4.1: Exponential functions and their derivatives

An **exponential function** is a function of the form

$$f(x) = a^x$$

where a is a positive constant. It is defined is the following manner:

- If $x = n$, a positive integer, then $a^n = \underbrace{a \cdot a \cdot \quad \cdots \quad \cdot a}_{n \text{ factors}}$
- If $x = 0$ then $a^0 = 1$.
- If $x = -n$, n is a positive integer, then $a^{-n} = \frac{1}{a^n}$.
- If x is a rational number, $x = \frac{p}{q}$, with p and q integers and $q > 0$, then

$$a^x = a^{\frac{p}{q}} = \sqrt[q]{a^p}.$$

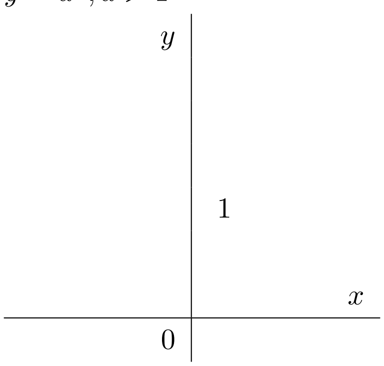
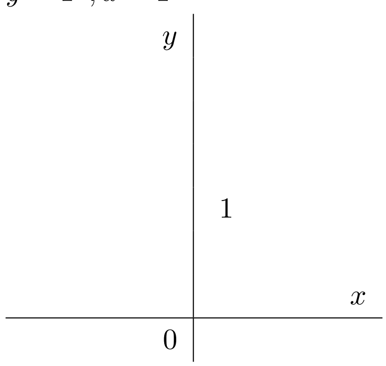
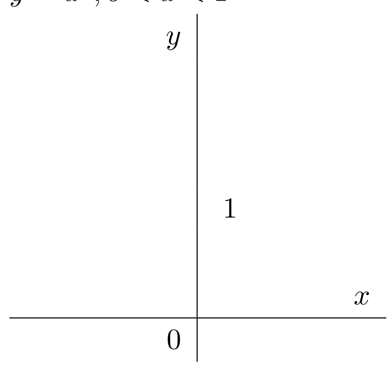
- If x is an irrational number then we define

$$a^x = \lim_{r \to x} a^r$$

  where r is a rational number.

It can be shown that this definition uniquely specifies a^x and makes the function $f(x) = a^x$ continuous.

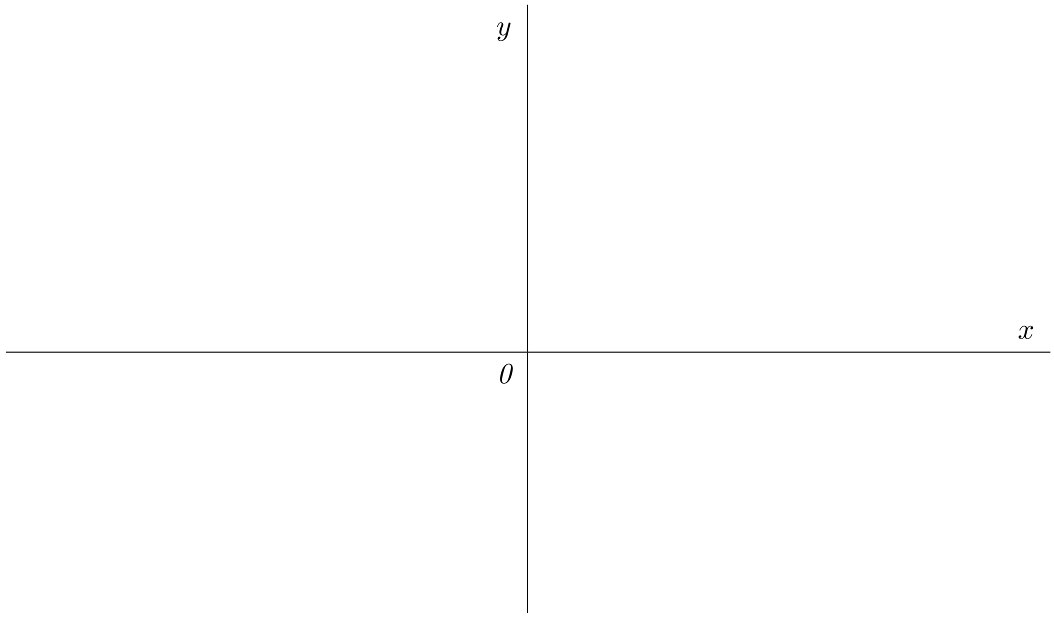
There are basically 3 kinds of exponential functions $y = a^x$:

Exponential growth	Constant	Exponential Decay
$y = a^x, a > 1$	$y = 1^x, a = 1$	$y = a^x, 0 < a < 1$
		
Domain:	Domain:	Domain:
Range:	Range:	Range:
$\lim_{x \to \infty} a^x =$		$\lim_{x \to \infty} a^x =$
$\lim_{x \to -\infty} a^x =$		$\lim_{x \to -\infty} a^x =$

EXAMPLE 1. **(a)** *Find*

$$\lim_{x \to \infty} (4^{-x} - 3)$$

(b) *Sketch the graph of the function* $y = 4^{-x} - 3$ *using transformations of graphs.*



PROPERTIES OF THE EXPONENTIAL FUNCTION: If $a, b > 0$ and x, y are real then

1. $a^{x+y} = a^x a^y$ 2. $a^{x-y} = \dfrac{a^x}{a^y}$ 3. $(a^x)^y = a^{xy}$ 4. $(ab)^x = a^x b^x$.

EXAMPLE 2. *Find the limit:*

(a) $\lim\limits_{x\to\infty} \left(\dfrac{\pi}{7}\right)^x$

(b) $\lim\limits_{x\to-\infty} \left(\pi^2 - 7\right)^x$

(c) $\lim\limits_{x\to3^+} \left(\dfrac{1}{7}\right)^{\frac{x}{x-3}}$

There are in fact a variety of ways to define e. Here are a two of them:

1. $e = \lim\limits_{n\to\infty} \left(1+\dfrac{1}{n}\right)^n$

2. e is the unique positive number for which $\lim\limits_{h\to 0} \dfrac{e^h-1}{h} = 1$.

It can be also shown that $e \approx 2.71828$.

Geometric interpretation:

EXAMPLE 3. *Find the limit:*

(a) $\lim\limits_{x\to 1^+} e^{\frac{4}{x-1}}$

(b) $\lim\limits_{x\to 1^-} e^{\frac{4}{x-1}}$

(c) $\lim\limits_{x\to\infty} \dfrac{e^{5x}-e^{-5x}}{e^{5x}+e^{-5x}}$

Derivative of exponential function.

EXAMPLE 4. *Find the derivative of* $f(x) = e^x$.

CONCLUSIONS:

1. e^x is differentiable function.
2. If $u(x)$ is a differentiable function then by Chain Rule: $\frac{\mathrm{d}}{\mathrm{d}x} e^{u(x)} = e^u \frac{\mathrm{d}u}{\mathrm{d}x}$.

EXAMPLE 5. *Find* y'' *for* e^{x^2}.

EXAMPLE 6. *Find the derivative:*

(a) $y = \sqrt{e^x + x^3}$

(b) $y = e^{x \sin x}$

EXAMPLE 7. *For what value(s) of* A *does the function* $y = e^{Ax}$ *satisfy the equation* $y'' + 2y' - 8y = 0$*?*