Homework #6. (Due March. 30)

Question 1

Let Ω, be a smooth bounded domain in $\mathbb{R}^3$. We denote $a{\cdot}b = \sum_{i=1}^3 a_i b_i$.

(i) Prove that if u and v are smooth vector fields, then

$$\int_\Omega (\nabla\times u){\cdot}v = \int_\Omega u{\cdot}(\nabla\times v) - \int_{\partial\Omega} (u\times n){\cdot}v.$$

(Hint: Use $\nabla{\cdot}(u\times v) = (\nabla\times u){\cdot}v - u{\cdot}(\nabla\times v)$.)
(ii) Prove that if u is a smooth vector field and p is a smooth scalar field, then

$$\int_\Omega u{\cdot}\nabla p = -\int_\Omega p\nabla{\cdot}u + \int_{\partial\Omega} p(u{\cdot}n).$$

(Hint: Use $\nabla{\cdot}(pu) = p\nabla{\cdot}u + u{\cdot}\nabla p$.)

Question 2

Let Ω, be a sooth bounded domain in $\mathbb{R}^3$. Let j be a smooth vector field in $[L^2(\Omega)]^3$. Let μ and σ be two scalar-valued, positive, and continuous functions on Ω. Assume that there are μ_0, μ_1, σ_0, and σ_1 such that

$$0 < \mu_0 \leq \mu(x) \leq \mu_1 < \infty, \qquad 0 < \sigma_1 \leq \sigma(x) \leq \sigma_1 < \infty.$$

Consider the problem:

$$\mu H + \nabla\times(\sigma\nabla\times H) = j; \quad H \times n|_\Gamma = 0.$$

(i) Write a weak formulation for the above problem. (Hint: use previous question).
(ii) Prove that your weak problem is wellposed.

Question 3

Let Ω, be a smooth bounded domain in $\mathbb{R}^d$. Let β be a smooth vector field in $[\mathcal{C}^1(\overline{\Omega})]^d$, and assume that $\nabla{\cdot}\beta = 0$. Let α and γ be two scalar-valued, positive, and continuous functions on Ω. Assume that there are α_0, α_1, γ_0, and γ_1 such that

$$0 < \alpha_0 \leq \alpha(x) \leq \alpha_1 < \infty, \qquad 0 < \gamma_1 \leq \gamma(x) \leq \gamma_1 < \infty.$$

Consider the following problem where u is a vector-valued function and p is scalar-valued:

$$\begin{aligned} \alpha u + (\beta\cdot\nabla)u - \nabla^2 u + \nabla p &= f, \\ \nabla\cdot u + \gamma p &= 0, \\ u|_{\partial\Omega} &= 0. \end{aligned}$$

(i) Write a weak formulation for the above problem. (Hint: use previous question).
(ii) Prove that your weak problem is wellposed.