Section 2.5:Continuity

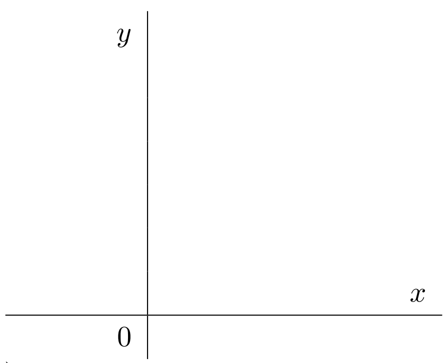
DEFINITION 1. *A function $f(x)$ is* **continuous** *at $x = a$ if $\lim_{x\to a} f(x) = f(a)$. More implicitly: if f is continuous at a then*

1. *$f(a)$ is defined (i.e. a is in the domain of f);*
2. *$\lim_{x\to a} f(x)$ exists.*
3. *$\lim_{x\to a} f(x) = f(a)$.*

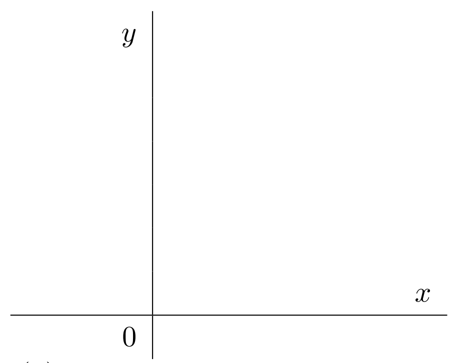
A function is said to be continuous on the interval $[a, b]$ if it is continuous at each point in the interval.

Geometrically, if f is continuous at any point in an interval then its graph has no break in it (i.e. can be drawn without removing your pen from the paper).

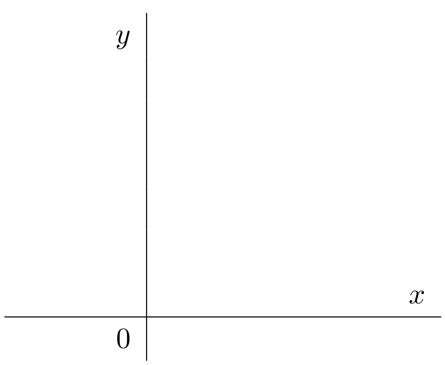
REASONS FOR BEING DISCONTINUOUS:



$f(a)$ is not defined
(i.e. a is not in the domain of f)



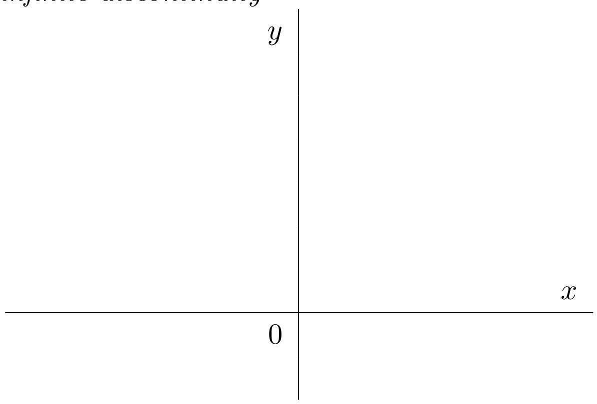
$f(a)$ is defined, but
the limit as $x \to a$ DNE



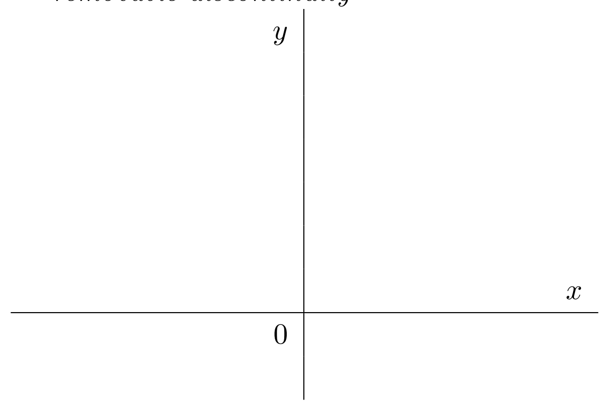
$f(a)$ is defined and $\lim_{x\to a} f(x)$ exists,
but $\lim_{x\to a} f(x) \neq f(a)$

Classification of discontinuities:

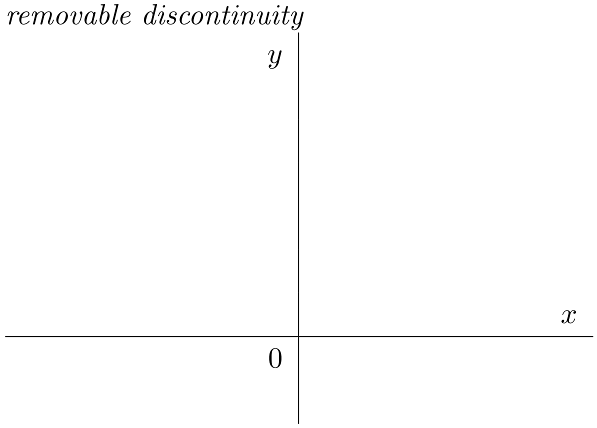
infinite discontinuity



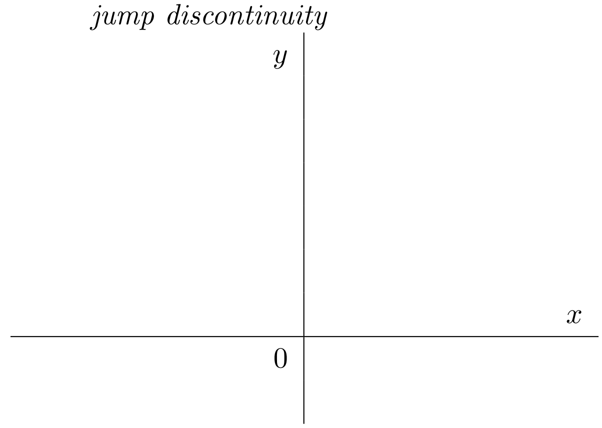
removable discontinuity



removable discontinuity



jump discontinuity



EXAMPLE 2. *Explain why each function is discontinuous at the given point:*

(a) $f(x) = \dfrac{2x}{x-3}, \quad x = 3$

(b) $f(x) = \begin{cases} \dfrac{x^2 - 2x + 1}{x-1} & \text{if} \quad x \neq 1 \\ 5 & \text{if} \quad x = 1, \end{cases} \qquad x = 1$

DEFINITION 3. *A function f is* **continuous from the right at** $x = a$ *if*

$$\lim_{x\to a^+} f(x) = f(a)$$

and f is **continuous from the left at** a *if*

$$\lim_{x\to a^-} f(x) = f(a).$$

REMARK 4. *Functions* ***continuous on an interval*** *if it is continuous at every number in the interval.* At the end point of the interval we understand continuous to mean continuous from the right or continuous from the left.

EXAMPLE 5. *Find the interval(s) where $f(x) = \sqrt{9 - x^2}$ is continuous.*

EXAMPLE 6. *Find the constant c that makes g continuous on $(-\infty, \infty)$:*

$$g(x) = \begin{cases} x^2 - c^2 & \text{if} \quad x < 4 \\ cx^2 - 1 & \text{if} \quad x \geq 4 \end{cases}$$

EXAMPLE 7. *For each of the following, find all discontinuities, classify them by using limits, give the continuity interval(s) for the corresponding function. If the discontinuity is removable, find a function g that agrees with the given function except of the discontinuity point and is continuous at that point.*

(a) $f(x) = \dfrac{x^2 - 9}{x^4 - 81}$

(b) $f(x) = \dfrac{7}{x + 12}$

(c) $f(x) = \begin{cases} x^2 + x & \textit{if} \quad x < 2 \\ 8 - x & \textit{if} \quad x > 2 \\ 4 & \textit{if} \quad x = 2 \end{cases}$

Intermediate Value Theorem: If $f(x)$ is continous on the closed interval $[a, b]$ and N is any number strictly between $f(a)$ and $f(b)$, then there is a number c, $a < c < b$, so that $f(c) = N$.

EXAMPLE 8. *If* $f(x) = x^5 - 2x^3 + x^2 + 2$, *show there a number* c *so that* $f(c) = 1$.

EXAMPLE 9. *Show that following equation has a solution (a root) between* 1 *and* 2*:*

$$3x^3 - 2x^2 - 2x - 5 = 0.$$