Section 3.1: Derivative

DEFINITION 1. The **Derivative** of a function f(x) at x = a is

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a} = \lim_{h \to 0} \frac{f(a + h) - f(a)}{h}.$$

Other common notations for the derivative of y = f(x) are f', $\frac{d}{dx}f(x)$.

It follows from the definition that the derivative f'(a) measures:

- The slope of the tangent line to the graph of f(x) at (a, f(a));
- The instantaneous rate of change of f(x) at x = a;
- The instantaneous velocity of the object at time at t = a (if f(t) is the position of an object at time t).

EXAMPLE 2. Given $f(x) = \frac{3}{x+5}$.

(a) Find the derivative of f(x) at x = -3.

(b) Find the equation of the tangent line of y = f(x) at x = -3.

Question: Where does a derivative not exist for a function?

DEFINITION 3. A function f(x) is said to be differentiable at x = a if f'(a) exists. EXAMPLE 4. Refer to the graph above to determine where f(x) is not differentiable.

CONCLUSION: A function f(x) is NOT differentiable at x = a if

- f(x) is not continuous at x = a;
- f(x) has a sharp turn at x = a (left and right derivatives are not the same);

• f(x) has a vertical tangent at x = a.

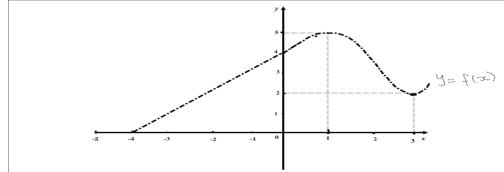
THEOREM 5. If f is differentiable at a then f is continuous at a.

The derivative as a function: If we replace a by x in Definition 1 we get:

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}.$$

A new function g(x) = f'(x) is called the **derivative of** f.

EXAMPLE 6. Use the graph of f(x) below to sketch the graph of the derivative f'(x).



EXAMPLE 7. Use the definition of the derivative to find f'(x) for $f(x) = \sqrt{1+3x}$.

EXAMPLE 8. Each limit below represents the derivative of function f(x) at x = a. State f and a in each case.

(a)
$$\lim_{h\to 0} \frac{(3+h)^4 - 81}{h}$$

(b)
$$\lim_{x \to \frac{3\pi}{2}} \frac{\sin x + 1}{x - \frac{3\pi}{2}}$$