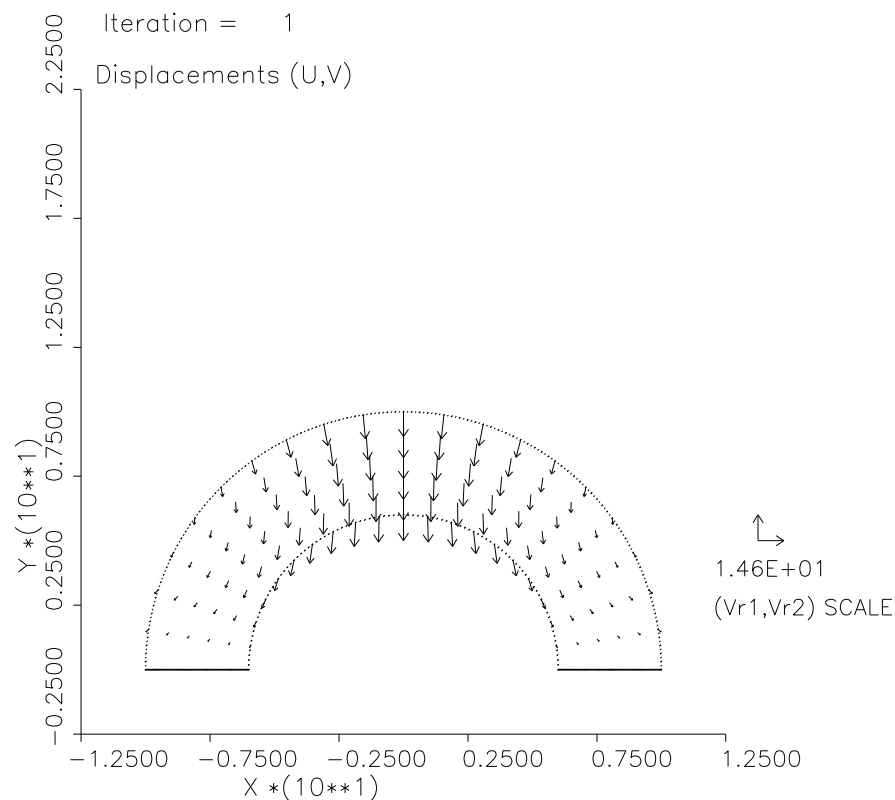


3. Consider an arch (half annulus, in 2D) described in polar coordinates as  $6 < r < 10, 0 < \theta < \pi$ . We want to solve the steady-state elasticity equations in this arch, that is, the equations 5.40 (p104), with the left hand side equal to 0 (see section 5.2 for definitions of operators used in this equation). Take  $E = 100$  and  $\nu = 0.2$  ( $E$  = elastic modulus,  $\nu$  = Poisson ratio), and take the external force vector to be  $(f_1, f_2) = (0, -10)$ , that is, there is a constant downward force, namely the weight of the uniform arch itself. On the two ends touching the "ground" ( $\theta = 0, \pi$ ), the displacement vector is zero,  $(U, V) = (0, 0)$ . On the top and bottom of the arch ( $r = 6, 10$ ), there are zero external forces, which means the following boundary conditions are satisfied:

$$\sigma_{xx}N_x + \sigma_{xy}N_y = g_1$$

$$\sigma_{xy}N_x + \sigma_{yy}N_y = g_2$$



where

$$\begin{aligned}\sigma_{xx} &= E \frac{(1-\nu)U_x + \nu V_y}{(1+\nu)(1-2\nu)} \\ \sigma_{xy} &= E \frac{U_y + V_x}{2(1+\nu)} \\ \sigma_{yy} &= E \frac{\nu U_x + (1-\nu)V_y}{(1+\nu)(1-2\nu)}\end{aligned}$$

are stresses,  $(N_x, N_y)$  is the unit outward normal to the boundary, and  $(g_1, g_2)$  is the external boundary force vector, in this case  $g_1 = g_2 = 0$ .  $N_x$  and  $N_y$  are referenced in the boundary conditions as NORMx and NORMy.

Plot the resulting displacement vector field,  $(U, V)$ , and calculate the integral of  $V$  in the entire arch. (Note: if you use the GUI, it will generate plots of the gradients  $(U_x, U_y)$  and  $(V_x, V_y)$  by default, so just change IVAR1 to 1 and IVAR2 to 4 on one of these (in the Fortran) to get a plot of  $(U, V)$ .)

If you use the Galerkin method instead of collocation, you need to write the equations in the form:

$$\begin{aligned}\frac{\partial}{\partial x}\sigma_{xx} + \frac{\partial}{\partial y}\sigma_{xy} + f_1 &= 0 \\ \frac{\partial}{\partial x}\sigma_{xy} + \frac{\partial}{\partial y}\sigma_{yy} + f_2 &= 0\end{aligned}$$

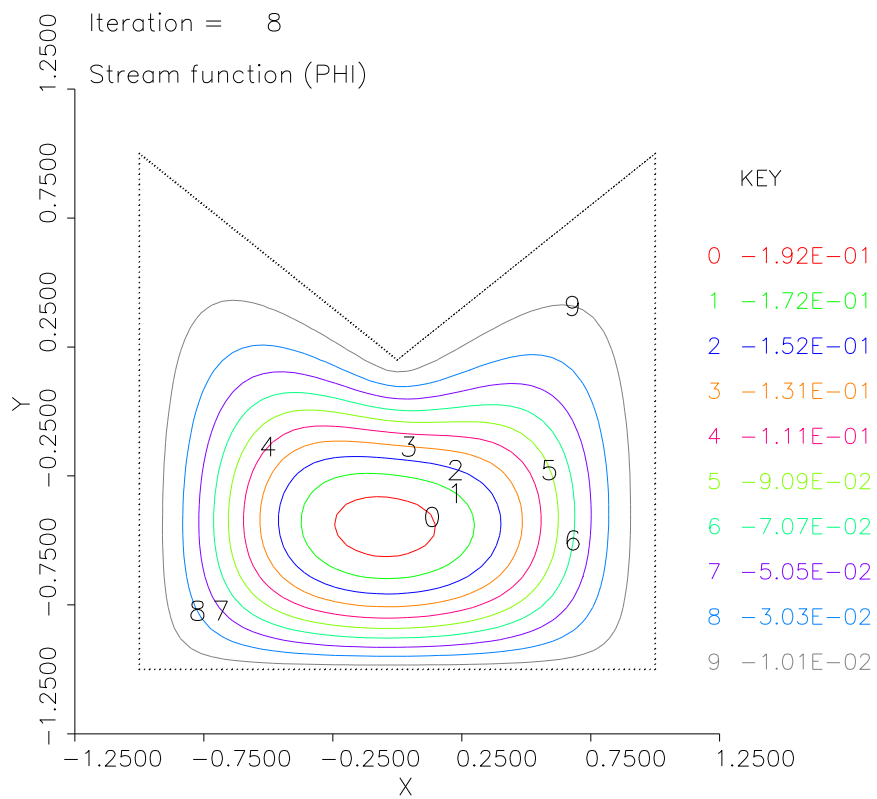
which is equivalent to 5.40, in the 2D case. If Galerkin is used, use the initial triangulation option ITRI = 2, and note that on the free boundary,  $(g_1, g_2) = (GB1, GB2)$ .

4. a. Consider the fluid flow equations 5.26 (p100). As noted on page 97 (for the 2D case), the fact that the divergence of the fluid velocity  $(U, V)$  is zero guarantees that there is a "stream function"  $\phi$  such that  $(U, V) = (\phi_y, -\phi_x)$ , and the divergence equation 5.26a ( $U_x + V_y = 0$ ) is automatically satisfied for any stream function. Now let's replace the gravity force term  $\rho g$  by a general external force vector  $f = (f_1, f_2)$ , and let us define the vorticity by  $\omega = U_y - V_x = \phi_{xx} + \phi_{yy}$ . Then write out the two components of 5.26b and differentiate the first with respect to  $y$ , and the second with respect to  $x$ , and subtract, and show (yourself) that the pressure terms disappear, and that we are left with the equation:

$$\rho \frac{\partial \omega}{\partial t} + \rho(\phi_y \omega_x - \phi_x \omega_y) + f_2 x - f_1 y = \mu(\omega_{xx} + \omega_{yy})$$

Together with  $\omega = \phi_{xx} + \phi_{yy}$ , we now have a system of two equations for the two unknowns  $\phi$  and  $\omega$ .

In this example we will find the steady-state flow (so the time derivative term is zero) in a pentagon with vertices at  $(-1, -1)$ ,  $(1, -1)$ ,  $(1, 1)$ ,  $(0, 0.2)$ ,  $(-1, 1)$ . We will assume an external force  $f = (y, -x)$ , which tends to rotate the fluid around the origin. On the bottom of the pentagon, we will apply "free-slip" boundary conditions,  $V = 0$ ,  $U_y = 0$ , and on the other four sides, we will apply "no-slip" boundary conditions,  $U = 0$ ,  $V = 0$ . Verify that the free-slip conditions are  $\phi = 0$ ,  $\omega = 0$ , and that the no-slip conditions are equivalent to setting  $\phi$  and its normal derivative ( $\phi_x N_x + \phi_y N_y$ ) to 0. Solve this PDE problem, with  $\rho = 1.1$ ,  $\mu = 0.1$  ( $\rho$  is the fluid density,  $\mu$  is the fluid viscosity), and make vector plots of the fluid velocity  $(\phi_y, -\phi_x)$  and a contour plot of the stream function. (Hint: set  $APRINT(1) = \phi_y$ ,  $BPRINT(1) = -\phi_x$  and plot (A1,B1).) Also compute the integral of  $\omega$ . Because of the geometry, you will have to use the Galerkin method, thus you cannot use the GUI.



- b. Increase  $\rho$  by a factor of 20 and re-solve the problem; this results in flow with a larger Reynold's number. If you have trouble getting convergence of Newton's method, you may need to multiply the nonlinear terms by  $beta = \min(1.d0, (T - 1)/5.d0)$ , which means the first iteration ( $T = 1$ ), you are solving a linear problem, and you are increasing the Reynold's number gradually over the next 5 iterations.

