Chain Rule & Implicit Differentiation

Kelly Day, Deanna Sessions,
Lisa Smith, & Andrew Wlazlo

INTRODUCTION

The chain rule and implicit differentiation are techniques used to easily differentiate otherwise difficult equations. Both use the rules for derivatives by applying them in slightly different ways to differentiate the complex equations without much hassle. In this presentation, both the chain rule and implicit differentiation will be shown with applications to real world problems.

DEFINITION

Chain Rule

A way to differentiate functions within functions.

Implicit Differentiation

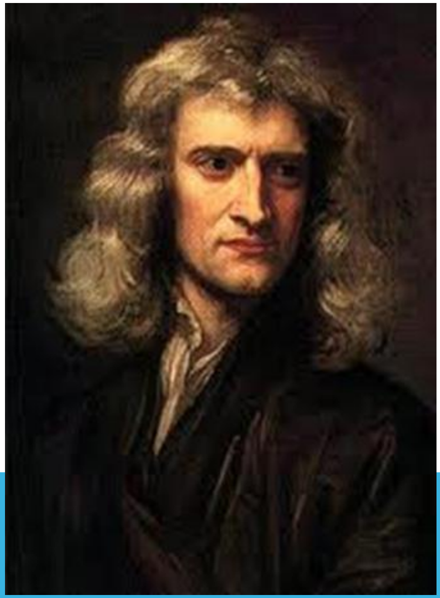
A way to take the derivative of a term with respect to another variable without having to isolate either variable.

HISTORY

The Chain Rule is thought to have first originated from the German mathematician Gottfried W. Leibniz. Although the memoir it was first found in contained various mistakes, it is apparent that he used chain rule in order to differentiate a polynomial inside of a square root. Guillaume de l'Hôpital, a French mathematician, also has traces of the chain rule in his *Analyse des infiniment petits*.

HISTORY

Implicit differentiation was developed by the famed physicist and mathematician Isaac Newton. He applied it to various physics problems he came across. In addition, the German mathematician Gottfried W. Leibniz also developed the technique independently of Newton around the same time period.



EXAMPLE 1: CHAIN RULE

Find the derivative of the following using chain rule

$$y=(x^2+5x^3-16)^{37}$$

EXAMPLE 1: CHAIN RULE

Step 1: Define inner and outer functions

$$y=(x^2+5x^3-16)^{37}$$

EXAMPLE 1: CHAIN RULE

Step 2: Differentiate outer function via power rule

$$y' = 37(x^2 + 5x^3 - 16)^{36}$$

EXAMPLE 1: CHAIN RULE

Step 3: Differentiate inner function and multiply by the answer from the previous step

$$y' = 37(x^2+5x^3-16)^{36}(2x+15x^2)$$

EXAMPLE 2: CHAIN RULE

A biologist must use the chain rule to determine how fast a given bacteria population is growing at a given point in time t days later.

The population grows at a rate of : $y(t) = 1000e^{5t-300}$

EXAMPLE 2: CHAIN RULE

Step 1: Identify the outer and inner functions

$$y = 1000e^{5t-300}$$

EXAMPLE 2: CHAIN RULE

Step 2: Derive the outer function

$$y' = 1000e^{5t-300}$$

EXAMPLE 2: CHAIN RULE

Step 3: Derive the inner function and multiply it by the outer function

$$y' = (5)1000e^{5t-300}$$

EXAMPLE 3: CHAIN RULE

For springs that obey Hooke's Law, the position of oscillation from the origin is given by $x=A\cos(\omega t)$ where A is the amplitude, ω is the frequency of oscillation, and t is time. A spring of spring constant $K=125$ N/m is attached to a block of mass $M=5$ kg. The spring is then compressed .25 meters and allowed to oscillate. Find the acceleration of the block with respect to time if $\omega=\sqrt{K/M}$. What is its acceleration at time $t=4$ seconds?

EXAMPLE 3: CHAIN RULE SOLUTIONS

$a(t)=\frac{d^2x}{dt^2}$

$x=A\cos(\omega t)$

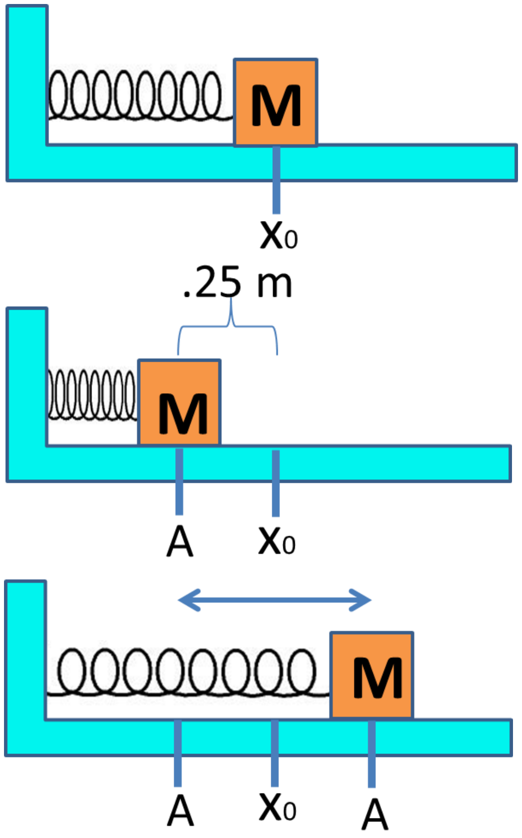
$\frac{dx}{dt}=-\omega A\sin(\omega t)$

$\frac{d^2x}{dt^2}=-\omega^2 A\cos(\omega t)$

$a(t)=-\omega^2 A\cos(\omega t)$

$\omega=\sqrt{K/M}=\sqrt{\frac{125}{5}}=5$

$a(4)=-(5)^2(.25)\cos(5*4)\approx-2.55\ m/s^2$



EXAMPLE 4: IMPLICIT DIFFERENTIATION

Find $\frac{dy}{dx}$ of:

$$x^2+y^2=27$$

EXAMPLE 4: IMPLICIT DIFFERENTIATION

Step 1: Notate y as y(x) to remember that y is a function of x

$$x^2+[y(x)]^2=27$$

EXAMPLE 4: IMPLICIT DIFFERENTIATION

Step 2: Differentiate function with respect to x. This means that every y must be multiplied by $\frac{dy}{dx}$

$$2x+2[y(x)]\frac{dy}{dx}=0$$

Remember to differentiate both sides of the function

EXAMPLE 4: IMPLICIT DIFFERENTIATION

Step 3: Isolate $\frac{dy}{dx}$ to one side of the equation.

$$\frac{dy}{dx} = \frac{-2x}{2[y(x)]}$$

EXAMPLE 4: IMPLICIT DIFFERENTIATION

Step 4: Replace y(x) with y and simplify the equation

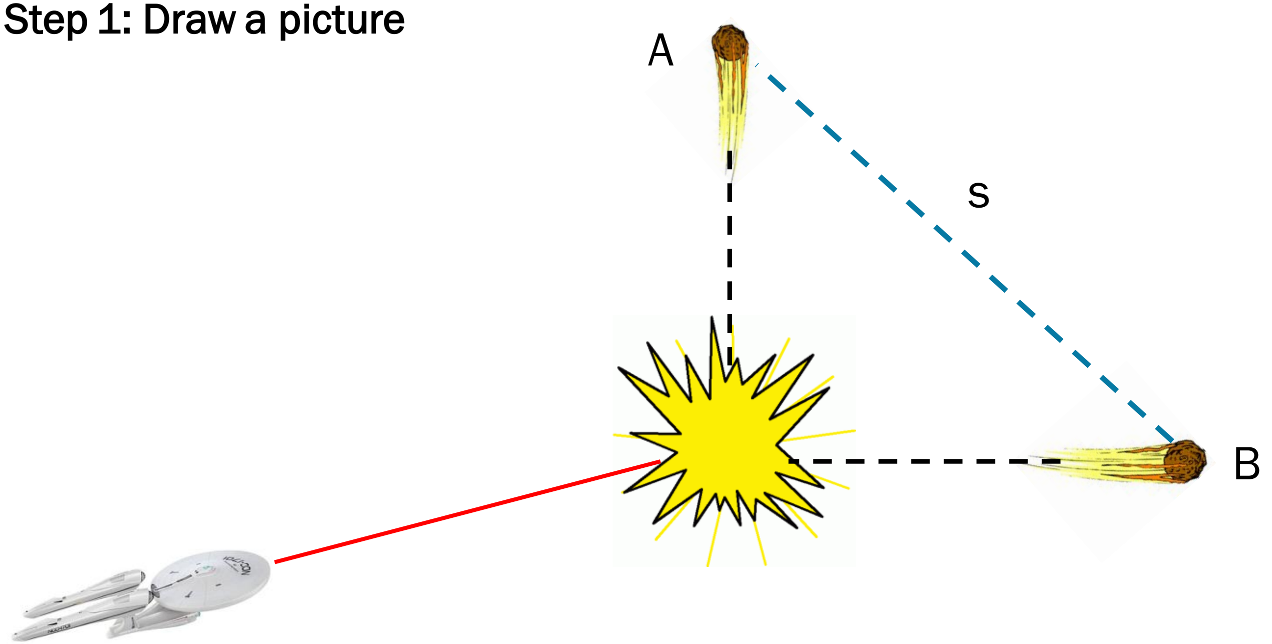
$$\frac{dy}{dx} = \frac{-x}{y}$$

EXAMPLE 5: IMPLICIT DIFFERENTIATION

Captain Kirk and the crew of the Starship Enterprise spot a meteor off in the distance. They decide it must be destroyed so they can live long and prosper, so they shoot the meteor in order to deter it from its earthbound path. After the explosion, the meteor divides in two, one piece heading due north, the other heading due east. While watching the rock split, Spock decides to measure the rate at which the distance between the two halves is increasing as the Starship Enterprise glides safely through the galaxy. He deduces that Meteor A is moving at a constant rate of 3000 mph north, and Meteor B is moving at a constant rate of 4000 mph east. At what rate is the distance between the two changing an hour after the explosion?

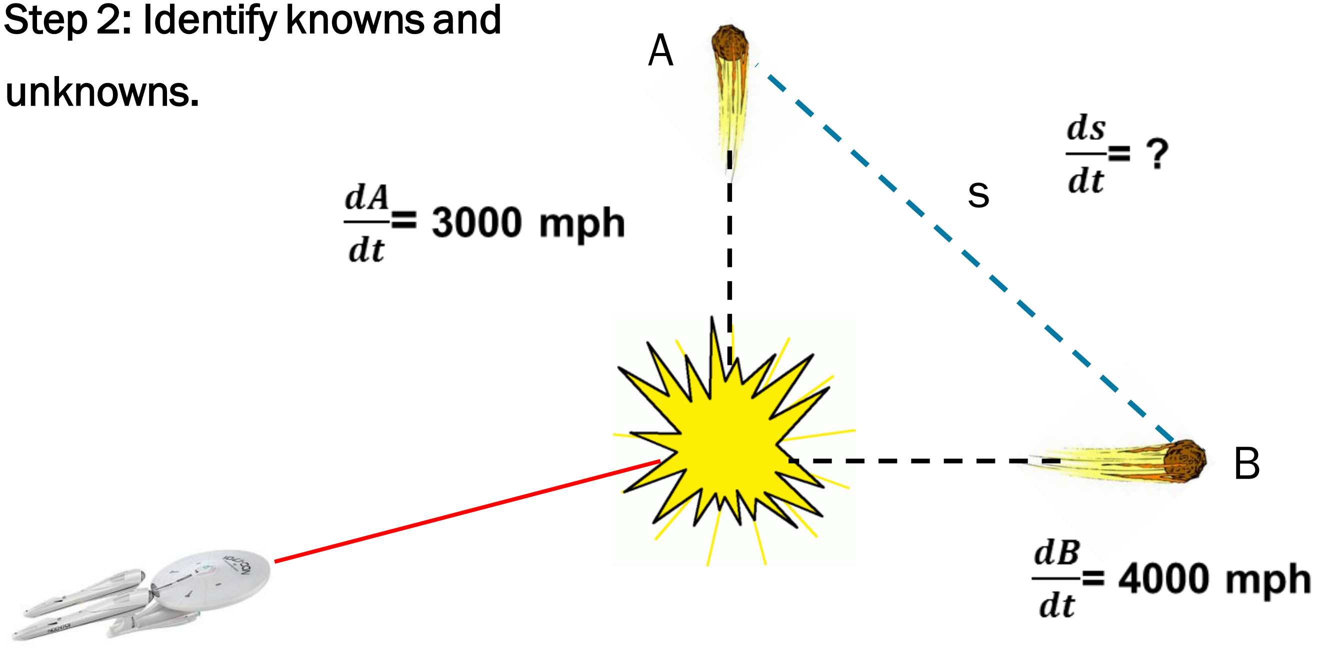
EXAMPLE 5: IMPLICIT DIFFERENTIATION

Step 1: Draw a picture



EXAMPLE 5: IMPLICIT DIFFERENTIATION

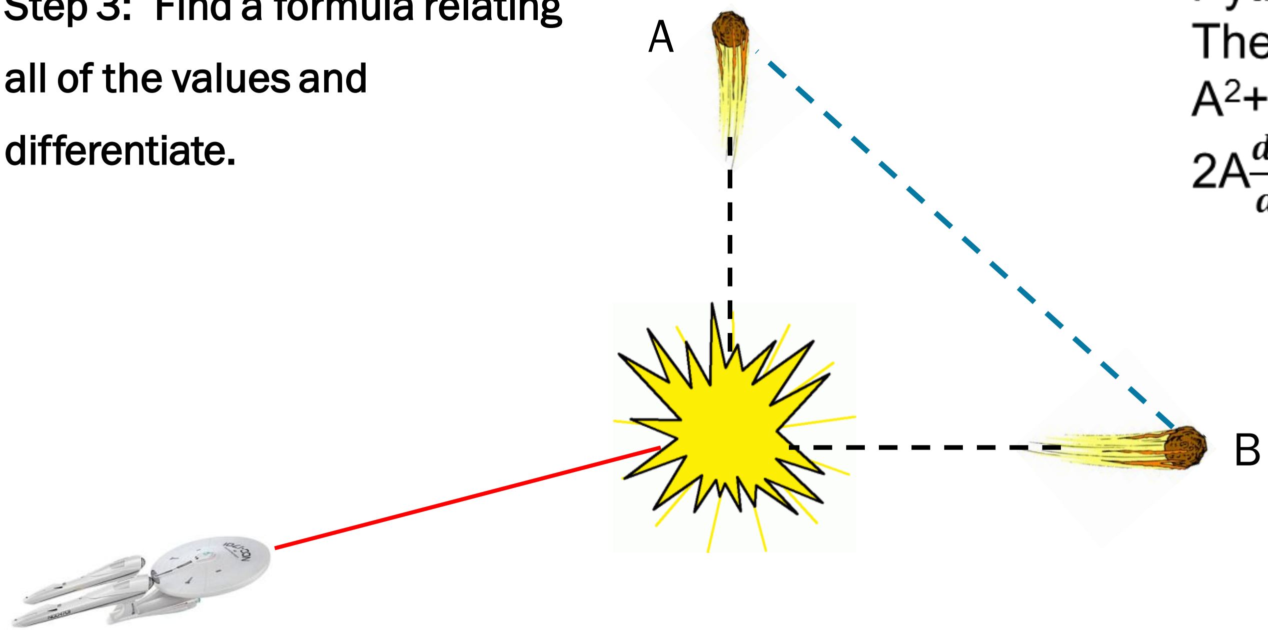
Step 2: Identify knowns and unknowns.



Using Pythagorean Theorem we find that at time t=1:
A= 3000
B=4000
S= 5000

EXAMPLE 5: IMPLICIT DIFFERENTIATION

Step 3: Find a formula relating all of the values and differentiate.



Pythagorean Theorem:

$A^2+B^2=S^2$

$2A\frac{dA}{dt}+2B\frac{dB}{dt}=2S\frac{dS}{dt}$

EXAMPLE 5: IMPLICIT DIFFERENTIATION

Step 4: Plug in values to derivative and solve for $\frac{dS}{dt}$

$2A\frac{dA}{dt}+2B\frac{dB}{dt}=2S\frac{dS}{dt}$

$2(3000)(3000)+2(4000)(4000)=2(5000)\ \frac{dS}{dt}$

$18000000+32000000=10000\ \frac{dS}{dt}$

$50000000=10000\ \frac{dS}{dt}$

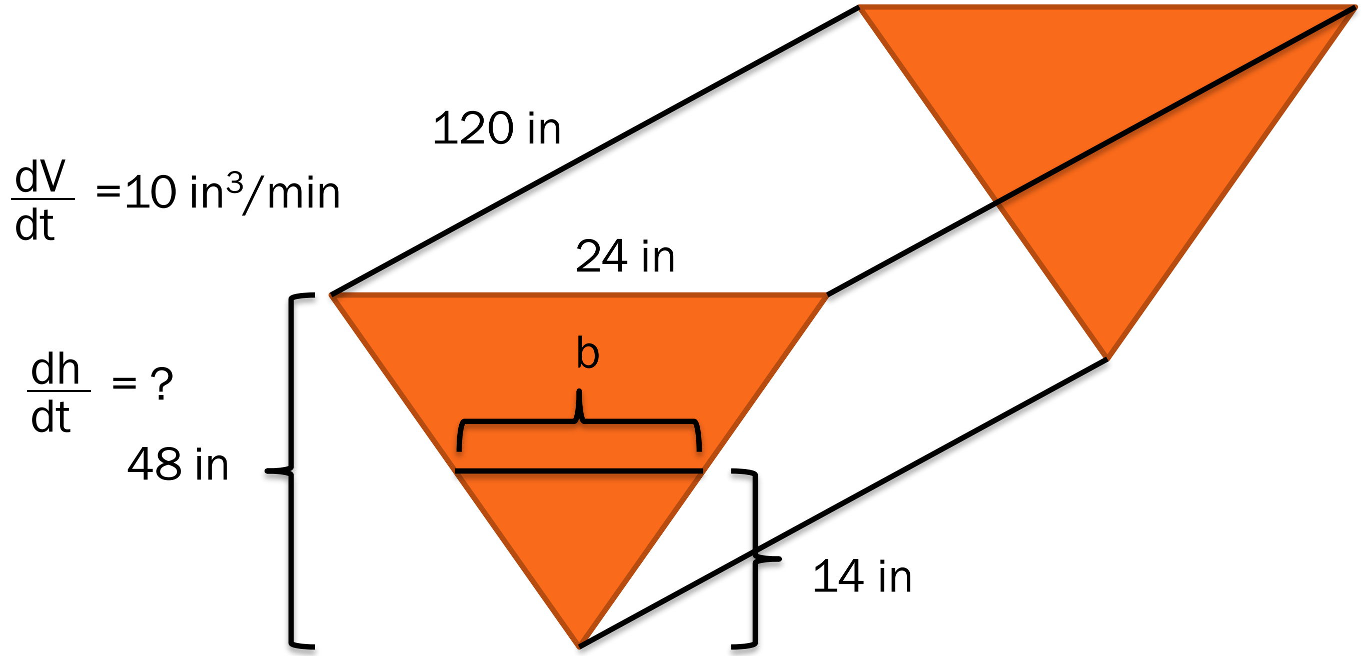
$\frac{dS}{dt}$ = 5000 mph

EXAMPLE 6: IMPLICIT DIFFERENTIATION

A trough is being filled with bird seed to fatten up turkeys for Thanksgiving. The trough is a triangular prism 10 feet long, 4 feet high, and 2 feet wide at the top. The trough is being filled at a rate of 10 inches^3/minute. How fast is the depth of the seed changing when the seed is 14 inches deep?

EXAMPLE 6: IMPLICIT DIFFERENTIATION

Step 1: Draw and label a picture.



Be sure all of the units are in agreement; the rate of change in height is given in inches, but the dimensions of the trough are given in feet.

EXAMPLE 6: IMPLICIT DIFFERENTIATION

Step 2: Find a formula relating the given values and differentiate.

$24/48 = b/h$

$48b = 24h$

$b = 24h/48$

$b = h/2$

$\frac{dV}{dt} = 10$ in/min

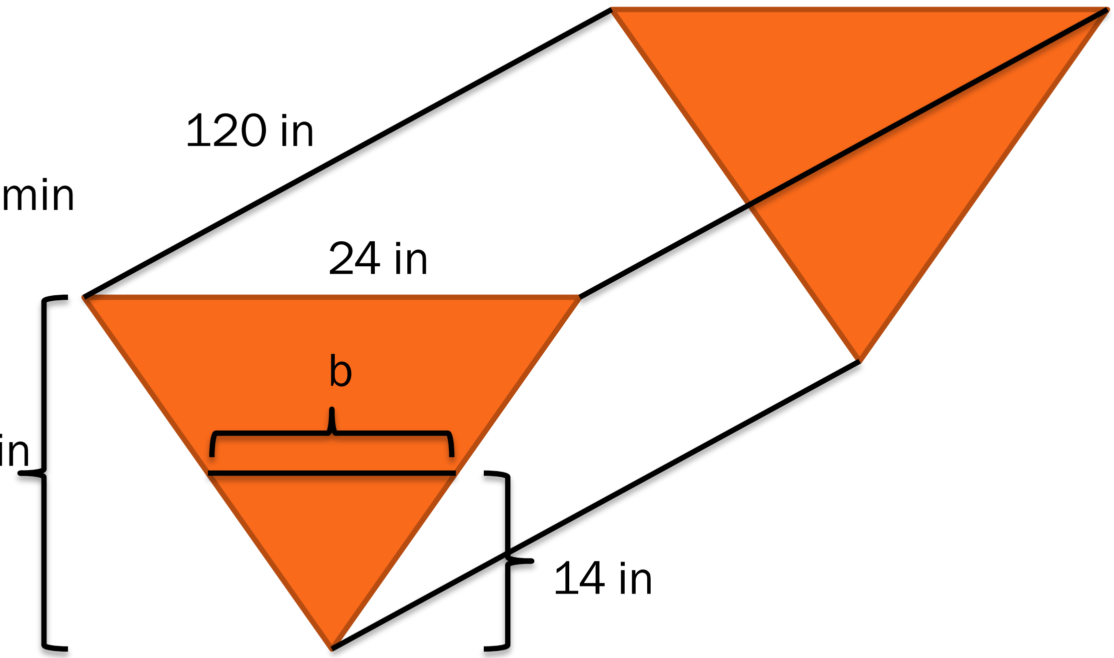
$V = (b*h)/2*L$

$V = [(h/2)*h]/2*L$

$V = [(h^2/2)/2]*L$

$dV/dt = (2h/4)*L*(dh/dt)$

$dV/dt = h/2*L*(dh/dt)$



Since you don't know the exact value of b, you need to use similar triangles to get b in terms of h.

EXAMPLE 6: IMPLICIT DIFFERENTIATION

Step 3: Plug in values and solve for dh/dt.

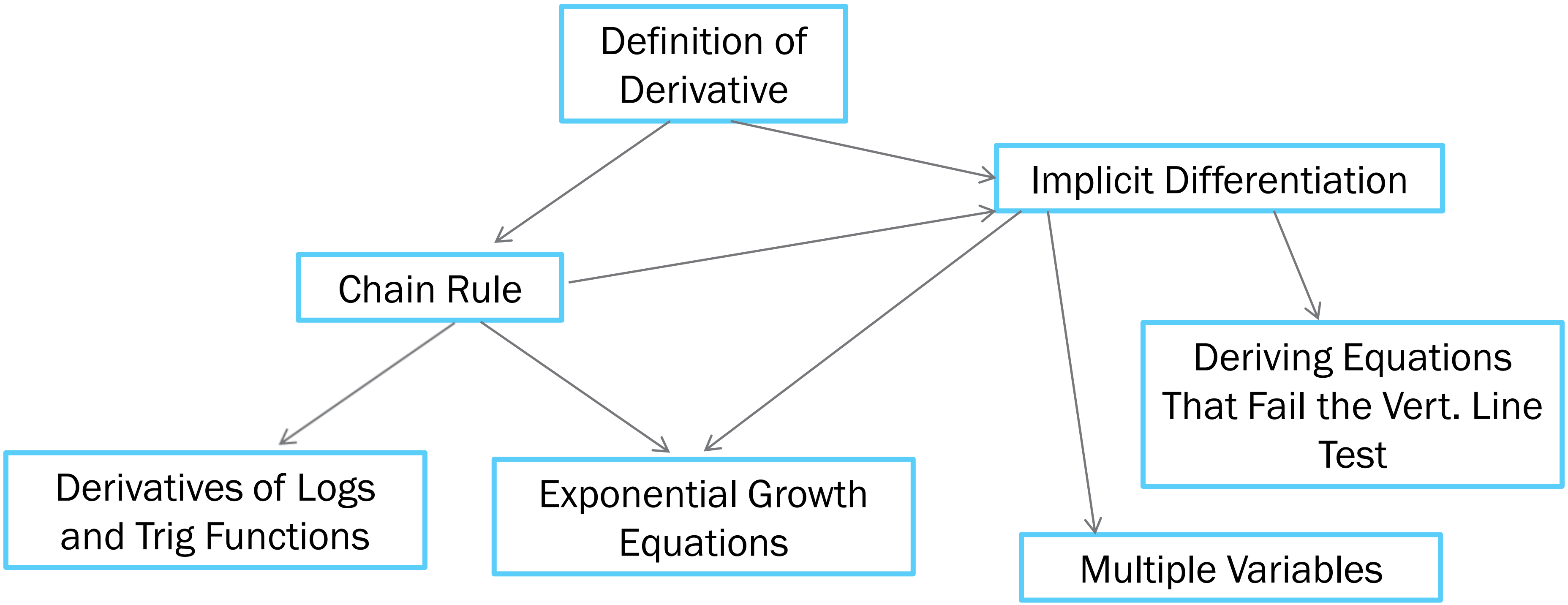
dV/dt = h/2*L*(dh/dt)

10 = (14/2)*(120)*(dh/dt)

10 = 840*(dh/dt)

dh/dt = .012 in/min

VISUAL MAP


Definition of Derivative
Implicit Differentiation
Chain Rule
Deriving Equations That Fail the Vert. Line Test
Derivatives of Logs and Trig Functions
Exponential Growth Equations
Multiple Variables