Homework #7. (Due April. 13)

Question 1

Let X be a real vector space and let a be a real-valued bilinear mapping. Assume that a is symmetric and positive. Prove that

$$\forall u, v \in X, \qquad a(u,v) \leq a(u,u)^{\frac{1}{2}} a(v,v)^{\frac{1}{2}}.$$

Question 2

Consider two finite-dimensional vector spaces W and V. Let $\{\psi_i, \dots, \psi_M\}$ and $\{\phi_i, \dots, \phi_N\}$ be bases of W_h and V_h, respectively. Let a be a real-valued bilinear mapping on $W_h \times V_h$. Define the matrix $\mathcal{A}$ such that $\mathcal{A}_{ij} = a(\psi_j, \phi_i)$ for $1 \leq i \leq N$, $1 \leq j \leq M$.

(i) Prove the following statement: Null($\mathcal{A}$) $= \{0\}$ if and only if

$$(1) \qquad \exists \alpha > 0, \qquad \inf_{w \in W} \sup_{v \in V} \frac{a(w,v)}{\|w\|_W \|v\|_V} \geq \alpha.$$

(ii) Prove the following statement: rank($\mathcal{A}$) $= \dim(V)$ if and only if

$$(2) \qquad \forall v \in V, \quad (\forall w \in W,\ a(w,v) = 0) \implies (v = 0).$$

(iii) Prove that if $\dim(W) = \dim(V)$, then (1) $\iff$ (2).

(iv) What can you say of the following problem:

$$\text{For } b \in \mathbb{R}^N, \text{ find } X \in \mathbb{R}^M \text{ such } \mathcal{A}X = b$$

when (1) hods?

(v) What can you say of the following problem:

$$\text{For } b \in \mathbb{R}^N, \text{ find } X \in \mathbb{R}^M \text{ such } \mathcal{A}X = b$$

when (2) holds?

(vi) Do you understand the BNB theorem now?

Question 3

Let $\mathcal{A}$ be a $N \times N$ real-valued matrix. Assume that $\mathcal{A}$ is invertible. Let F be a vector in $\mathbb{R}^N$ and let $U \in \mathbb{R}^N$ be such $AU = F$.

(i) Prove that $U \in \mathbb{R}^N$ solves $AU = F$ if and only if

$$U^T \mathcal{A}^T \mathcal{A} V = V^T \mathcal{A}^T F, \quad \forall V \in \mathbb{R}^N.$$

(ii) Define $J(V) = \frac{1}{2}V^T\mathcal{A}^T\mathcal{A}V - V^T\mathcal{A}^TF$. Prove that $U \in \mathbb{R}^N$ solves $AU = F$ if and only if U solve the following optimization problem:

$$J(U) = \inf_{V\in\mathbb{R}^N} J(V).$$