1. Let $I = (0,1)$. Consider the boundary value problem

$$u^{(4)}(x) = f(x), \qquad x \in I$$
$$u(0) = u''(0) = u(1) = u''(1) = 0.$$

   (a) Give a weak formulation of the boundary value problem of the form:
   Find $u \in V$ such that

   $$a(u,v) = (f,v), \quad \forall v \in V.$$

   Specify the energy space V and the bilinear form $a(u,v)$.

   (b) Show that

   $$||v'||_{L^2(I)} \leq ||v''||_{L^2(I)}, \quad \forall v \in C^2(I) \quad \text{with} \quad v(0) = v(1) = 0.$$

   Using this inequality show that the biliear form $a(u,v)$ is coercive in V.

2. Consider the fourth order problem:

$$u^{(4)}(x) = f(x), \qquad x \in I$$
$$u(0) = u'(0) = u(1) = u'(1) = 0.$$

   (a) Derive a weak formulation of the above problem in a subspace of $H^2(\Omega)$. The above boundary conditions are essential and so the problem is posed on a proper subspace V. Define V so that the boundary terms disappear.

   (b) Show that the weak problem is well-posed.

3. Consider the pure Neumann problem:

$$-u''(x) = f(x), \qquad x \in \Omega = (0,1)$$
$$u'(0) = u'(1) = 0$$

   and set

   $$V = \{\phi \in H^1(\Omega), \bar{\phi} = 0\}.$$

   Here $\bar{\phi}$ denotes the mean value of ϕ on $(0,1)$. The V is a Hilbert space. Show that

   (a) Derive a weak formulation base on V.

   (b) Show that the weak problem is well-posed.