Programming assignment #4. (Due April 20)

Let $\Omega = (0,1)^2$ be the unit square in $\mathbb{R}^2$. Consider the vector field $\beta = (\alpha, 0)$. Consider the problem:

$$-\epsilon\nabla^2 u + u + \beta\cdot\nabla u = 1; \quad u|_{\partial\Omega} = 0.$$

Let N be a positive integer. Let $h = \frac{1}{N+1}$ and define the following vertices $x_{ij} = (ih, jh)$ for $0 \le i,j \le N+1$. Denote by S_{ij} the square whose vertices are x_{ij}, $x_{i+1,j}$, $x_{i+1,j+1}$, and $x_{i,j+1}$ for $0 \le i,j \le N$. Divide S_{ij} into two triangles so that the vertices of these triangles are x_{ij}, $x_{i+1,j}$, $x_{i+1,j+1}$, and x_{ij}, $x_{i+1,j+1}$, and $x_{i,j+1}$. Denote by $\mathcal{T}_h$ the mesh composed of the triangles above defined. The interior nodes are globally numbered from left to right and bottom to left.

(i) Write a weak formulation for the above problem.

(ii) Write the discrete form of the above problem using $\mathbb{P}_1$ finite elements.

(iii) Write the Galerkin Least-Squares formulation of the problem using $\mathbb{P}_1$ finite elements. Use the stabilizing parameter

$$\delta(h) = (\frac{\alpha}{h} + \frac{\epsilon}{h^2})^{-1}.$$

(iii) Assemble the matrix and the right-hand side arising from the discrete GaLS problem.

(iv) Make sure you assembled the matrix correctly by solving the discrete problem whose source term f corresponds to the following solution $u = x(1-x)y(1-y)$. Use $\alpha = 1$. Plot in log-log scale $\max_{0\le i,j\le N} |u(x_i j) - u_h(x_{ij})|$ versus h, where u_h denotes the approximate solution. Use $N = 10$, 20, 40, 80, and $N = 100$.

(v) Solve the problem with $f = 1$, $\alpha = 1$, and $N = 50$ using successively $\epsilon = 10^{-1}$, $\epsilon = 10^{-2}$, $\epsilon = 10^{-3}$, and $\epsilon = 10^{-4}$. Report what you observe. Explain.