Homework #4. (Due Feb. 23)

Question 1

Let $\Omega =]0,1[$, let $f \in L^2(\Omega)$, and let $k \in \mathbb{R}$. Consider the problem:

$$\begin{cases} \text{Seek } u \in H_0^1(\Omega) \text{ such that} \\ \int_0^1 u'v' + k\int_0^1 u'v + \int_0^1 uv = \int_0^1 fv, \quad \forall v \in H_0^1(\Omega). \end{cases}$$

(i) Write the corresponding PDE and boundary conditions.

(ii) Prove that the problem is well-posed. (*Hint*: Use the Lax–Milgram Lemma.)

Question 2

Let Ω be a polygonal domain in $\mathbb{R}^2$ and let $\mathcal{T}_h$ be an affine mesh of Ω composed of triangles. Assume that all the angles of the triangles in $\mathcal{T}_h$ are acute. Let $\{\varphi_1, \dots, \varphi_N\}$ be the global Lagrange shape functions asociated with the vertices of the mesh. Let $\mathcal{A}$ be the stiffness matrix associated with the Laplace operator, i.e., $\mathcal{A}_{ij} = \int_\Omega \nabla\varphi_i \cdot \nabla\varphi_j$ for $1 \le i, j \le N$.

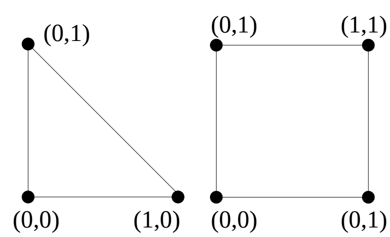
(i) Show that $\mathcal{A}$ is an *M-matrix*, i.e., all its off-diagonal entries are non-positive and its row-wise sums are non-negative.

(ii) Prove the following discrete maximum principle: If $f \in L^2(\Omega)$ is such that $f \le 0$ in Ω, the finite element solution u_h to the homogeneous Dirichlet problem with right-hand side f is such that $u_h \le 0$ in Ω.

Question 3

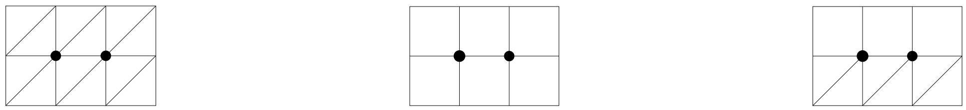
Let $\Omega =]0,3[\times]0,2[$. Consider the problem $-\nabla^2 u = 1$ in Ω and $u_{|\partial\Omega} = 0$. Approximate its solution with $\mathbb{P}_1$ H^1-conforming finite elements.

(i) Consider the reference simplex $\widehat{T}$ and the reference square $\widehat{K}$ shown in the figure.



The nodes are numbered anticlockwise from $(0,0)$. Let $\{\widehat{\lambda}_1, \widehat{\lambda}_2, \widehat{\lambda}_3\}$ and $\{\widehat{\theta}_1, \widehat{\theta}_2, \widehat{\theta}_3, \widehat{\theta}_4\}$ be the local shape functions on $\widehat{T}$ and $\widehat{K}$, respectively. Compute the matrices $\left(\int_{\widehat{T}} \nabla\widehat{\lambda}_i \cdot \nabla\widehat{\lambda}_j\right)_{1\le i,j\le 3}$ and $\left(\int_{\widehat{K}} \nabla\widehat{\theta}_i \cdot \nabla\widehat{\theta}_j\right)_{1\le i,j\le 4}$.

(ii) Consider the meshes shown in the figure.



Assemble the stiffness matrix for each of these three meshes.