Homework #7. (Due April. 20)

Question 1

Let $\Omega =]0,1[$. Henceforth $L^1(\Omega)$ denotes the space of the scalar-valued functions that are integrable over Ω. $W^{1,1}(\Omega)$ is the space of the scalar-valued functions in $L^1(\Omega)$ whose first weak derivatives are in $L^1(\Omega)$. We denote

$$\|v\|_{L^1} = \int_0^1 |v|, \qquad \|v\|_{W^{1,1}} = \|v\|_{L^1} + \|v'\|_{L^1}.$$

Let $f \in L^1(\Omega)$, and consider the following problem:

$$\begin{cases} \mu u + u_x = f, \\ u(0) = 0, \end{cases}$$

where μ is a nonnegative constant. Accept as a fact that for all $f \in L^1(\Omega)$ this problem has a unique solution in $W = \{w \in W^{1,1}(\Omega);\, w(0) = 0\}$.

Let $\mathcal{T}_h$ be a mesh of Ω composed of N segments. Define the finite element spaces

$$W_h = \{w_h \in \mathcal{C}^0(\overline{\Omega});\, \forall K \in \mathcal{T}_h,\, w_{h|K} \in \mathbb{P}_1;\, w_h(0) = 0\},$$
$$V_h = \{v_h \in L^1(\Omega);\, \forall K \in \mathcal{T}_h,\, v_{h|K} \in \mathbb{P}_0\}.$$

The trial space W_h is equipped with the norm of $W^{1,1}(\Omega)$ and the test space V_h is equipped with the maximum norm: $\|v_h\|_{L^\infty} = \sup_{K\in\mathcal{T}_h; x\in K} |v_h(x)|$. Introduce the bilinear form

$$a(w,v) = \int_0^1 (\mu w + w_x)v,$$

and the following discrete problem:

$$(1) \qquad \begin{cases} \text{Seek } u_h \in W_h \text{ such that} \\ a(u_h, v_h) = \int_0^1 f v_h, \quad \forall v_h \in V_h. \end{cases}$$

(i) What are the dimensions of W_h and V_h?

(ii) Show that a is bounded on $W_h \times V_h$.

(iii) For $w_h \in W_h$, let $\overline{w}_h \in V_h$ be the function such that the restriction of $\overline{w}_h$ to each mesh cell K is the mean value of w_h over this mesh cell. Show that there is $c_1 > 0$, independent of h, such that

$$\|w_h - \overline{w}_h\|_{L^1} \le c_1 h \|w_h\|_{W^{1,1}}.$$

(iv) Denote by sg the sign function, i.e., $\mathrm{sg}(x) = \frac{x}{|x|}$ if x is not zero and $\mathrm{sg}(0) = 0$. Let w_h be a nonzero function in W_h. Set $z_h = \mathrm{sg}(\mu \overline{w}_h + w_{h,x})$. Show that

$$\frac{a(w_h, z_h)}{\|z_h\|_{L^\infty(\Omega)}} \geq \|\mu w_h + w_{h,x}\|_{L^1(\Omega)} - c_1 \mu h \|w_h\|_{W^{1,1}(\Omega)}$$

(v) Accept as a fact that there exists $\alpha > 0$ such that

$$\forall w \in W, \qquad \|\mu w + w_x\|_{L^1(\Omega)} \geq \alpha \|w\|_{W^{1,1}(\Omega)}.$$

Prove that there is $\gamma > 0$ and h_0 such that for all $h \leq h_0$,

$$\inf_{w_h \in W_h} \sup_{v_h \in V_h} \frac{a(w_h, v_h)}{\|w_h\|_{W^{1,1}(\Omega)} \|v_h\|_{L^\infty(\Omega)}} \geq \gamma.$$

(vi) Show that (1) has a unique solution.

(vii) Prove that the solution to (1) satisfies the following error estimate

$$\|u - u_h\|_{W^{1,1}} \leq c \inf_{w_h \in W_h} \|u - w_h\|_{W^{1,1}}.$$