3.8: Higher Derivatives

The derivative of a differentiable function f is also a function and it may have a derivative of its own:

$$(f')' = f'' \qquad \textbf{second derivative}$$

$$f''(x) = \frac{\mathrm{d}}{\mathrm{d}x}(f'(x)) = \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\mathrm{d}}{\mathrm{d}x}f(x)\right)$$

`Alternative Notation:` If $y = f(x)$ then

$$y'' = f''(x) = \frac{\mathrm{d}^2 y}{\mathrm{d}x^2} = D^2 f(x).$$

Similarly, the **third derivative** $f''' = (f'')'$ or

$$y''' = f'''(x) = \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\mathrm{d}^2 y}{\mathrm{d}x^2}\right) = \frac{\mathrm{d}^3 y}{\mathrm{d}x^3} = D^3 f(x).$$

In general, the n^{th} derivative of $y = f(x)$ is denoted by $f^{(n)}(x)$:

$$y^{(n)} = f^{(n)}(x) = \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\mathrm{d}^{n-1} y}{\mathrm{d}x^{n-1}}\right) = D^n f(x).$$

EXAMPLE 1. *If* $y = x^5 + 3x + 1$ *find* $f^{(n)}(x)$

CONCLUSION: If $p(x)$ *is a polynomial of degree* n *then,* $p^{(k)}(x) = 0$ *for* $k \geq n + 1$.

EXAMPLE 2. *Find the second derivative of* $f(x) = \tan(x^3)$.

EXAMPLE 3. *Let* $h(x)$ *be a twice differentiable function. Find* $f''(x)$ *if*

$$f(x) = h(\sqrt{x}) + \sqrt{h(x)}.$$

EXAMPLE 4. *Find* $D^{2013} \sin x$.

EXAMPLE 5. *If* $f(x) = \dfrac{1}{x}$ *find a general formula for its* n^{th} *derivative.*

Acceleration: If $s(t)$ is the position of an object then the acceleration of the object is the first derivative of the velocity (consequently, the acceleration is the second derivative of the position function.)

$$a(t) = v'(t) = s''(t).$$

EXAMPLE 6. *If* $s(t) = t^3 - \frac{9}{2}t^2 - 10t + 12$ *is the position of a moving object at time* t *(where* $s(t)$ *is measured in feet and* t *is measured in seconds) find the acceleration at the times when the velocity is zero.*

EXAMPLE 7. *Sketch the curve traced by* $\mathbf{r}(t) = \langle \cos t, \sin t \rangle$ *and plot the position, tangent and acceleration vectors at* $t = \dfrac{\pi}{4}$.

Implicit second derivatives:

EXAMPLE 8. *Find* $y''(x)$ *if* $x^6 + y^6 = 66$.