Homework #1. (Due Feb. 2)

Let $\widehat{K} = [0,1]$ and set $\widehat{P} = \mathbb{P}_2(\widehat{K})$. Define $\widehat{\xi}_1 = 0$, $\widehat{\xi}_2 = 1$, $\widehat{\xi}_3 = \frac{1}{2}$ and consider the following linear mappings (i.e. reference degrees of freedom) from $\widehat{P}$ to $\mathbb{R}$ defined as follows: for all $\widehat{p} \in \widehat{P}$,

$$\widehat{\sigma}_1(\widehat{p}) = \widehat{p}(\widehat{\xi}_1), \quad \widehat{\sigma}_2(\widehat{p}) = \widehat{p}(\widehat{\xi}_2), \quad \widehat{\sigma}_1(\widehat{p}) = \widehat{p}(\widehat{\xi}_1) + \widehat{p}(\widehat{\xi}_2) - 2\widehat{p}(\widehat{\xi}_3).$$

Set $\widehat{\Sigma} = \{\widehat{\sigma}_1, \widehat{\sigma}_2, \widehat{\sigma}_3\}$.

Question 1

Prove that $\{\widehat{K}, \widehat{P}, \widehat{\Sigma}\}$ is a finite element.

Question 2

Compute the local shape functions, i.e., the functions $\widehat{\phi}_1, \widehat{\phi}_2, \widehat{\phi}_3 \in \widehat{P}$ such that $\widehat{\sigma}_i(\widehat{\phi}_j) = \delta_{ij}$ for $1 \leq i, j \leq 3$.

Question 3

Define the mapping $\mathcal{I}_{\widehat{K}} : \mathcal{C}^0(\widehat{K}) \longrightarrow \widehat{P}$ be such that

$$\mathcal{I}_{\widehat{K}}(\widehat{v}) = \sum_{i=1}^{3} \widehat{\sigma}_i(\widehat{v})\widehat{\phi}_i, \quad \forall \widehat{v} \in \mathcal{C}^0(\widehat{K}).$$

Prove that $\mathcal{I}_{\widehat{K}}\widehat{p} = \widehat{p}$ for all $\widehat{p} \in \widehat{P}$.

Question 4

Let $\mathcal{I}^L_{\widehat{K}} : \mathcal{C}^0(\widehat{K}) \longrightarrow \widehat{P}$ be the Lagrange interpolation operator based on the nodes $\{\widehat{\xi}_1, \widehat{\xi}_2, \widehat{\xi}_3\}$. What is the relation between $\mathcal{I}^L_{\widehat{K}}$ and $\mathcal{I}_{\widehat{K}}$?

Question 5

Let (a, b) be an interval in $\mathbb{R}$. Let $\mathcal{T}_h = \cup_{m=1}^{M} I_m$ be a mesh of the interval (a, b). Denote by $x_{1,m}$ and $x_{2,m}$ the extremities of I_m and let $x_{3,m}$ be the midpoint of I_m. Construct a linear mapping that maps $\widehat{K}$ to I_m. Denote by T_m this mapping.

Question 6

Starting from 1, give a number to all the nodes of the mesh. Observe that there are $J = 2M + 1$ nodes. Denote by $j : \{1, 2, 3\} \times \{1, \ldots, M\} \longrightarrow$

$\{1, \dots J\}$ the connectivity array. For $k \in \{1, \dots J\}$, define the functions

$$\phi_k | I_m = \begin{cases} \widehat{\phi}_l \circ T_m^{-1} & \text{if there is } l \in \{1,2,3\} \text{ s.t. } k = j(l,m) \\ 0 & \text{otherwise.} \end{cases}$$

Prove that functions $\{\phi_1, \dots \phi_J\}$ are continuous.

Question 7

Prove that $\{\phi_1, \dots \phi_J\}$ is linearly independent.

Question 8

Prove that $\{\phi_1, \dots \phi_J\}$ is a basis for

$$P_h^2 \{v_h \in \mathcal{C}^0(\overline{\Omega}); v_h|_{I_m} \in \mathbb{P}_2(I_m), \forall I_m \in \mathcal{T}_h\}.$$