Cauchy-Schwartz Inequality

Proof of Cauchy-Schwartz inequality by induction.

$$|\sum_{i=1}^{n} a_i b_i| \leq \sqrt{\sum_{i=1}^{n} a_i^2}\sqrt{\sum_{i=1}^{n} b_i^2}$$

In terms of the inner product, CS says $|(A,B)| \leq \|A\|\|B\|$

We will assume all a_i and b_i are nonnegative, for the moment.

CS is trivial for n=1. Now assume the n-term CS holds, and try to prove n+1-term CS holds.

$$\left(b_{n+1}\sqrt{a_1^2 + ... + a_n^2} - a_{n+1}\sqrt{b_1^2 + ... + b_n^2}\right)^2 \geq 0$$

$$b_{n+1}^2(a_1^2 + ... + a_n^2) + a_{n+1}^2(b_1^2 + ... + b_n^2) \geq 2a_{n+1}b_{n+1}\sqrt{a_1^2 + ... + a_n^2}\sqrt{b_1^2 + ... + b_n^2}$$

Using the assumption that the n-term CS inequality holds:

$$b_{n+1}^2(a_1^2 + ... + a_n^2) + a_{n+1}^2(b_1^2 + ... + b_n^2) \geq 2a_{n+1}b_{n+1}(a_1b_1 + ... + a_nb_n)........(A)$$

Now we simply restate the n-term CS inequality:

$$(a_1^2 + ... + a_n^2)(b_1^2 + ... + b_n^2) \geq (a_1b_1 + ... + a_nb_n)^2........(B)$$

and throw in an identity:

$$a_{n+1}^2 b_{n+1}^2 = a_{n+1}^2 b_{n+1}^2........(C)$$

Then we add the left and right sides of inequalities A,B and C together:

$$\left((a_1^2 + ... + a_n^2) + a_{n+1}^2\right)\left((b_1^2 + ... + b_n^2) + b_{n+1}^2\right) \geq ((a_1b_1 + ... + a_nb_n) + a_{n+1}b_{n+1})^2$$

which is just CS for n+1 terms. QED

If a_i,b_i are not assumed to be nonnegative, then

$$|\sum_{i=1}^{n} a_i b_i| \leq \sum_{i=1}^{n} |a_i||b_i| \leq \sqrt{\sum_{i=1}^{n} |a_i|^2}\sqrt{\sum_{i=1}^{n} |b_i|^2} = \sqrt{\sum_{i=1}^{n} a_i^2}\sqrt{\sum_{i=1}^{n} b_i^2}$$

Now the triangle inequality follows from CS:

$$\|A+B\| \leq \|A\| + \|B\|$$

$$\sqrt{\sum_{i=1}^{n}(a_i+b_i)^2} \leq \sqrt{\sum_{i=1}^{n} a_i^2} + \sqrt{\sum_{i=1}^{n} b_i^2}$$

$$\sum_{i=1}^{n}(a_i+b_i)^2 \leq \left(\sqrt{\sum_{i=1}^{n} a_i^2} + \sqrt{\sum_{i=1}^{n} b_i^2}\right)^2$$

$$\sum_{i=1}^{n} a_i^2 + \sum_{i=1}^{n} b_i^2 + 2\sum_{i+1}^{n} a_i b_i \leq \sum_{i=1}^{n} a_i^2 + \sum_{i=1}^{n} b_i^2 + 2\sqrt{\sum_{i=1}^{n} a_i^2}\sqrt{\sum_{i=1}^{n} b_i^2}$$

$$2\sum_{i+1}^{n} a_i b_i \leq 2\sqrt{\sum_{i=1}^{n} a_i^2}\sqrt{\sum_{i=1}^{n} b_i^2}$$

which follows from the CS inequality. (So we were actually working backwards, but all steps are reversible.)