right) = (cz+d)^k f(z), \quad \text{for all } \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}).$$

It has a Fourier expansion $f(z) = \sum_{n=0}^{\infty} a(n) e^{2\pi i n z}$. For a cusp form f (i.e. $a(0) = 0$), one could associate it to an L-function which is defined by $L(s,f) = \sum_{n=1}^{\infty} \frac{a(n)}{n^s}$ for $\mathrm{Re}(s)$ large enough. The L-function $L(s,f)$ has several properties in common with $\zeta(s)$. For example, it satisfies a functional equation and has analytic continuation to the whole complex plane. Moreover, it encodes rich arithmetic information. My research is to study the L-functions $L(s,f)$ and their convolutions, where f is a modular form on $SL_2(\mathbb{Z})$ or more generally an automorphic form on GL_n.

2. Moments of L-functions and their applications

An L-function does not live in isolation; it is connected to other L-functions in subtle and mysterious ways. A strategy in analytic number theory is to deduce a result about an individual L-function from sufficient information about the family. My current research is to understand the families of L-functions and their applications.

2.1. **Determining cusp forms by central values of L-functions.** Let $L(s) = \sum_{n=1}^{\infty} \frac{a(n)}{n^s}$ be a Dirichlet L-series. If $L(s)$ satisfies certain "*good*" properties, (functional equation, analytic continuation, etc.) then it is believed that $L(s)$ is equal to $L(s,f)$ for some automorphic form f on GL_n. A key step in proving Fermat's Last Theorem was to show that this is true for the Hasse-Weil L-series associated to an elliptic curve, i.e. to show this L-series is modular. The converse theorem of Weil [We], Cogdell and Piatetski-Shapiro [CPS1, CPS2] asserts that the modularity of a L-series can be determined by infinitely many twisted L-series. Assuming the modularity of a GL_2 L-function, Luo and Ramakrishnan [LR] proved that the corresponding cusp form on $SL_2(\mathbb{Z})$ is uniquely determined by the special values of infinitely many $GL(2)\times GL(1)$ L-functions. In [Liu1, Liu2], I have generalized this result to GL_3 in which we consider the $GL(3)\times GL(2)$ L-functions. More precisely, we have the following theorem.

Theorem 1 (Liu, [Liu1, Liu2])**.** *Let f and g be self-dual Hecke-Maass forms for $SL(3,\mathbb{Z})$ of types (ν,ν) and (μ,μ) respectively.*

(1) *If $L\left(\frac{1}{2}, f\times h\right) = L\left(\frac{1}{2}, g\times h\right)$ for all Hecke eigenforms h for $SL_2(\mathbb{Z})$ of weight k, for all $k\equiv 0 \pmod 4$, then $f=g$.*
(2) *If $L\left(\frac{1}{2}, f\times h\right) = L\left(\frac{1}{2}, g\times h\right)$ for all Hecke eigenforms h for $\Gamma_0(q)$ of weight 10, for infinitely many primes q, then $f=g$.*

This theorem is actually an application of knowing the asymptotics of the first moments of twisted L-functions at the critical point, combined with the multiplicity one theorem. In [Liu1, Theorem 2] and [Liu2, Theorem 3], we evaluated asymptotically the average of critical values $L\left(\frac{1}{2}, f\times h\right)$ over h in the family under consideration. In an ongoing project, we investigate an analogous problem for a holomorphic Hilbert modular form.

The vanishing or nonvanishing of an L-function at the critical point could have deep arithmetic meaning. For example the Birch and Swinnerton-Dyer conjecture relates the order of vanishing of an L-function at the critical point to the rank of an elliptic curve. As a consequence of our asymptotic formulas of the first moment of $GL_3\times GL_2$ L-functions, we have the following corollary.

Corollary 2. *Let f be a fixed self-dual Hecke-Maass form for $SL(3,\mathbb{Z})$.*

(1) *For each K large enough, there exists a Hecke eigenform h for $SL_2(\mathbb{Z})$ of weight k, with $K\le k\le 2K$ such that $L(\frac{1}{2}, f\times h)>0$.*
(2) *For each prime q large enough, there exists a Hecke eigenform h for $\Gamma_0(q)$ of weight 10 such that $L(\frac{1}{2}, f\times h)>0$.*

In [Liu3], we proved a result on simultaneous nonvanishing of $GL(2)\times GL(2)$ and $GL(2)$ L-functions at the critical point which is also an application of the first moment of L-functions. The main result in [Liu3] is the following theorem.

Theorem 3 (Liu [Liu3])**.** *Let f be a fixed holomorphic Hecke cusp form for $SL_2(\mathbb{Z})$. For each K large enough, there exists $g\in H_k$ with $K\le k\le 2K$ such that*

$$L\left(\frac{1}{2}, g\right) L\left(\frac{1}{2}, f\otimes g\right)\neq 0.$$

2.2. **The L^2 restriction norms of Saito-Kurokawa lifts.** A Siegel modular form F is a holomorphic function defined on the Siegel upper half plane

$$\mathcal{H}_2 = \left\{ Z = \begin{pmatrix} \tau & z \\ z & \tau' \end{pmatrix} \in Mat_{2\times 2}(\mathbb{C}) : \tau,\tau'\in\mathbb{H} \text{ and } \mathrm{Im}(z)^2 < \mathrm{Im}(\tau)\mathrm{Im}(\tau')\right\}$$

which satisfies a transformation law under the sympletic group $Sp_4(\mathbb{Z})$. When we restrict F to $z=0$, $F\begin{pmatrix} \tau & 0 \\ 0 & \tau' \end{pmatrix}$ is a modular form on $SL_2(\mathbb{Z})\times SL_2(\mathbb{Z})$ in τ and in τ'. In joint work with Matt Young, we study the L^2 norm of this restriction.

Given a modular form f of weight $2k$ on $SL_2(\mathbb{Z})$, one can associate it to a Siegel modular form F_f of weight $k+1$, called the Saito-Kurokawa lift of f. A beautiful result of Ichino [Ic] links the behavior of restricted L^2 norm to a central value of a $GL_3 \times GL_2$ L-function in the case that F_f arises as a Saito-Kurokawa lift of a Hecke eigenform. If we let $N(F_f)$ be the normalized L^2 norm, then as a consequence of Ichino's formula, we have

$$N(F_f) = \frac{\mathrm{vol}(SL_2(\mathbb{Z})\backslash\mathbb{H})^{-2}}{\mathrm{vol}(Sp_4(\mathbb{Z})\backslash\mathcal{H}_2)^{-1}} \frac{24\pi}{L(3/2,f)L(1,\mathrm{sym}^2 f)} \sum_{g\in B_{k+1}} \frac{1}{k} L\left(\tfrac{1}{2}, \mathrm{sym}^2 g\otimes f\right), \tag{2.1}$$

which involves the first moment of $GL_3 \times GL_2$ L-functions. We conjecture that $N(F_f) \sim 2$ as $k \to \infty$. In [LY] we prove that the conjecture is true on average. More precisely, we have

Theorem 4 (Liu and Young [LY])**.** *Suppose that w is a function satisfying*

$$\begin{cases} w \text{ is smooth with compact support on } [K, 2K] \\ |w^{(j)}(x)| \le C_j K^{-j} \text{ for some } C_j > 0, \quad j = 0, 1, 2, \dots. \end{cases}$$

Then as $K \to \infty$,

$$\sum_{k \text{ odd}} w(k) \sum_{f\in B_{2k}} N(F_f) \sim \sum_{k \text{ odd}} w(k) \sum_{f\in B_{2k}} 2.$$

On the other hand, we also fix the weight and study the nonvanishing of the restricted L^2 norm in two ways. One is using the structure of the theory of Jacobi forms while the other is from a lower bound on twisted L-functions. We proved the theorems below.

Theorem 5. *Of the $\dim(S_{2\ell-2}) = \frac{\ell}{6} + O(1)$ Hecke eigenform lifting to Siegel modular forms of weight ℓ under the Saito-Kurokawa correspondence, no more than $\dim(M_{\ell-10}) = \frac{\ell}{12} + O(1)$ have vanishing restricted L^2 norm $N(F_f)$. Consequently, for such f we have that there exists a Hecke eigenform g of weight ℓ such that $L(1/2, \mathrm{sym}^2 g \otimes f) \neq 0$.*

Theorem 6. *Suppose that for ℓ even large enough and for given f a Hecke eigenform of weight $2\ell - 2$, there exists a fundamental discriminant $D < 0$, $4|D$, such that $|D| \ll \ell^{1-\varepsilon}$ and satisfying $L(\frac{1}{2}, f \otimes \chi_D) \ge |D|^{-100}$. Then $N(F_f) \neq 0$.*

The Saito-Kurokawa lift was generalized to modular forms with level and a formula analgous to (2.1) is known (due to Böcherer, Furusawa and Schulze-Pillot). It would be interesting to establish our theorems for the forms with level. I plan to investigate this in the future.

3. Bergman kernel and mass equidistribution of Hecke eigenforms

In 1994, Rudnick and Sarnak [RS] formulated a conjecture predicting the behavior of Maass-Hecke eigenforms on arithmetic surfaces as their corresponding Laplace eigenvalues tend to infinity. The conjecture is known as the *Arithmetic Quantum Unique Ergodicity* (AQUE) conjecture for modular surfaces. This conjecture was proved by Lindenstrauss [Lin] in the compact case and by Soundararajan [S] in the non-compact case. A holomorphic analog of AQUE in the non-compact case was proved by Holowinsky and Soundararajan [HS]. In [Liu4] I consider an analogous AQUE for holomorphic Hilbert modular forms. The method of Holowinsky and Soundararajan was generalized by Marshall [M] to Hilbert modular forms. However it is still interesting to consider an average over the forms since it links to the Bergman kernel and we can prove a sharp convergence rate.

Let F be a totally real number field of degree n over $\mathbb{Q}$ with ring of integers $\mathcal{O}$ and let $\sigma_1, \sigma_2, \cdots, \sigma_n$ be the real embeddings of F. Then the full Hilbert modular group $\Gamma =$

$SL_2(\mathcal{O})$ can be embedded into $SL_2(\mathbb{R})^n$ and acts discontinuously on $\mathbb{H}^n$. Denote by $S_{2k}(\Gamma)$ the space of holomorphic Hilbert modular cusp form of weight $(2k, 2k, \cdots, 2k)$, If we let $J_k = \dim_{\mathbb{C}} S_{2k}(\Gamma)$, then it was shown by Shimizu (using the Selberg trace formula) that

$$J_k = \frac{vol(\Gamma\backslash\mathbb{H}^n)}{(4\pi)^n}(2k-1)^n + O(1).$$

Let $d\mu = \dfrac{1}{vol(\Gamma\backslash\mathbb{H}^n)}\dfrac{dxdy}{(Ny)^2}$ be the normalized invariant measure on $\Gamma\backslash\mathbb{H}^n$. One expects that the following mass equidistribution conjecture (analogue of AQUE) on the Hilbert modular variety should be true.

Conjecture: (Mass equidistribution) *For any Schwartz function $h(z)$ on $\Gamma\backslash\mathbb{H}^n$, one has*

$$\lim_{k\to\infty}\max_{1\le i\le J_k}\left|\int_{\Gamma\backslash\mathbb{H}^n} h(z)|f_k|^2 y^k d\mu - \int_{\Gamma\backslash\mathbb{H}^n} h(z)d\mu\right| = 0$$

where $\{f_{i,k}\}_{i=1}^{J_k}$ is the orthonormal Hecke basis of $S_{2k}(\Gamma)$.

The Bergman kernel on the Hilbert modular variety $\Gamma\backslash\mathbb{H}^n$ is given by

$$B_k(z,w) = \sum_{\gamma\in\Gamma} N(\gamma z - \overline{w})^{-2k} j(\gamma, z)^{-2k}$$

where $N(\gamma z - \overline{w}) = \prod_{i=1}^{n}(\sigma_i(\gamma)z_i - \overline{w_i})$ and $j(\gamma, z) = N(cz+d)$, $\gamma = \begin{pmatrix} a & b \\ c & d\end{pmatrix}$. We apply Selberg trace formula on the Bergman kernel and obtain the following theorem which shows that the equidistribution conjecture is true on the average and gives a sharp bound of the rate of convergence.

Theorem 7 (Liu [Liu4])**.** *Let $\{f_{i,k}\}_{i=1}^{J_k}$ be an orthonormal basis of $S_{2k}(\Gamma)$. Set*

$$d\mu_k = \frac{1}{J_k}\left(\sum_{i=1}^{J_k}|f_{i,k}(z)|^2\right)(Ny)^{2k}d\mu.$$

Then for any compact subset $A \subset \Gamma\backslash\mathbb{H}^n$ and any $0 < \epsilon < 1$, we have

$$\int_A d\mu_k = \int_A d\mu + O_{\epsilon,A}\left((k^{-1+\epsilon})^n\right)$$

as $k \longrightarrow \infty$.

4. Future Research

Sarnak and Watson proved that the L^4 norm of a Maass-Hecke eigenform f is bounded by λ^{ε} where λ is the corresponding Laplacian eigenvalue of f. It would be interesting to investigate the L^4 norm in the level aspect. By using a spectral decomposition, it leads to study the first moment of L-functions $L(1/2, \mathrm{sym}^2 f \otimes g)$ where f and g have the same level. One possible approach is using a GL_3 Voronoi formula for forms with level. However this is not established yet. It would be nice to develop such a formula and also have an application for it.

Another approach to AQUE is using the period formula. In his Ph.D. thesis, Watson proved an identity relating the AQUE conjecture to the triple product L-function. This connects AQUE conjecture to the subconvexity bound of L-functions. It would be of great interest to consider the subconvexity bound for higher dimensional L-functions, especially for the family of L-functions which occur in Watson's formula.

References

[CPS1] J.W. Cogdell and I.I. Piatetski-Shapiro, *Converse Theorems for GL_n*, Publ. Math. IHES **79** (1994), 157-214.

[CPS2] J.W. Cogdell and I.I. Piatetski-Shapiro, *Converse Theorems for GL_n, II*, J. reine angew. Math. **507** (1999), 165-188.

[HS] Holowinsky R. and Soundararajan, K. *Mass equidistribution for Hecke eigenforms*, Ann. of Math. (2) **172** (2010), no. 2, 1517-1528.

[Ic] A. Ichino, *Pullbacks of Saito-Kurokawa lifts*, Invent. Math. **162** (2005), 551–647.

[Liu1] Sheng-Chi Liu, *Determination of $GL(3)$ cusp forms by central values of $GL(3) \times GL(2)$ L-functions, level aspect*, Journal of Number Theory **131** (2011), 1397-1408.

[Liu2] Sheng-Chi Liu, *Determination of $GL(3)$ cusp forms by central values of $GL(3) \times GL(2)$ L-functions*, Int. Math. Res. Not. 2010, Art. ID rnq021, 17pp.

[Liu3] Sheng-Chi Liu, *Simultaneous nonvanishing of $GL(2) \times GL(2)$ and $GL(2)$ L-functions*, submitted

[Liu4] Sheng-Chi Liu, *Equidistribution of Hecke eigenforms on the Hilbert modular varieties*, Journal of Number Theory **127** (2007), 1-9.

[LY] Sheng-Chi Liu and M. Young, *Growth and nonvanishing of restricted Siegel modular forms arising as Saito-Kurokawa lifts*, submitted, available at http://arxiv.org/abs/1108.0723

[LR] W. Luo and D. Ramakrishnan, *Determination of modular forms by twists of critical L-values*, Invent. math. **130** (1997), 371-398.

[Lin] Lindenstrauss, E. *Invariant measures and arithmetic quantum unique ergodicity,* Ann. of Math. **163** (2006), 165-219.

[RS] Rudnick, Z. and Sarnak, P. *The behaviour of eigenstates of arithmetic hyperbolic manifold,* Comm. Math. Phys. **161**(1994), 195-213.

[M] Marshall, S. *Mass equidistribution for automorphic forms of cohomological type on GL_2*, J. Amer. Math. Soc. **24** (2011), no. 4, 1051-1103.

[S] Soundararajan, K. *Quantum unique ergodicity for $SL_2(\mathbb{Z})\backslash\mathbb{H}$,* Ann. of Math. (2) **172** (2010), no. 2, 1529-1538.

[We] A. Weil, *Üder die Bestimmung Dirichletscher Reihen durch Funktionalgleichungen*, Math. Ann. **168**, (1967), 149-156.