

4.5: Exponential Growth and Decay

If $y(t)$ is the value of a quantity y at time t and if the rate of change of y with respect to t is proportional to its size $y(t)$ at any time, then

$$\frac{dy}{dt} = ky$$

where k is a constant.

THEOREM 1. *The only solution of the differential equation $\frac{dy}{dt} = ky$ are the exponential functions*

$$y(t) = y_0 e^{kt}$$

where $y_0 = y(0)$ is the initial quantity.

In the problems below your primary goal is to find the constant k .

EXAMPLE 2. *A bacteria culture starts with 4000 bacteria and the population triples every half-hour.*

(a) *Find an expression for the number of bacteria after t hours.*

(b) *Find the number of bacteria after 20 minutes.*

EXAMPLE 3. *After 3 days a sample of radon-222 decayed to 58% of its original amount.*

(a) *What is the half-life of radon-222?*

(b) *How long would it take the sample to decay to 10% of its original amount?*

EXAMPLE 4. A tank contains 1500L of brine with a concentration of 0.3 kg of salt per liter. In order to dilute the solution, pure water is run into the tank at a rate of 20L/min. The resulting solution, which is kept mixed, runs out at the same rate.

(a) How many kilograms of salt will remain after half an hour?

(b) When will the concentration be reduced to 0.2 kg of salt per liter?

EXAMPLE 5. A curve passes through the point $(0, 5)$ and has the property that the slope of the curve at every point P is twice the y -coordinate of P . Find the equation of the curve.

Newton's Law of Cooling states that *the rate of cooling of an object is proportional to the temperature difference between the object and its surroundings*. For example, if $y(t)$ is the temperature of the object at time t and T is the the temperature of the room in which the object is cooling then

$$\frac{dy}{dt} = k(y - T).$$

The solution of this equation, which gives the temperature of the object at time t , is

$$y(t) = (y_0 - T)e^{kt} + T,$$

where $y_0 = y(0)$ is the initial temperature of the object. Indeed,

EXAMPLE 6. A thermometer is taken from a room where the temperature is 20°C to the outdoors, where the temperature is 5°C . After one minute, the temperature reads 12°C .

(a) What will the reading of the thermometer be after one more minute?

(b) When will the thermometer read 6°C ?