Section 2.2: The Limit of a function

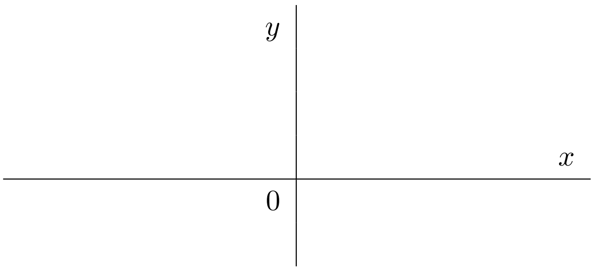
A limit is a way to discuss how the values of a function $f(x)$ behave when x approaches a number a, whether or not $f(a)$ is defined.

Let's consider the following function:

$$f(x) = \frac{\sin x}{x} \qquad (x \text{ in radians}).$$

Note that $f(0) = \dfrac{\sin 0}{0}$ is undefined. However, one can compute the values of $f(x)$ for values of x close to 0.

x	$f(x)$
$\pm$ 0.1	0.99833417
$\pm$ 0.05	0.99958339
$\pm$ 0.01	0.99998333
$\pm$ 0.005	0.99999583
$\pm$ 0.001	0.99999983



The table allows us to guess (correctly) that that our function gets closer and closer to 1 as x approaches 0 through positive and negative values. In limit notation it can be written as

$$\lim_{x\to 0^-} \frac{\sin x}{x} = \lim_{x\to 0^+} \frac{\sin x}{x} = 1$$

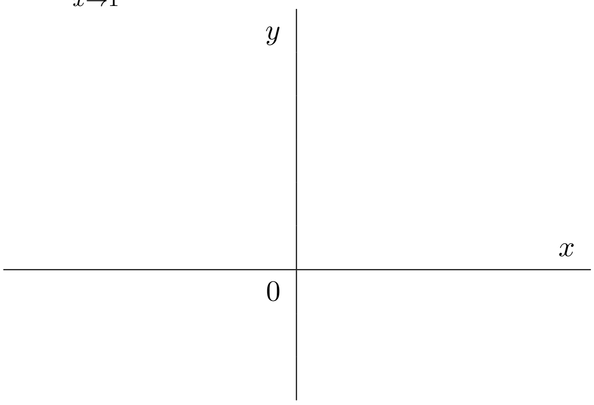
which implies that

$$\lim_{x\to 0} \frac{\sin x}{x} = 1.$$

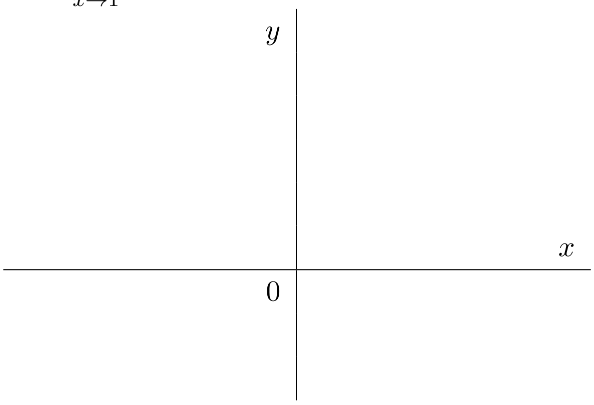
DEFINITION 1.

- *If* $\lim\limits_{x\to a^-} f(x) = \lim\limits_{x\to a^+} f(x) = L$ *then* $\lim\limits_{x\to a} f(x)$ *exists and* $\lim\limits_{x\to a} f(x) = L$;
- *If* $\lim\limits_{x\to a^-} f(x) \neq \lim\limits_{x\to a^+} f(x) = L$ *then* $\lim\limits_{x\to a} f(x)$ *does not exist.*

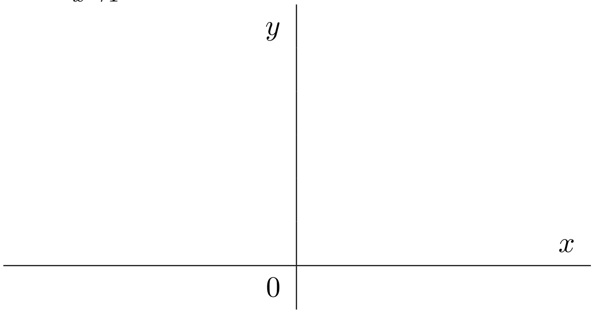
$\lim\limits_{x\to 1} f(x) =$



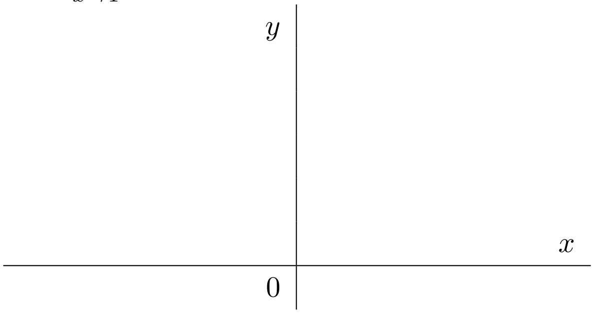
$\lim\limits_{x\to 1} f(x) =$



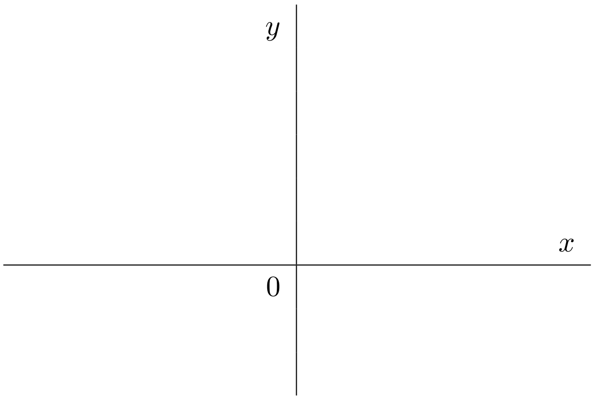
$\lim_{x \to 1} f(x) =$



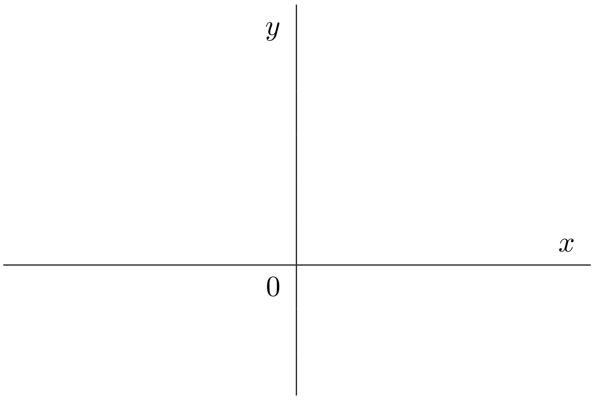
$\lim_{x \to 1} f(x) =$



$\lim_{x \to -2} f(x) =$



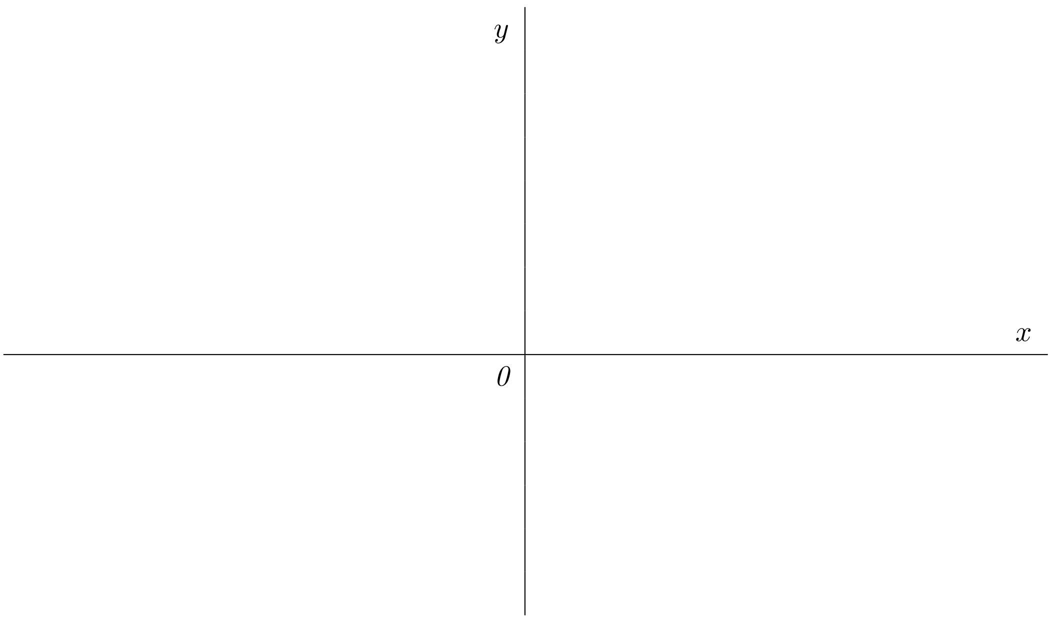
$\lim_{x \to -2} f(x) =$



Limits of piecewise defined function.

EXAMPLE 2. *Plot the graph of the function*

$$f(x) = \begin{cases} -3 - x & \text{if} \quad x \le -2 \\ 2x & \text{if} \quad -2 < x < 2 \\ x^2 - 4x + 3 & \text{if} \quad x \ge 2 \end{cases}$$



Find the limits (using the graph above):

$\lim_{x\to 0^-} f(x) =$ $\lim_{x\to 0^+} f(x) =$ $\lim_{x\to 0} f(x) =$

$\lim_{x\to -2^-} f(x) =$ $\lim_{x\to -2^+} f(x) =$ $\lim_{x\to -2} f(x) =$

$\lim_{x\to 2^-} f(x) =$ $\lim_{x\to 2^+} f(x) =$ $\lim_{x\to 2} f(x) =$

Limints involving infinity:

DEFINITION 3. *The line $x = a$ is said to be a* **vertical asymptote** *of the curve $y = f(x)$ if at least one of the following six statements is true:*

$\lim_{x\to a^-} f(x) = \infty$ $\lim_{x\to a^+} f(x) = \infty$ $\lim_{x\to a} f(x) = \infty$

$\lim_{x\to a^-} f(x) = -\infty$ $\lim_{x\to a^+} f(x) = -\infty$ $\lim_{x\to a} f(x) = -\infty$

REMARK 4. The vertical asymptotes of a rational function come from the zeroes of the denominator.

EXAMPLE 5. *Determine the infinite limit:*

(a) $\lim_{x\to 4^-} \frac{7}{x-4} =$

(b) $\lim_{x\to 4^+} \frac{7}{x-4} =$

(c) $\lim\limits_{x\to 4} \dfrac{7}{x-4} =$

(d) $\lim\limits_{x\to 0^-} \dfrac{3-x}{x^4(x+4)} =$

(e) $\lim\limits_{x\to 0^+} \dfrac{3-x}{x^4(x+4)} =$

(f) $\lim\limits_{x\to 0} \dfrac{3-x}{x^4(x+4)} =$

(g) $\lim\limits_{x\to \pi^-} \csc x =$

EXAMPLE 6. *Given:* $f(x) = \dfrac{x-4}{x^2-5x+4}$.

(a) *What are the vertical asymptotes of* $f(x)$*?*

(b) *How does* $f(x)$ *behave near the asymptotes?*