**Notes on Chapter 4**

1. Some review:
$< x, y >= \sum_{i=1}^N x_i \bar{y}_i$
$< x, x >= \sum_{i=1}^N |x_i|^2 = \|x\|^2$
$A^* = \bar{A}^T$
$(AB)^* = (\bar{A}\bar{B})^T = \bar{B}^T \bar{A}^T = B^* A^*$
$< x, y >= y^* x$
U is unitary if $U^*U = I$, which means that the columns form an orthonormal basis for $C^N$ (so do the rows). If U is real and unitary, it is called orthogonal, and then $U^T U = I$.

2. If U is unitary, $< Ux, Uy >= (Uy)^*(Ux) = y^*U^*Ux = y^*x =< x, y >$. Thus U is "norm-preserving", that is $\|Ux\| = \|x\|$. U is also "angle-preserving" because the angle between two vectors is defined as $\frac{<x,y>}{\|x\|\|y\|}$ and so the angle between Ux and Uy is the same as the angle between x and y.

3. If U is unitary, all eigenvalues have $|\lambda| = 1$. Proof: If $Uz = \lambda z$, then
$|\lambda|^2 z^* z = \bar{\lambda}\lambda z^* z = (\lambda z)^*(\lambda z) = (Uz)^*(Uz) = z^*U^*Uz = z^*z$
and since $z^*z = \|z\|^2 \neq 0, |\lambda| = 1$.

4. If U is unitary, we prove that U is "unitarily similar" (unitarily equivalent) to a diagonal matrix, that is, that $U = SDS^{-1}$, where D is diagonal and S is unitary. Note that since $US = SD$, this means that U has an orthonormal basis of eigenvectors. The proof is by induction. Obviously it is true for any 1 by 1 matrix, just take S=I. Assume the claim is true for every n-1 by n-1 matrix. Then let U be an n by n unitary matrix; and let $\lambda_1$ be an eigenvalue, with eigenvector $u_1$, of norm (2-norm) one. Then put $u_1$ in the first column of S, and pick the other columns of S to complete an "orthonormal" basis for $C^n$, so that S is unitary, and then US = SB, or $U = SBS^{-1}$ where B has the form:

$$\begin{bmatrix} \lambda_1 & w^T \\ 0 & B_1 \end{bmatrix}$$

Now since U is unitary, $B = S^{-1}US = S^*US$ is also unitary, because $B^*B = (S^*US)^*(S^*US) = S^*U^*SS^*US = I$. Thus the first column of B is orthogonal to the other columns, which means $w = 0$, and $B_1$ must be a unitary n-1 by n-1 matrix. Thus by assumption, $B_1 = P_1 D_1 P_1^{-1} = P_1 D_1 P_1^*$ where $D_1$ is diagonal and $P_1$ is unitary. Then:

$$B = \begin{bmatrix} \lambda_1 & 0 \\ 0 & B_1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & P_1 \end{bmatrix} \begin{bmatrix} \lambda_1 & 0 \\ 0 & D_1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & P_1^* \end{bmatrix}$$

or $B = PDP^*$, where P is unitary.

Now since $U = SBS^* = S(PDP^*)S^* = (SP)D(SP)^*$, U is unitarily similar to a diagonal matrix (note that SP is unitary).

5. A permutation matrix is a special unitary matrix, which is just the identity matrix with rows permuted. When a vector x is multiplied by P, the result is just a permutation of the elements of x.

6. A Householder matrix is $H = I - 2ww^*$, where $\|w\| = 1$. Then $H^*H = (I - 2ww^*)^*(I - 2ww^*) = (I - 2ww^*)(I - 2ww^*) = I - 4ww^* + 4w(w^*w)w^* = I$, so a Householder matrix is unitary.

7. Schur's theorem says that every matrix is unitarily similar to an upper triangular matrix.