Fall 2012 Math 152

Week in Review 12

courtesy: *Oksana Shatalov*

Final Exam Practice

1. Find the average value of the function $f(x) = \cos(ax + \pi/4)$ on the interval $\left[0, \frac{\pi}{a}\right]$, where a is a positive real parameter.

$$f_{ave} = \frac{1}{b-a}\int_a^b f(x)\,dx$$

$$f_{ave} = \frac{1}{\frac{\pi}{a} - 0}\int_0^{\pi/a} \cos\left(ax + \frac{\pi}{4}\right) dx$$

$$= \cancel{\frac{a}{\pi}}\,\frac{1}{\cancel{a}}\sin\left(ax + \frac{\pi}{4}\right)\Bigg|_0^{\pi/a}$$

$$= \frac{1}{\pi}\left(\underbrace{\sin\left(\cancel{a}\cdot\frac{\pi}{\cancel{a}} + \frac{\pi}{4}\right)}_{\sin\frac{5\pi}{4}} - \sin\frac{\pi}{4}\right) = \frac{1}{\pi}\left(-\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}\right) =$$

$$= -\frac{\sqrt{2}}{\pi}$$

2. Evaluate the integral $\int_0^1 xe^x \, dx$

$u = x$, $dv = e^x dx$

$du = dx$, $v = e^x$

Think also about

$\int x e^{x^2} dx$

u-sub

$$\int_0^1 x e^x dx = uv\Big|_0^1 - \int_0^1 v \, du$$

$$= xe^x\Big|_0^1 - \int_0^1 e^x dx$$

$$= 1 \cdot e^1 - 0 \cdot e^0 - e^x\Big|_0^1 = e - (e-1) = \boxed{1}$$

IMPROPER

3. Evaluate the integral $\displaystyle\int_0^1 \frac{4}{(3x+1)(x-1)}\,dx$

$$\frac{4}{(3x+1)(x-1)} = \frac{A}{3x+1} + \frac{B}{x-1} = \frac{A(x-1)+B(3x+1)}{(3x+1)(x-1)}$$

$$4 = A(x-1) + B(3x+1)$$

$x=1$ $\quad 4 = 0 + 4B \Rightarrow \boxed{B=1}$

$x=-\frac{1}{3}$ $\quad 4 = A\left(-\frac{1}{3}-1\right) \Rightarrow 4 = -\frac{4A}{3}$

$\Downarrow$

$\boxed{A=-3}$

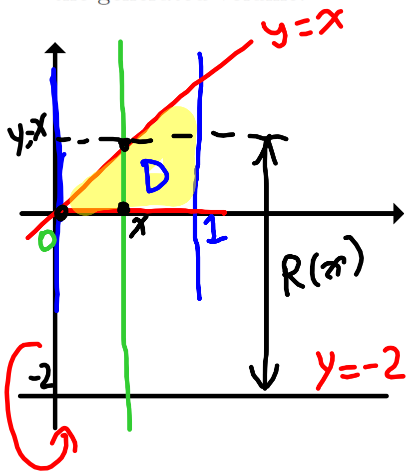
$$\int_0^1 f\,dx = \lim_{t\to 1^-}\int_0^t -\frac{3}{3x+1} + \frac{1}{x-1}\,dx = -\infty$$

$\boxed{\text{divergent}}$

$$\left(-3\cdot\frac{1}{3}\ln|3x+1| + \ln|x-1|\right)$$

Note that this is an antiderivati of the integrand

4. The region $D = \{(x, y) : 0 \le x \le 1, 0 \le y \le x\}$ is rotated about the horizontal line $y = -2$. Find the generated volume.



Apply Washer Method

$$V = \int_0^1 A(x)\,dx$$

$$= \int_0^1 \pi\left(R^2(x) - r^2(x)\right)dx$$

$$R(x) = x - (-2) = x + 2$$

$$r(x) = 2$$

$$V = \pi \int_0^1 (x+2)^2 - 2^2 \, dx$$

$$= \pi \int_0^1 x^2 + 4x + \cancel{4} - \cancel{4} \, dx = \boxed{\frac{7\pi}{3}}$$

5. Evaluate the integral $\displaystyle\int_{-8}^{0} \frac{3x}{\sqrt{x+9}}\,\mathrm{d}x$

$u = x+9 \Rightarrow du = dx$

$1 \le u \le 9 \Rightarrow x = u-9$

$$\int_{-8}^{0} \frac{3x\,dx}{\sqrt{x+9}} = 3\int_{1}^{9} \frac{u-9}{\sqrt{u}}\,du$$

$$= 3\int_{1}^{9} \frac{u}{\sqrt{u}} - \frac{9}{\sqrt{u}}\,du = 3\int_{1}^{9} u^{\frac{1}{2}} - 9u^{-\frac{1}{2}}\,du$$

$$= 3\left[\frac{u^{3/2}}{3/2} - \frac{9u^{1/2}}{1/2}\right]\Bigg|_{1}^{9}$$

$$= 3\left[\frac{2}{3}\cdot 3^{3} - 2\cdot 9\cdot 3 - \left(\frac{2}{3} - 18\right)\right]$$

$$= \ldots$$

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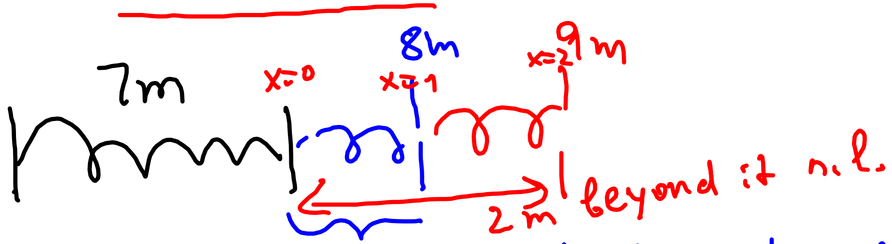
6. When a spring of natural length 7m is extended to 8m, the force required to hold it in position is 20N. Find the work done (in Joules) when the spring is extended from 8m to 9m.



$x = 1$m beyond its natural length.

$x = 1 \Rightarrow F = 20$ N

$F = kx$

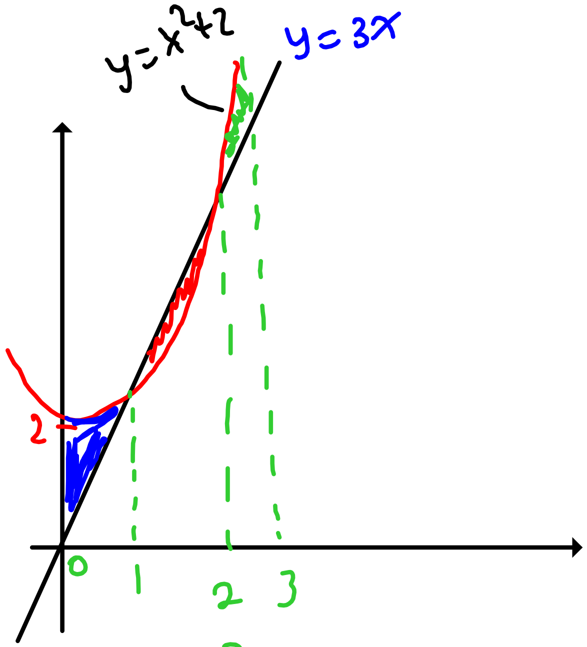
$20 = k \cdot 1 \Rightarrow \boxed{k = 20}$

$$W = \int_1^2 F(x)\,dx = \int_1^2 kx\,dx = 20\int_1^2 x\,dx$$

$$= 20 \cdot \frac{x^2}{2}\Big|_1^2$$

$$= 10\left(2^2 - 1^2\right) = 10 \cdot 3 = 30\text{J}$$

7. Find the area bounded by the curves $y = 3x$ and $y = x^2 + 2$ from $x = 0$ to $x = 3$.



Intersection point

$3x = x^2 + 2$

$x^2 - 3x + 2 = 0$

$(x-1)(x-2) = 0$

$x = 1, \quad x = 2$

$$A = \int_0^3 (\text{upper funct.}) - (\text{lower funct.})\, dx$$

$$= \underbrace{\int_0^1 (x^2+2) - 3x\, dx}_{\frac{5}{6}} + \underbrace{\int_1^2 3x - (x^2+2)\, dx}_{\frac{1}{6}} + \underbrace{\int_2^3 (x^2+2) - 3x\, dx}_{\frac{5}{6}}$$

$$= \boxed{\frac{11}{6}}$$

8. Evaluate the integral $\int_0^{\pi/2} \sin^4 x \cos^3 x \, dx$.

$$= \int_0^{\pi/2} \underbrace{\sin^4 x}_{u^4} \underbrace{\cos^2 x}_{\underbrace{1-\sin^2 x}_{1-u^2}} \underbrace{\cos x \, dx}_{d(\sin x) = du}$$

$$\boxed{\sin x = u}$$

$x = 0 \Rightarrow u = \sin 0 = 0$

$x = \frac{\pi}{2} \Rightarrow u = \sin \frac{\pi}{2} = 1$

$$\int_0^1 u^4 (1-u^2)\, du = \int_0^1 u^4 - u^6 \, du$$

$$= \frac{u^5}{5} - \frac{u^7}{7} \Big|_0^1 = \frac{1}{5} - \frac{1}{7} = \boxed{\frac{2}{35}}$$

9. Find the integral $\int \frac{1}{\sqrt{4x - x^2 - 3}}\, dx$

Completing square $(a \pm b)^2 = a^2 \pm 2ab + b^2$

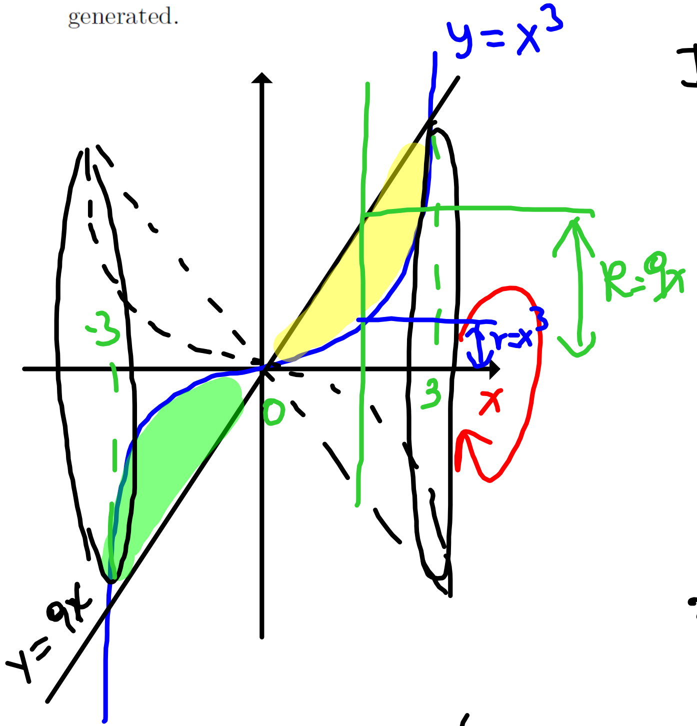
$$4x - x^2 - 3 = -(x^2 - 4x + 3)$$

$$= -(\underbrace{x^2 - 4x + 4}_{(x-2)^2} - 4 + 3)$$

$$= -[(x-2)^2 - 1] = 1 - (x-2)^2$$

$$\int \frac{dx}{\sqrt{1 - (x-2)^2}} = \arcsin(x-2) + C$$

10. The region bounded by the curves $y = x^3$ and $y = 9x$ is rotated about the x-axis. Find the volume generated.



Intersection points

$$x^3 = 9x$$

$$x^3 - 9x = 0$$

$$x(x^2 - 9) = 0$$

$$x(x-3)(x+3) = 0$$

$$x = 0,\ x = 3,\ x = -3$$

$$V = 2\left(\begin{array}{l}\text{Volume generated}\\ \text{by yellow region}\end{array}\right)$$

use washers

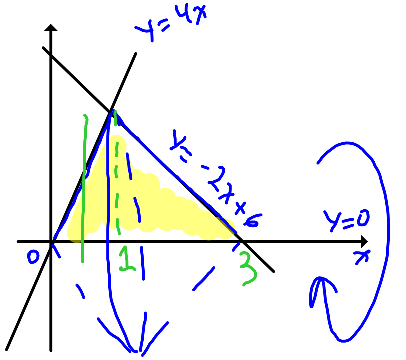
$$V = 2\pi \int_0^3 R^2(x) - r^2(x)\,dx$$

$$= 2\pi \int_0^3 (9x)^2 - (x^3)^2\,dx = \ldots = \frac{5832\pi}{7}$$

11. The region bounded by the lines $y = 0$, $y = -2x + 6$ and $y = 4x$ is rotated about the x-axis. Set up, but don't evaluate, integrals which give the volume generated using

(a) the washer/disk method

(b) the cylindrical shells method.



Int. point

$4x = -2x + 6$

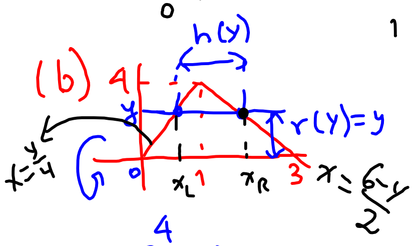
$x = 1$

(a) $V = \int_0^3 A(x)\,dx = \pi \int_0^3 R^2(x)\,dx$

$$R(x) = y = \begin{cases} 4x, & 0 \le x \le 1 \\ -2x+6, & 1 \le x \le 3 \end{cases}$$

$$V = \int_0^1 (4x)^2\,dx + \int_1^3 (-2x+6)^2\,dx$$

(b)



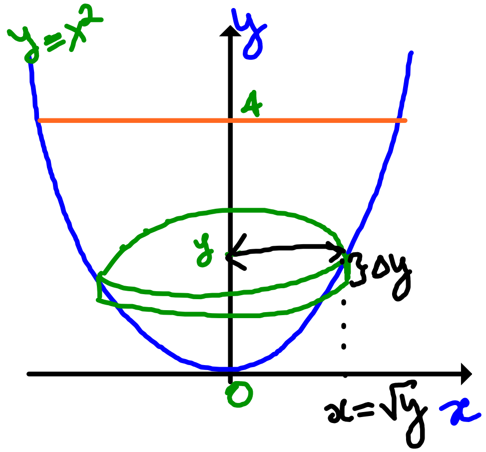
$h(y) = x_R - x_L$

$= \frac{6-y}{2} - \frac{y}{4}$

$$V = \int_0^4 A(y)\,dy = 2\pi \int_0^4 r(y)\,h(y)\,dy$$

$$V = 2\pi \int_0^4 y\left(\frac{6-y}{2} - \frac{y}{4}\right)dy$$

12. A tank was constructed by rotating about the y-axis the part of the parabola $y = x^2$ such that depth of the tank is $4ft$. The tank is then filled with a liquid solution weighing $60lb/ft^3$. Find the work done in pumping out the tank.



Given $\rho g = 60$

$$W = \rho g \int_0^4 \underbrace{A(y)}_{\text{crossect. area}} \underbrace{dist(y)}_{\text{distance from top}} \, dy$$

$$A(y) = \pi[r(y)]^2 = \pi(\sqrt{y})^2 = \pi y$$

$$dist(y) = 4 - y$$

$$W = 60 \int_0^4 \pi y(4-y)dy = 60\pi \int_0^4 (4y - y^2)dy$$

$$= 60\pi \left(2y^2 - \frac{y^3}{3}\right)\Big|_0^4$$

$$= 60\pi\left(2\cdot 16 - \frac{64}{3}\right) = 60\pi \cdot 32\left(1 - \frac{2}{3}\right)$$

$$= \cancel{60}^{20}\pi \cdot 32 \cdot \frac{1}{\cancel{3}}$$

$$= \boxed{640\pi \text{ ft-lb}}$$

13. Which of these integrals represents the length of the curve $y = x^4$ from $x = 0$ to $x = 1$?

(a) $\int_0^1 \sqrt{1 + x^8}\, dx$

(b) $\int_0^1 \sqrt{1 + 4x^3}\, dx$

(c) $\int_0^1 \sqrt{1 + 16x^6}\, dx$

(d) $2\pi \int_0^1 x^4 \sqrt{1 + 16x^6}\, dx$

(e) $\int_0^1 \sqrt{1 + x^4}\, dx$

$$L = \int ds = \int_0^1 \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

$$= \int_0^1 \sqrt{1 + \left(4x^3\right)^2}\, dx$$

$$= \int_0^1 \sqrt{1 + 16x^6}\, dx$$

14. By comparing the functions $\frac{1}{1+x^5}$ and $\frac{1}{x^5}$ what conclusion can be drawn about $\int_1^\infty \frac{1}{1+x^5}\,dx$?

(a) Its value is 1.

(b) Its value is 1/2.

(c) It diverges.

(d) It converges.

(e) No conclusion is possible.

$$\int_1^\infty \frac{dx}{1+x^5}$$

$$1+x^5 > x^5 \quad , x \geq 1$$

$$\frac{1}{1+x^5} < \frac{1}{x^5}$$

$$\int_1^\infty \frac{dx}{x^5}$$ converges $p = 5 > 1$

By Comp. Theorem the integral converges

15. Does the integral $\int_0^1 \frac{1+x}{\sqrt{x}}\,dx$ diverge? NO

Improper $\int$-l of 2nd type

First find anti-derivative

$$\int \frac{1+x}{\sqrt{x}}\,dx = \int x^{-\frac{1}{2}} + x^{\frac{1}{2}}\,dx$$

$$= 2x^{\frac{1}{2}} + \frac{2}{3}x^{3/2}.$$

$$= 2\sqrt{x} + \frac{2}{3}x\sqrt{x}$$

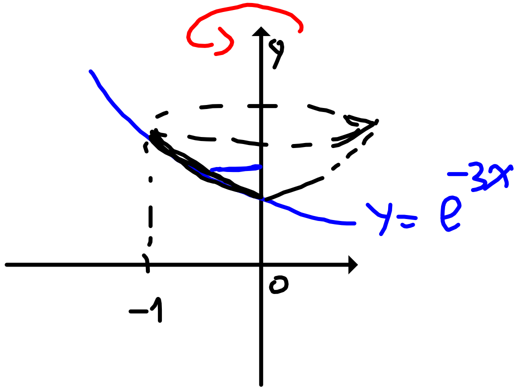
$$\int_0^1 \frac{1+x}{\sqrt{x}}\,dx = \lim_{t\to 0^+} \int_t^1 \frac{1+x}{\sqrt{x}}\,dx$$

$$= \lim_{t\to 0^+} \left(2\sqrt{x} + \frac{2}{3}x\sqrt{x}\right)\Big|_t^1$$

$$= 2 + \frac{2}{3} - (0+0) = \boxed{\frac{8}{3}}$$

Convergent

16. The curve $y = e^{-3x}$ from $x = -1$ to $x = 0$ is rotated about the y-axis. Set up, but don't evaluate, integral which gives the generated surface area.



$$SA = 2\pi \int x \, ds$$

$$= 2\pi \int_{-1}^{0} x \sqrt{1 + \left(\frac{dy}{dx}\right)^2} \, dx$$

$$= 2\pi \int_{-1}^{0} x \sqrt{1 + \left(-3e^{-3x}\right)^2} \, dx$$

$$SA = 2\pi \int_{-1}^{0} x \sqrt{1 + 9e^{-6x}} \, dx$$

$\sum_{}^{\infty} a_n$

17. Given a positive series with general term a_n.

(a) **TRUE** **FALSE** If $\lim\limits_{n\to\infty} a_n = 0$ then the series converges.

(b) **TRUE** **FALSE** If $a_n \geq \frac{1}{n^4}$ then the series converges.

(c) **TRUE** **FALSE** If $\lim\limits_{n\to\infty} \frac{a_{n+1}}{a_n} = 1$ then the series converges.

(d) **TRUE** **FALSE** If $a_n \leq \frac{1}{\sqrt{n}}$ then the series diverges.

(a) False

because $\lim\limits_{n\to\infty} a_n = 0$

$\Downarrow$

Div Test Fails

(b) $\sum \frac{1}{n^4}$ conv. , $p=4>1$

(c) $\lim\limits_{n\to\infty} \frac{a_{n+1}}{a_n} = 1 \Rightarrow$ RT fails

(d) $\sum \frac{1}{\sqrt{n}}$ divergent , $p=\frac{1}{2}<1$

18. Find the interval of convergence of $\sum_{n=1}^{\infty} \frac{(-1)^{n+1} x^n}{\sqrt[5]{n^2}}$.

$|a_n| = \frac{|x|^n}{n^{2/5}}$ $\qquad (n^2)^{\frac{1}{5}} = n^{2/5}$

$|a_{n+1}| = \frac{|x|^{n+1}}{(n+1)^{2/5}}$

RT:

$L = \lim_{n\to\infty} |a_{n+1}| \cdot \frac{1}{|a_n|} = \lim_{n\to\infty} \frac{|x|^{\cancel{n+1}}}{(n+1)^{2/5}} \cdot \frac{n^{2/5}}{\cancel{|x|^n}}$

$= |x| \lim_{n\to\infty} \left(\frac{n}{n+1}\right)^{2/5} = |x| < 1$

$R = 1$

$L = 1 \Rightarrow x = \pm 1$ (end points)

$x = -1$

$\sum \frac{(-1)^{n+1} \cdot (-1)^n}{n^{2/5}}$

$\sum \frac{(-1)^{2n+1}}{n^{2/5}}$

$\sum -\frac{1}{n^{2/5}}$

divergent p-series

$p = \frac{2}{5} < 1$

$x = 1$

$\sum \frac{(-1)^{n+1} 1^n}{n^{2/5}}$

$\sum \frac{(-1)^{n+1}}{n^{2/5}}$

Alternating Series

$\lim_{n\to\infty} \frac{1}{n^{2/5}} = 0$

$\left\{\frac{1}{n^{2/5}}\right\}$ decreasing

convergent by AST

$\boxed{(-1, 1]}$

19. Compute $\lim_{x\to 0} \dfrac{x^4 - \sin x^4}{1-\cos(x^6)}$.

We know

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \dots$$

Thus.

$$\sin x^4 = x^4 - \frac{(x^4)^3}{3!} + \frac{(x^4)^5}{5!} - \dots = x^4 - \frac{x^{12}}{3!} + \frac{x^{20}}{5!} - \dots$$

$$\cos x^6 = 1 - \frac{(x^6)^2}{2} + \frac{(x^6)^4}{4!} - \dots = 1 - \frac{x^{12}}{2} + \frac{x^{24}}{4!} - \dots$$

$$\lim_{x\to 0} \frac{\cancel{x^4} - \left(\cancel{x^4} - \frac{x^{12}}{3!} + \frac{x^{20}}{5!} - \dots\right)}{\cancel{1} - \left(\cancel{1} - \frac{x^{12}}{2} + \frac{x^{24}}{4!} - \dots\right)}$$

$$\lim_{x\to 0} \frac{\frac{x^{12}}{6} - \frac{x^{20}}{5!} + \dots}{\frac{x^{12}}{2} - \frac{x^{24}}{4!} + \dots} = \lim_{x\to 0} \frac{\cancel{x^{12}}\left(\frac{1}{6} - \frac{x^8}{5!} + \dots\right)}{\cancel{x^{12}}\left(\frac{1}{2} - \frac{x^{12}}{4!} + \dots\right)} =$$

$$\frac{\frac{1}{6}+0}{\frac{1}{2}+0} = \frac{2}{6} = \boxed{\frac{1}{3}}$$

20. The series $\displaystyle\sum_{n=2012}^{\infty} \frac{(n!)^6}{((3n)!)^6} = \sum_{n=2012}^{\infty} \left(\frac{n!}{(3n)!}\right)^6$

(a) Diverges by the Integral Test

(b) Diverges by the Comparison Test

(c) Diverges by the Ratio Test

(d) Converges by the Ratio Test

(e) Diverges because $\lim_{n\to\infty} a_n = 0$.

$$\lim_{n\to\infty} |a_{n+1}| \cdot \frac{1}{|a_n|} =$$

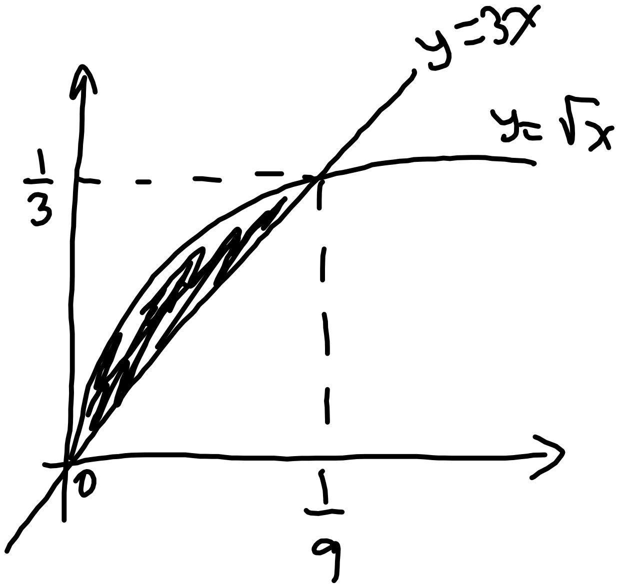
$$\lim_{n\to\infty} \left[\frac{(n+1)!}{(3(n+1))!}\right]^6 \cdot \left[\frac{(3n)!}{n!}\right]^6$$

$$= \lim_{n\to\infty} \left[\frac{\cancel{n!}\,(n+1)}{\cancel{(3n)!}\,(3n+1)(3n+2)(3n+3)} \cdot \frac{\cancel{(3n)!}}{\cancel{n!}}\right]^6 = 0 < 1$$

$\Downarrow$

converg.

21. Find the area bounded by the curves $y = 3x$ and $y = \sqrt{x}$.



Int. point

$3x = \sqrt{x}$

$9x^2 = x \quad , \quad x > 0$

$x(9x-1) = 0$

$x = 0$ OR $x = \frac{1}{9}$

$$A = \int_0^{1/9} (\sqrt{x} - 3x)\,dx$$

OR

$$\int_0^{1/3} \left(\frac{y}{3} - y^2\right) dy$$

$$\boxed{\frac{1}{162}}$$

22. A trigonometric substitution converts the integral $\int \sqrt{x^2+20x+75}\,dx$ to

(a) $5\int \tan^3\theta\, d\theta$

(b) $25\int \tan^2\theta \sec\theta\, d\theta$

(c) $25\int \sin^3\theta\, d\theta$

(d) $5\int \sin^2\theta \cos\theta\, d\theta$

(e) $5\int \tan\theta \sec^2\theta\, d\theta$

Completing squares

$$x^2+20x+75$$

$$= x^2 + \overset{2\cdot x\cdot 10}{20x} + 10^2 - 10^2 + 75$$

$$= \underbrace{(x+10)^2}_{} - 25 = (x+10)^2 - 5^2$$

$$= 5^2\left[\left(\frac{x+10}{5}\right)^2 - 1\right]$$

$$\sec^2\theta - 1 = \tan^2\theta$$

$$\boxed{\frac{x+10}{5} = \sec\theta}$$

$$= 25(\sec^2\theta - 1)$$

$$= 25\tan^2\theta$$

continued on next page

$$\frac{x+10}{5} = \sec\theta$$

$$\frac{dx}{5} = \sec\theta\tan\theta\, d\theta$$

$$\sqrt{x^2+20x+75} = \sqrt{25\tan^2\theta} = 5\tan\theta$$

$$\int \sqrt{x^2+20x+75}\, dx =$$

$$\int 5\tan\theta \cdot 5\sec\theta\tan\theta\, d\theta$$

$$= 25\int \tan^2\theta\sec\theta\, d\theta$$

23. Find the average value of the function $f(x) = \sin^3 x$ on the interval $\left[0, \frac{\pi}{2}\right]$.

odd

$$f_{ave} = \frac{1}{\frac{\pi}{2} - 0} \int_0^{\pi/2} \sin^3 x \, dx$$

$$= \frac{2}{\pi} \int_0^{\pi/2} \underbrace{\underbrace{\sin^2 x}_{1-\cos^2 x}}_{1-u^2} \; \underbrace{\underbrace{\sin x \, dx}_{-d(\cos x)}}_{-du}$$

$u = \cos x$

$x = 0 \Rightarrow u = 1$

$x = \pi/2 \Rightarrow u = 0$

$$f_{ave} = \frac{2}{\pi} \int_1^0 (1 - u^2)(-du) = \frac{4\pi}{3}$$

24. Evaluate the integral $\int x^3 \sin 3x \, dx$.

	diff. u	int. dV
$+$	x^3	$\sin 3x$
$-$	$3x^2$	$-\frac{1}{3}\cos 3x$
$+$	$6x$	$-\frac{1}{9}\sin 3x$
$-$	6	$\frac{1}{27}\cos 3x$
$+$	0	$\frac{1}{81}\sin 3x$

Integration by parts

$$\int x^3 \sin 3x\,dx = -\frac{1}{3}x^3\cos 3x - 3x^2\left(-\frac{1}{9}\sin 3x\right) + 6x\left(\frac{1}{27}\cos 3x\right) - 6\cdot\frac{1}{81}\sin 3x + C$$

$$= -\frac{x^3}{3}\cos 3x + \frac{x^2}{3}\sin 3x + \frac{2x}{9}\cos 3x - \frac{2}{27}\sin 3x + C$$

25. The improper integral $\displaystyle\int_5^{\infty} \frac{5+\sin x}{x^{10}}\,\mathrm{d}x$

(a) converges to the value $1/50$.

(b) converges, because the integrand oscillates.

(c) diverges to ∞

(d) converges by comparison with $\displaystyle\int_5^{\infty} \frac{6}{x^{10}}\,\mathrm{d}x$

(e) diverges but doesn't approach ∞, because the integrand oscillates.

Comparison Theorem

$$\frac{5+\sin x}{x^{10}} < \frac{5+1}{x^{10}} = \frac{6}{x^{10}}$$

$$\int_5^{\infty} \frac{6\,dx}{x^{10}} \quad \text{conv.} \quad p=10>1$$

26. Set up, but don't evaluate, integral which gives the arc length of the curve

$$x = 2012 + \cos(2t), \quad y = t - \sin(2t), \quad 0 \le t \le \pi/2.$$

Circle the correct answer:

(a) $\int_0^{\pi/2} \sqrt{2013 + t^2 + 2\cos(2t) - 2\sin(2t)}\,dt$

(b) $\int_0^{\pi/2} \sqrt{2012 + t^2 + 2\cos(2t) - 2\sin(2t)}\,dt$

(c) $\int_0^{\pi/2} \sqrt{2012 - 4\cos(2t) + 4\sin(2t)}\,dt$

(d) $\int_0^{\pi/2} \sqrt{5 - 4\cos(2t)}\,dt$

(e) $\int_0^{\pi/2} \sqrt{6 - 4\cos(2t)}\,dt$

$$L = \int ds$$

$$\int_0^{\pi/2} ds = \int_0^{\pi/2} \sqrt{\underbrace{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}}\, dt$$

$$\left(-2\sin(2t)\right)^2 + \left(1 - 2\cos(2t)\right)^2$$

$$= 4\sin^2(2t) + 1 - 4\cos(2t) + 4\cos^2(2t)$$

$$4\underbrace{\left(\sin^2(2t) + \cos^2(2t)\right)}_{1} + 1 - 4\cos(2t)$$

$$= 5 - 4\cos 2t$$

27. Determine whether the integral $\int_1^{\infty} \frac{1}{(x-3)^4}\,dx$ is divergent or convergent.

infinite integral → TYPE I

discontinuity at $x=3$ → TYPE II

$$\int_1^{\infty} \frac{dx}{(x-3)^4} = \underbrace{\int_1^{3} \frac{dx}{(x-3)^4}}_{\text{TYPE II}} + \underbrace{\int_3^{4} \frac{dx}{(x-3)^4}}_{\text{TYPE II}} + \underbrace{\int_4^{\infty} \frac{dx}{(x-3)^4}}_{\text{TYPE I}}$$

$u = x-3$

$$\underbrace{\int_1^{\infty} \frac{du}{u^4}}_{\text{conv.}}$$

$$\lim_{t \to 3^-} \int_1^{t} \frac{dx}{(x-3)^4}$$

$$= \lim_{t \to 3^-} -\frac{1}{3(x-3)^3}\Bigg|_1^t$$

$$= -\frac{1}{3}\lim_{t \to 3^-}\left(\frac{1}{(t-3)^3} - \frac{1}{(-2)^3}\right) = \infty \text{ divergent}$$

$\Downarrow$

the sum of 3 integrals is divergent

28. Find $\displaystyle\int \frac{x^2+1}{x^3+2x^2+x}\,dx$

$$\frac{x^2+1}{x(x^2+2x+1)} = \frac{x^2+1}{x(x+1)^2} = \frac{A}{x} + \frac{B}{x+1} + \frac{C}{(x+1)^2}$$

$$x^2+1 = A(x+1)^2 + Bx(x+1) + Cx$$

$$x=-1 \Rightarrow 2 = -C \Rightarrow \boxed{C=-2}$$

$$x=0 \Rightarrow \boxed{1 = A}$$

$$x^2: \quad 1 = A + B$$

$$\Rightarrow B = 0$$

$$\int = \int \frac{dx}{x} - \frac{2\,dx}{(x+1)^2}$$

$$= \ln|x| + \frac{2}{x+1} + C$$

29. Describe the surface having the equation $x^2 + y^2 + z^2 - 10x + 2z + 1 = 0$.

$$x^2 - 10x + 5^2$$
$$+ y^2$$
$$+ z^2 + 2z + 1 = -1 + 5^2 + 1$$

$$(x-5)^2 + y^2 + (z+1)^2 = 5^2$$

sphere centered at $(5, 0, -1)$

with radius $r = 5$

30. Compute $\int_{-1}^{1} \frac{1}{x^6}\, dx$ improper of TYPE $\underline{\mathrm{II}}$

$x=0$ discontinuity point

$$\int_{-1}^{0} \frac{dx}{x^6} + \int_{0}^{1} \frac{dx}{x^6}$$

$$\underbrace{\lim_{t \to 0^-} \int_{-1}^{t} \frac{dx}{x^6}} + \lim_{t \to 0^+} \int \frac{dx}{x^6}$$

$$\lim_{t \to 0^-} -\frac{1}{5x^5}\Big|_{-1}^{t}$$

$$-\frac{1}{5} \lim_{t \to 0^-} \left(\frac{1}{t^5} - \frac{1}{(-1)^5} \right) = -\infty$$

divergent

the given integral diverges

31. Which of the following series are convergent?

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{2012^n}{n!}$ convergent

Ratio Test

$$\lim_{n\to\infty} \frac{2012^{\cancel{n+1}}}{\cancel{(n+1)!}} \cdot \frac{\cancel{n!}}{\cancel{2012^n}} = 0 < 1$$

$n+1$

(b) $\displaystyle\sum_{n=1}^{\infty} \frac{2012^n}{n+2013^n}$ $\sim \sum \frac{2012^n}{2013^n} = \sum \left(\frac{2012}{2013}\right)^n$ conv. geom. series $r<1$

$$\frac{2012^n}{n+2013^n} < \frac{2012^n}{2013^n}$$

conv. by Comp. Test

32. Compute $\displaystyle\sum_{n=0}^{\infty} \frac{2012^{n-1}}{2011^n} = \underbrace{\sum_{n=0}^{\infty} \frac{1}{2011} \cdot \left(\frac{2012}{2011}\right)^{n-1}}$

divergent geometric series with common ratio

$$r = \frac{2012}{2011} > 1$$

33. The series $\sum_{k=1}^{\infty} \frac{(-1)^k}{\sqrt[4]{k}}$ is

(a) divergent to ∞

(b) divergent to $-\infty$

(c) divergent but not to $\pm\infty$

(d) absolutely convergent

(e) conditionally convergent

$$\underbrace{\sum \frac{1}{\sqrt[4]{k}} = \sum \frac{1}{k^{1/4}}}_{\text{divergent p-series with } p=\frac{1}{4}<1}$$

Thus, the original series doesn't converges absolutely.

However $\sum_{k=1}^{\infty} \frac{(-1)^k}{\sqrt[4]{k}}$ converges as alternating series ($\lim_{k\to\infty} \frac{1}{\sqrt[4]{k}} = 0$ and $\left\{\frac{1}{\sqrt[4]{k}}\right\}$ is decreasing sequence).

34. Find the value(s) of x such that the vectors $\langle x, -1, 1\rangle$ and $\langle 1, -x^2, x^3\rangle$ are orthogonal.

$$\vec{a} \perp \vec{b} \iff \vec{a} \cdot \vec{b} = 0$$

$$\langle x, -1, 1\rangle \cdot \langle 1, -x^2, x^3\rangle = 0$$

$$x \cdot 1 + (-1)(-x^2) + 1 \cdot x^3 = 0$$

$$x + x^2 + x^3 = 0 \Rightarrow x(x^2 + x + 1) = 0$$

$$x = 0 \qquad x^2 + x + 1 = 0$$

no solutions

Final answer: $\boxed{x = 0}$

35. Find the Taylor series for $f(x) = x^3 + x^2 + 3$ about $x = 3$.

$f(x) = x^3 + x^2 + 3$

$f'(x) = 3x^2 + 2x$

$f''(x) = 6x + 2$

$f'''(x) = 6$

$f^{(4)}(x) = 0$

$f(3) = 27 + 9 + 3 = 39$

$f'(3) = 27 + 6 = 33$

$f''(x) = 18 + 2 = 20$

$f'''(x) = 6$

$f^{(n)}(x) = 0$ for $n \geq 4$

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(3)}{n!}(x-3)^n$$

$$f(x) = 39 + 33(x-3) + \frac{20}{2!}(x-3)^2 + \frac{6}{3!}(x-3)^3$$

$$f(x) = 39 + 33(x-3) + 10(x-3)^2 + (x-3)^3$$

36. Find a power series centered at $x = 0$ for the function $f(x) = \frac{x}{1-8x^3}$, and determine the radius of convergence.

$$f(x) = x \frac{1}{1-8x^3} = x \underbrace{\sum_{n=0}^{\infty} (8x^3)^n}_{\text{converges when}} = \boxed{\sum_{n=0}^{\infty} 8^n x^{3n+1}}$$

sum of geom. series with common ratio $r = 8x^3$

converges when $|r|<1$, i.e.

$|8x^3|<1$

$\underbrace{|x|<\frac{1}{2}}$

$\Downarrow$

$\boxed{R = \frac{1}{2}}$

37. Find the angle between the vectors $\underbrace{\langle 1, 2, 1\rangle}_{=\vec{a}}$ and $\underbrace{\langle 3, 3, 0\rangle}_{=\vec{b}}$.

$$\cos\widehat{\vec{a},\vec{b}} = \frac{\vec{a}\cdot\vec{b}}{|\vec{a}|\cdot|\vec{b}|} = \frac{\langle 1,2,1\rangle \cdot \langle 3,3,0\rangle}{\sqrt{1^2+2^2+1^2}\ \sqrt{3^2+3^2+0^2}}$$

$$= \frac{1\cdot 3 + 2\cdot 3 + 1\cdot 0}{\sqrt{6}\ \sqrt{18}} = \frac{9}{\sqrt{6}\ \sqrt{6\cdot 3}}$$

$$= \frac{\cancel{9}\,\cancel{3}\,\sqrt{3}}{2\,\cancel{6}\,\cancel{\sqrt{3}}} = \frac{\sqrt{3}}{2}$$

$$\Downarrow$$

$$\widehat{\vec{a},\vec{b}} = \boxed{\frac{\pi}{6}}$$

38. Evaluate the integral $\int_0^{1/3} \frac{1}{1+x^7}\,dx$ as infinite series.

$$\int_0^{1/3} \frac{1}{1+x^7}\,dx = \int_0^{1/3} \sum_{n=0}^{\infty} (-1)^n (x^7)^n \,dx =$$

sum of geom. series with common ratio $r=-x^7$

converges when $|r|<1$, i.e. $|-x^7|<1 \Rightarrow |x|<1$ OR $\underbrace{-1<x<1}$

Note that this interval contains the integration interval $[0, 1/3]$

$$= \sum_{n=0}^{\infty} (-1)^n \int_0^{1/3} x^{7n}\,dx$$

$$= \sum_{n=0}^{\infty} (-1)^n \left.\frac{x^{7n+1}}{7n+1}\right|_0^{1/3} = \sum_{n=0}^{\infty} \frac{(-1)^n}{3^{7n+1}(7n+1)}$$

39. Let $f(x) = \ln x$.

(a) Find the third degree Taylor polynomial for $f(x)$ about $x = 5$.

(b) If $T_3(x)$ is used to approximate $f(x)$ on the interval $[4, 6]$, estimate the maximum error in this approximation.

(a)

$f(x) = \ln x$ | $f(5) = \ln 5$

$f'(x) = \frac{1}{x}$ | $f'(5) = 1/5$

$f''(x) = -\frac{1}{x^2}$ | $f''(5) = -1/25$

$f'''(x) = \frac{2}{x^3}$ | $f'''(5) = 2/125$

$$T_3(x) = \ln 5 + \frac{1}{5}(x-5) - \frac{1}{25}\frac{(x-5)^2}{2!} + \frac{2}{125}\frac{(x-5)^3}{3!}$$

$$T_3(x) = \ln 5 + \frac{1}{5}(x-5) - \frac{1}{50}(x-5)^2 + \frac{1}{375}(x-5)^3$$

see next page for (b)

(b) $f(x) \approx T_3(x) \qquad 4 \le x \le 6, \quad a=5$

Apply Taylor inequality for $n=3$

$$|R_3(x)| \le \frac{M}{(3+1)!}|x-5|^{3+1} = \frac{M}{4!}|x-5|^4 \quad ☺$$

where $\displaystyle M = \max_{4 \le x \le 6} |f^{(4)}(x)|$.

Estimate $|x-5|^4$ on $[4,6]$

We have $4 \le x \le 6 \Rightarrow 4-5 \le x-5 \le 6-5$

$$-1 \le x-5 \le 1$$

$$|x-5|^4 \le 1 \quad ☆$$

Find M. Using part (a):

$$f^{(4)}(x) = \left(f^{(3)}(x)\right)' = \left(\frac{2}{x^3}\right)' = -\frac{2\cdot 3}{x^4} = -\frac{6}{x^4}$$

$|f^{(4)}(x)| = \dfrac{6}{x^4}$ monotonically decreasing on $[4,6]$, thus

$$M = \max_{[4;6]} |f^{(4)}(x)| = |f^{(4)}(4)| = \frac{6}{4^4} = \frac{6}{256} = \frac{3}{128}$$

Combining ☺, ☆, & we get

$$|R_3(x)| \le \frac{M}{4!}|x-5|^4 \le \frac{\cancel{3}^{\,1}}{128 \cdot \cancel{24}_{\,8}} = \boxed{\frac{1}{1024} \approx 9.8 \cdot 10^{-4}}$$

40. Consider the points $A(0,1,4)$, $B(2,1,3)$, and $C(1,-1,0)$.

(a) Find a unit vector orthogonal to the plane determined by the given points.

(b) Find the area of the triangle with vertices A,B, and C.

(a) If $\vec{c} = \vec{a} \times b$ then $\vec{c} \perp \vec{a}$ & $\vec{c} \perp \vec{b}$ which implies that $\vec{c}$ is orthogonal to the plane determined by $\vec{a}$ & $\vec{b}$.

In our case, let

$$\vec{a} = \overrightarrow{AB} = \langle 2-0, 1-1, 3-4 \rangle = \langle 2, 0, -1 \rangle$$

$$\vec{b} = \overrightarrow{AC} = \langle 1-0, -1-1, 0-4 \rangle = \langle 1, -2, -4 \rangle$$

Then $\vec{c} = \vec{a} \times \vec{b} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & 0 & -1 \\ 1 & -2 & -4 \end{vmatrix} = \begin{vmatrix} 0 & -1 \\ -2 & -4 \end{vmatrix} \vec{i} - \begin{vmatrix} 2 & -1 \\ 1 & -4 \end{vmatrix} \vec{j} +$

$$+ \begin{vmatrix} 2 & 0 \\ 1 & -2 \end{vmatrix} \vec{k} = -2\vec{i} + 7\vec{j} - 4\vec{k}$$

Then $|\vec{c}| = |\vec{a} \times \vec{b}| = |\langle -2, 7, -4 \rangle|$

$$= \sqrt{(-2)^2 + 7^2 + (-4)^2} = \sqrt{69}$$

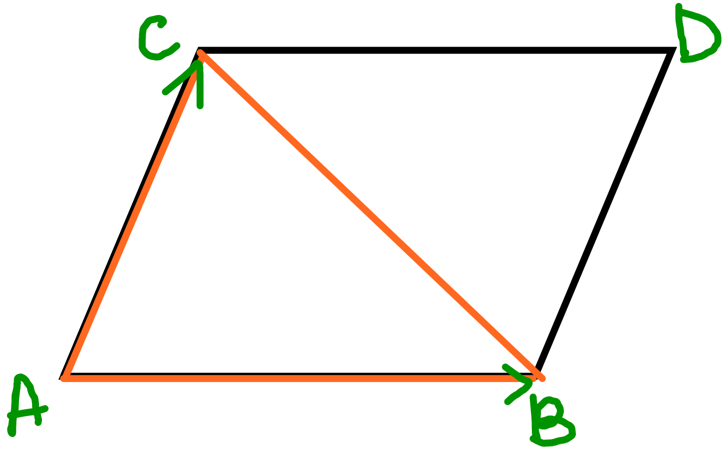
The unit $\vec{c}$ is

$$\hat{c} = \frac{\vec{c}}{|\vec{c}|} = \frac{\langle -2, 7, -4 \rangle}{\sqrt{69}} = \boxed{\left\langle -\frac{2}{\sqrt{69}}, \frac{7}{\sqrt{69}}, -\frac{4}{\sqrt{69}} \right\rangle}$$

Note that the opposite vector $\left\langle \frac{2}{\sqrt{69}}, -\frac{7}{\sqrt{69}}, \frac{4}{\sqrt{69}} \right\rangle$ also satisfies the conditions of the problem

see next page for (b)

(b)



ABCD is parallelogram s.t.

$\vec{AB} = \vec{CD}$

$\vec{AC} = \vec{BD}$

$$\text{Area}(ABC) = \frac{1}{2}\text{Area}(ABCD)$$

$$= \frac{1}{2}\left|\vec{AB} \times \vec{AC}\right| = \boxed{\frac{\sqrt{69}}{2}}$$

by part (a)

41. Which of the following statements is most accurately describes the convergence or divergence of the improper integral $\int_1^\infty \frac{x}{\sqrt{x^7+77}}\,dx$?

(a) The integral converges because $\frac{x}{\sqrt{x^7+77}} < \frac{1}{x^{7/2}}$ and the integral $\int_1^\infty \frac{1}{x^{7/2}}\,dx$ converges.

(b) The integral converges because $\frac{x}{\sqrt{x^7+77}} < \frac{1}{x^6}$ and the integral $\int_1^\infty \frac{1}{x^6}\,dx$ converges.

(c) The integral converges because $\frac{x}{\sqrt{x^7+77}} < \frac{1}{x^{5/2}}$ and the integral $\int_1^\infty \frac{1}{x^{5/2}}\,dx$ converges.

(d) The integral diverges because $\frac{x}{\sqrt{x^7+77}} \geq \frac{1}{x^{5/2}}$ and the integral $\int_1^\infty \frac{1}{x^{5/2}}\,dx = \infty$.

(e) The integral diverges because $\frac{x}{\sqrt{x^7+77}} \geq \frac{1}{x^6}$ and the integral $\int_1^\infty \frac{1}{x^6}\,dx = \infty$.

Note that $\sqrt{x^7+77} > \sqrt{x^7}$

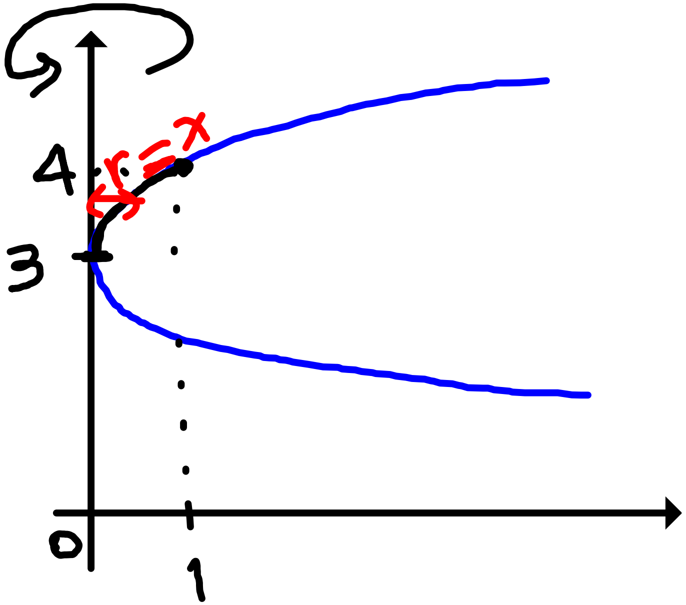
for $x \geq 1$ it implies

$$\frac{x}{\sqrt{x^7+77}} < \frac{x}{x^{7/2}} = \frac{1}{x^{5/2}}$$

But $\int_1^\infty \frac{dx}{x^{5/2}}$ converges $\left(p = \frac{5}{2} > 1\right)$

Thus By Comparison theorem the given integral converges.

42. Set up the integral that will compute the area of the surface obtained by revolving the curve $x = (y-3)^2$ from $(0, 3)$ to $(1, 4)$ about the y-axis.



$$SA = 2\pi \int x \, ds$$

We have

$$x = (y-3)^2, \quad 1 \leq y \leq 4$$

$$ds = \sqrt{1 + \left(\frac{dx}{dy}\right)^2} \, dy$$

$$= \sqrt{1 + [2(y-3)]^2} \, dy$$

$$= \sqrt{1 + 4(y-3)^2} \, dy$$

$$SA = 2\pi \int_3^4 (y-3)^2 \sqrt{1 + 4(y-3)^2} \, dy$$

43. If $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ are vectors in $\mathbb{R}^3$, which of the following expressions has no meaning?

(a) $(\mathbf{a} \times (3\mathbf{b})) \cdot \mathbf{c}$ = (vector) $\cdot$ vector = scalar

(b) $((5\mathbf{a}) \cdot (3\mathbf{b})) \times (-\mathbf{c})$ = (scalar) $\times$ vector = no meaning

(c) $(\mathbf{a} \cdot \mathbf{b})\mathbf{c}$ = (scalar) vector = vector

(d) $(\mathbf{a} \times (3\mathbf{b})) \times \mathbf{c}$ = (vector) $\times$ vector = vector

(e) $-\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})$

$\downarrow$

vector $\cdot$ (vector) = scalar