5.3: Derivatives and Shapes of Curves

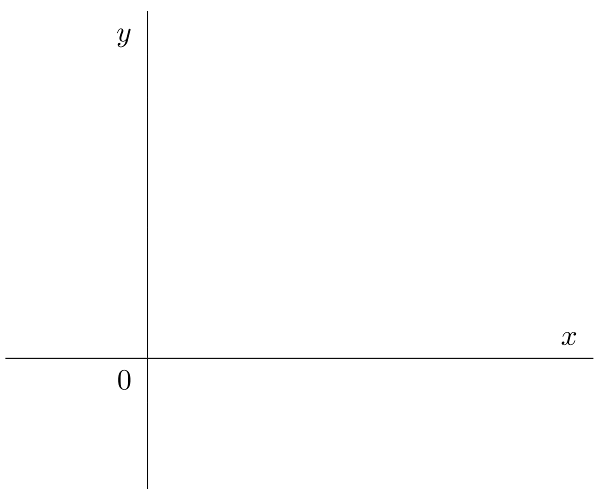
Mean Value Theorem: Suppose a function f is continuous on the (closed) interval $[a, b]$ and differentiable on the (open) interval (a, b). Then there is a number c such that $a < c < b$ and

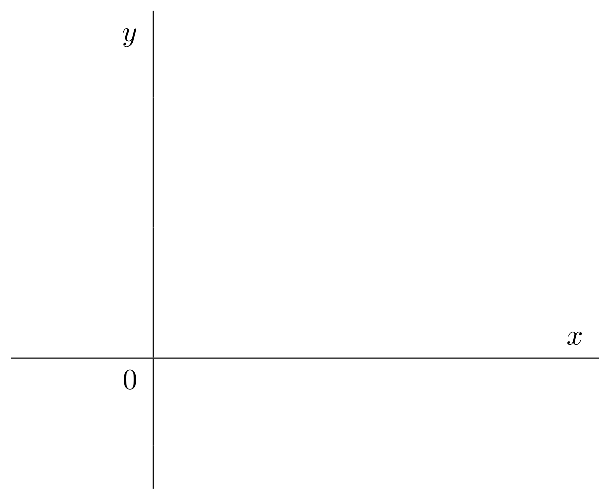
$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

or, equivalently,

$$f(b) - f(a) = f'(c)(b - a).$$

Illustration: $m_{AB} = \dfrac{f(b) - f(a)}{b - a}$





EXAMPLE 1. *At* $2:00$ *pm a car's speedometer reads* 30 mi/h. *At* $2:10$ *pm it reads* 50 mi/h. *Show that at some time between* $2:00$ *and* $2:10$ *the acceleration is exactly* $120 \text{mi}/\text{h}^2$.

EXAMPLE 2. *Find a number c that satisfies the conclusion of the Mean Value Theorem on the interval* $[0, 2]$ *when* $f(x) = x^3 + x - 1$.

EXAMPLE 3. *Suppose* $1 \le f'(x) \le 4$ *for all* x *in the [2, 5]. Show that* $3 \le f(5) - f(2) \le 12$.

Test for increasing/decreasing

- If $f'(x) > 0$ on an interval , then f is increasing on that interval.
- If $f'(x) < 0$ on an interval , then f is decreasing on that interval.
- If $f'(x) = 0$ on an interval , then f is constant on that interval.

EXAMPLE 4. *Determine all intervals where the following function*

$$f(x) = x^5 - \frac{5}{2}x^4 - \frac{40}{3}x^3 - 12$$

is increasing or decreasing.

First Derivative Test: Suppose that $x = c$ is a critical point of a continuous function f.

- If $f'(x)$ changes from negative to positive at $x = c$, then f has a local minimum at c.
- If $f'(x)$ changes from positive to negative at $x = c$, then f has a local maximum at c.
- If $f'(x)$ does not change sign at $x = c$, then f has no local maximum or minimum at c.

REMARK 5. The first derivative test only classifies critical points as local extrema and not as absolute extrema.

EXAMPLE 6. *For function from Example 4 identify all local extrema.*

EXAMPLE 7. *Find all intervals of increase and decrease of f and identify all local extrema.*

(a) $f(x) = xe^{2x}$

(b) $f(x) = x\sqrt[3]{x^2 - \frac{5}{3}}$

Recall here the **Second derivative test for concavity.** (see Section 5.1):

- If $f''(x) > 0$ for all x on an interval, then f is concave up on that interval.
- If $f''(x) < 0$ for all x on an interval, then f is concave down on that interval.

In addition, if f changes concavity at $x = a$, and $x = a$ is in the domain of f, then $x = a$ is an inflection point of f.

EXAMPLE 8. *Find intervals of concavity and inflection points of f, if $f'(x) = 4x^3 - 12x^2$.*

EXAMPLE 9. *Sketch the graph of f by locating intervals of increase/decrease, local extrema, concavity and inflection points.*

(a) $f(x) = x^4 - x^2$

(b) $f(x) = \dfrac{x}{(x-1)^2}$

Second derivative test for local extrema: Suppose f'' is continuous near c.

- If $f'(c) = 0$ and $f''(c) > 0$ then f has a local minimum at c.
- If $f'(c) = 0$ and $f''(c) < 0$ then f has a local maximum at c.

REMARK 10. If $f'(c) = 0$ and $f''(c) = 0$ or does not exist, then the test fails. In the case $f''(c)$ does not exist we use the first derivative test to find the local extrema.

EXAMPLE 11. *Find the local extrema for* $f(x) = 1 - 3x + 5x^2 - x^3$.