Fall 2012 Math 152

Week in Review 1

courtesy: *Oksana Shatalov*

(covering Sections 6.5&7.1)

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6.5: The substitution rule

Key Points

If $u = g(x)$ is a differentiable function, then

$$\int f(\overbrace{g(x)}^{u})\overbrace{g'(x)\,\mathrm{d}x}^{du} = \int f(u)\,\mathrm{d}u$$

$$\int_a^b f(g(x))g'(x)\,\mathrm{d}x = \int_{g(a)}^{g(b)} f(u)\,\mathrm{d}u$$

You should make sure that the old variable x has disappeared from the integral.

$u = g(x)$

$du = g'(x)dx$

Examples

Evaluate the following integrals

1. $\int e^{2012x}\,\mathrm{d}x = \int e^u \frac{du}{2012} = \frac{1}{2012}\int e^u du =$

$u = 2012x$

$du = (2012x)'dx = 2012dx$

$dx = \frac{du}{2012}$

$= \frac{1}{2012}e^u + C =$

$= \frac{1}{2012}e^{2012x} + C$

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2. $\int_2^4 \sin(4\pi x)\,dx = \int_{8\pi}^{16\pi} \sin u \, \frac{du}{4\pi} = \frac{1}{4\pi}(-\cos u)\Big|_{8\pi}^{16\pi}$

$u = 4\pi x$
$du = 4\pi\, dx$
$dx = \frac{1}{4\pi} du$

$x = 2 \Rightarrow u = 4\pi \cdot 2 = 8\pi$
$x = 4 \Rightarrow u = 16\pi$

new bounds

$= \frac{1}{4\pi}(\cos 16\pi - \cos 8\pi) = -\frac{1}{4\pi}(1-1) = 0$

$$3.\ \int_0^{\pi/2} \cos^7 x \underbrace{\sin x \, dx}_{-du} = -\int_1^0 u^7 \, du = -\frac{u^8}{8}\Bigg|_1^0 = -\left(0-\frac{1}{8}\right) = \frac{1}{8}$$

$$\boxed{\begin{array}{l} u = \cos x \\ du = -\sin x \, dx \end{array}}$$

$x = 0 \Rightarrow u = \cos 0 = 1$

$x = \frac{\pi}{2} \Rightarrow u = \cos \frac{\pi}{2} = 0$

4. $\int_0^1 x^4 e^{9x^5-8}\,dx = \int_0^1 \underbrace{e^{9x^5-8}}_{e^u} \underbrace{x^4 dx}_{\frac{du}{45}} = \int_{-8}^{1} \frac{e^u du}{45} =$

$$\boxed{\begin{aligned} &u = 9x^5 - 8 \\ &du = (9x^5-8)' dx = 45x^4 dx \\ &x^4 dx = \frac{du}{45} \end{aligned}}$$

$x=0 \Rightarrow u = -8$

$x=1 \Rightarrow u = 9-8 = 1$

$$= \frac{1}{45} e^u \Big|_{-8}^{1} = \frac{1}{45}\left(e^1 - e^{-8}\right) =$$

$$= \frac{1}{45}\left(e - e^{-8}\right)$$

5. $\int \frac{x^{10}}{\underbrace{x^{11}+11}_{u}} dx = \frac{1}{11}\int \frac{du}{u} = \frac{1}{11}\ln|u| + C =$

$= \frac{1}{11}\ln|x^{11}+11| + C$

$u = x^{11}+11$

$du = 11x^{10}\,dx$

$x^{10}\,dx = \frac{du}{11}$

6. $\int \frac{3}{\sqrt{3y+1}}\,\mathrm{d}y = \int \frac{du}{\sqrt{u}} = \int u^{-\frac{1}{2}} du =$

$u = 3y+1$
$du = 3dy$

$$= \frac{u^{-\frac{1}{2}+1}}{-\frac{1}{2}+1} + C =$$

$$= 2u^{\frac{1}{2}} + C = 2\sqrt{u} + C =$$

$$= 2\sqrt{3y+1} + C$$

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7. $\displaystyle\int \frac{\tan(\ln x)}{x}\,dx = \int \tan u\,du = \int \frac{\sin u}{\underbrace{\cos u}_{v}}\,du$ $\quad \frac{\sin u}{\cos u}du = -\frac{dv}{v}$

$\boxed{\begin{aligned} u &= \ln x \\ du &= \frac{1}{x}\,dx \end{aligned}}$

$v = \cos u$

$dv = (\cos u)'\,du = -\sin u\,du$

$$-\int \frac{dv}{v} = -\ln|v| + C = -\ln|\cos u| + C =$$

$$= -\ln|\cos(\ln x)| + C$$

8. $\displaystyle\int \frac{e^x}{\sqrt{1-e^{2x}}}\,dx$ du $= \displaystyle\int \frac{du}{\sqrt{1-u^2}} = \arcsin u + C =$

$= \arcsin e^x + C$

$u = e^x$

$du = e^x\,dx$

$e^{2x} = (e^x)^2 = u^2$

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9. $\int x^5 \sqrt{4+x^3}\,dx = \int \underbrace{x^3}_{u-4} \underbrace{\sqrt{4+x^3}}_{\sqrt{u}} \underbrace{x^2\,dx}_{\frac{du}{3}} = \int (u-4)\sqrt{u}\,\frac{du}{3} =$

$u = 4 + x^3 \longrightarrow u - 4 = x^3$

$du = 3x^2\,dx$

$x^2\,dx = \frac{du}{3}$

$= \frac{1}{3}\int (u-4)u^{\frac{1}{2}}\,du =$

$= \frac{1}{3}\int u^{\frac{3}{2}} - 4u^{\frac{1}{2}}\,du =$

$= \frac{1}{3}\left(\frac{u^{\frac{3}{2}+1}}{\frac{3}{2}+1} - 4\,\frac{u^{\frac{1}{2}+1}}{\frac{1}{2}+1} \right) + C =$

$= \frac{1}{3}\left(\frac{2}{5} u^{5/2} - 4\cdot\frac{2}{3} u^{3/2} \right) + C =$

$= \frac{1}{3}\left(\frac{2}{5}\,(4+x^3)^{5/2} - \frac{8}{3}\,(4+x^3)^{3/2} \right) + C$

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10. $\displaystyle\int \frac{x-1}{x^2+1}\,\mathrm{d}x = \int \underbrace{\frac{x}{x^2+1}} - \frac{1}{x^2+1}\,dx =$

$u = x^2+1$

$du = 2x\,dx$

$x\,dx = \frac{1}{2}du$

$$= \int \frac{\overset{du/2}{x\,dx}}{x^2+1} - \int \frac{dx}{x^2+1} =$$

$$= \frac{1}{2}\int \frac{du}{u} - \arctan x + C =$$

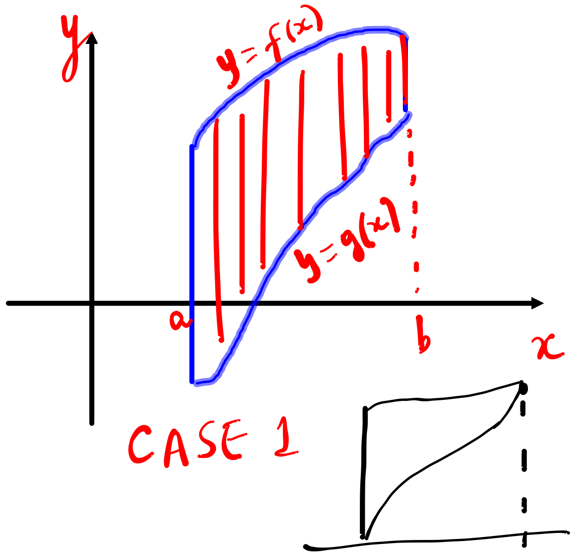
$$= \frac{1}{2}\ln|u| - \arctan x + C =$$

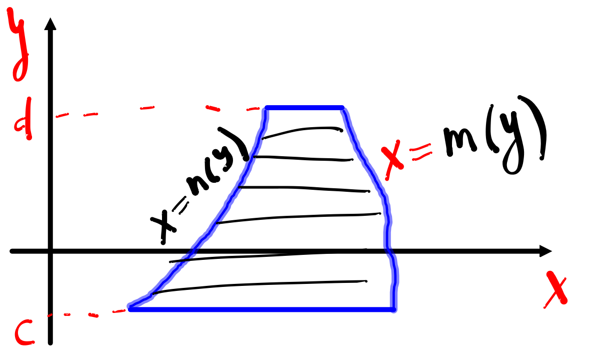
$$= \frac{1}{2}\ln(x^2+1) - \arctan x + C$$

Key Points

CASE I $A = \displaystyle\int_a^b \left(\begin{array}{c}\text{upper}\\ \text{function}\end{array}\right)(x) - \left(\begin{array}{c}\text{lower}\\ \text{function}\end{array}\right)(x)\,dx = \int_a^b f(x) - g(x)\,dx$

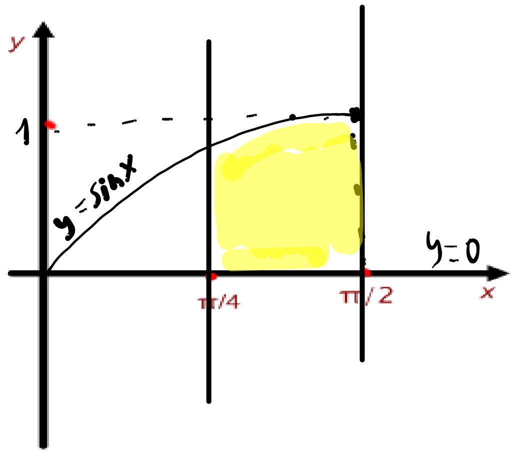
CASE II $A = \displaystyle\int_c^d \left(\begin{array}{c}\text{right}\\ \text{function}\end{array}\right)(y) - \left(\begin{array}{c}\text{left}\\ \text{function}\end{array}\right)(y)\,dy = \int_c^d m(y) - n(y)\,dy$





- In some cases the limits of integration can be determined as the intersection points of two curves.
- Sketch of a graph of the region is highly recommended.
- The area between two curves will always be positive.

11. Find the area of the region bounded by $y = \sin x$, $y = 0$, $x = \pi/4$, $x = \pi/2$.



$$A = \int_{\frac{\pi}{4}}^{\pi/2} (\sin x - 0)\, dx =$$

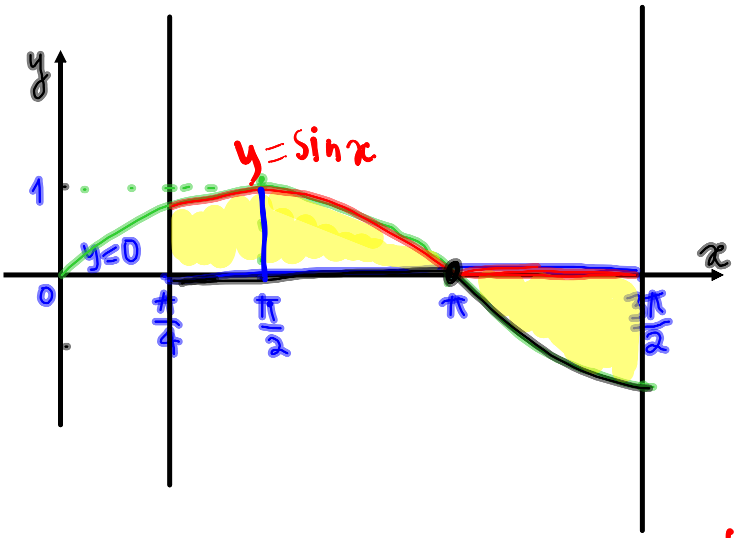
$$-\cos x \Big|_{\frac{\pi}{4}}^{\pi/2} = -\left(\cos\frac{\pi}{2} - \cos\frac{\pi}{4}\right) =$$

$$= -\left(0 - \frac{\sqrt{2}}{2}\right) = \frac{\sqrt{2}}{2}$$

$$\boxed{\frac{\sqrt{2}}{2} \cdot \frac{\sqrt{2}}{\sqrt{2}} = \frac{2}{2\sqrt{2}} = \frac{1}{\sqrt{2}}}$$

12. Find the area of the region

$$D = \{(x, y) : \pi/4 \le x \le 3\pi/2, 0 \le y \le \sin x\}.$$



$$A = \int_{\pi/4}^{3\pi/2} (U.F.) - (L.F.)\, dx =$$

$$= \int_{\pi/4}^{\pi} (\sin x - 0)\, dx + \int_{\pi}^{3\pi/2} (0 - \sin x)\, dx =$$

$$= -\cos x \Big|_{\frac{\pi}{4}}^{\pi} + \cos x \Big|_{\pi}^{3\pi/2} =$$

$$= -\left(\cos\pi - \cos\frac{\pi}{4}\right) + \left(\cos\frac{3\pi}{2} - \cos\pi\right)$$

$$= -\left(-1 - \frac{\sqrt{2}}{2}\right) + 0 - (-1) =$$

$$= 2 + \frac{\sqrt{2}}{2}$$

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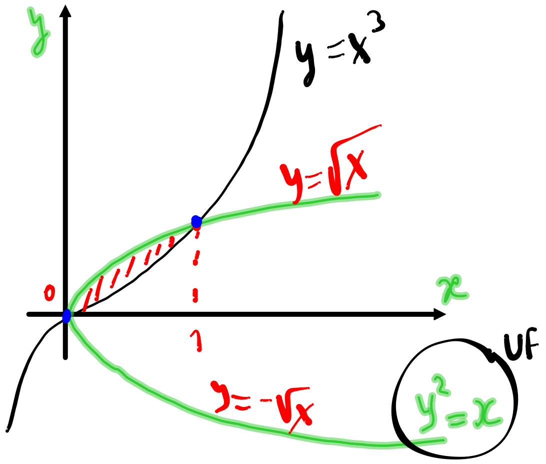
13. Find the area of the region bounded by $y = x^3$ and $y^2 = x$.



$y = \pm\sqrt{x}$

$$\begin{cases} y = x^3 \\ y^2 = x \end{cases}$$

$$(x^3)^2 = x$$

$$x^6 = x$$

$$x^6 - x = 0$$

$$x(x^5 - 1) = 0$$

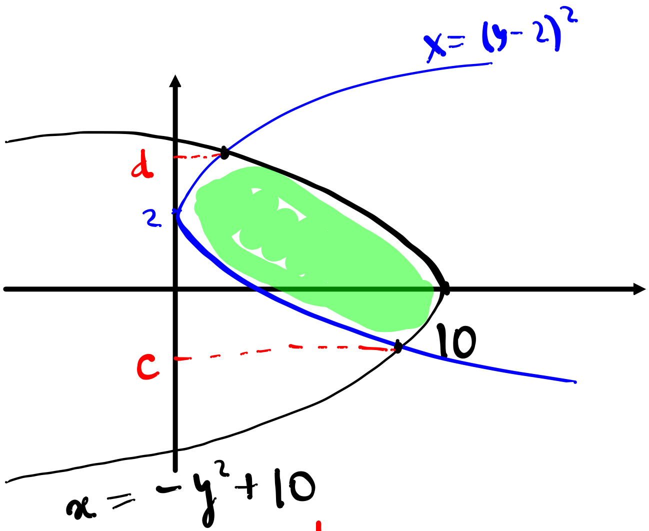
$x = 0$ or $x^5 = 1$

$x = 1$

$$A = \int_0^1 (UF)-(LF)\,dx = \int_0^1 \sqrt{x} - x^3\,dx =$$

$$= \frac{x^{3/2}}{3/2} - \frac{x^4}{4}\Big|_0^1 = \frac{2}{3} - \frac{1}{4} = \frac{5}{12}$$

14. Determine the area of the region enclosed by $x = -y^2 + 10$, $x = (y-2)^2$.



Find ~~intersection points~~ c, d

$$-y^2+10=(y-2)^2$$

$$-y^2+10=y^2-4y+4$$

$$2y^2-4y-6=0 \quad \left(\times\tfrac{1}{2}\right)$$

$$y^2-2y-3=0$$

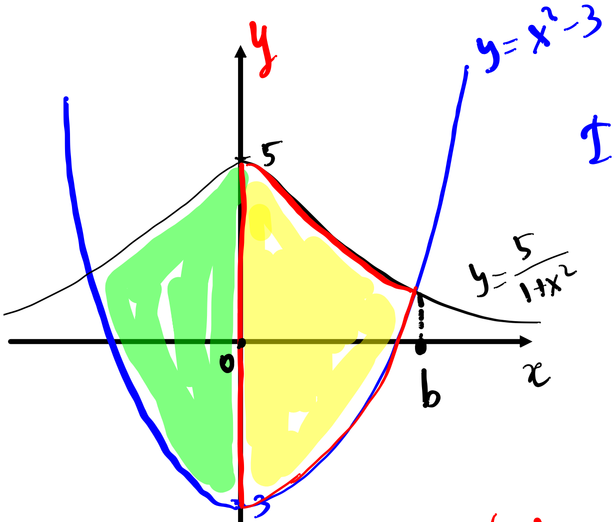
$$(y+1)(y-3)=0 < \begin{matrix} y=-1 \\ y=3 \end{matrix}$$

$$A = \int_c^d (RF)-(LF)\,dy = \int_{c=-1}^{d=3} -y^2+10-(y-2)^2 =$$

$$= \int_{-1}^{3} -y^2+10-y^2+4y-4\,dy = \int_{-1}^{3} -2y^2+4y+6\,dy =$$

$$= \dots = \frac{64}{3}$$

15. Find the area of the region bounded by $y = x^2 - 3$ and $y = \frac{5}{1+x^2}$. positive even



If $x>0$ $y' = \left(\frac{5}{1+x^2}\right)' = -\frac{5\cdot 2x}{(1+x^2)^2} < 0$

$y = \frac{5}{1+x^2}$ is monotonically decreasing for all $x > 0$

Region is symmetric w.r.t. y-axis.

$$A = 2\int_0^b \frac{5}{1+x^2} - (x^2-3)\,dx =$$

$$= 2\int_0^2 \frac{5}{1+x^2} - x^2 + 3\,dx =$$

$$= 2\left(5\arctan x - \frac{x^3}{3} + 3x\right)\Big|_0^2 =$$

$$= 2\left(5\arctan 2 - \frac{8}{3} + 6\right) =$$

$$= \boxed{10\arctan 2 - \frac{20}{3}}$$

Find b

$$x^2 - 3 = \frac{5}{1+x^2} \quad (\times(1+x^2))$$

$$(x^2-3)(1+x^2) = 5$$

$$x^2 + x^4 - 3 - 3x^2 = 5$$

$$x^4 - 2x^2 - 8 = 0$$

$x^2 = w$

$$w^2 - 2w - 8 = 0$$

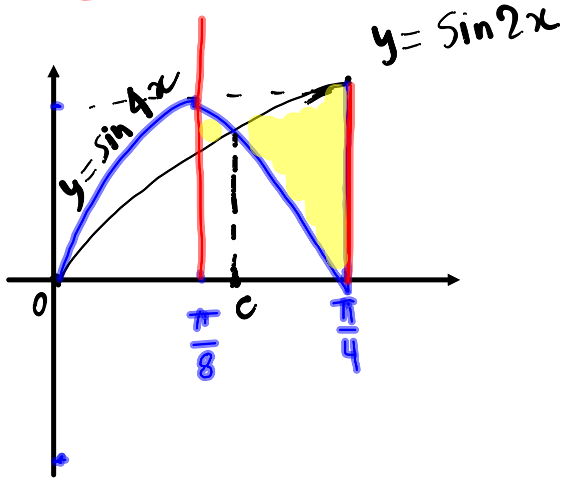
$$(w+2)(w-4) = 0$$

$\underbrace{x^2 = w = -2}_{\text{impossible}}$ OR $x^2 = w = 4$

$x = \pm 2$

$b = 2$

16. Find the area of the region bounded by $y = \sin 2x$, $y = \sin 4x$, $x = \pi/8$, and $x = \pi/4$. Do not evaluate. Just set up the integral.



$$A = \int_{\pi/8}^{c=\pi/6} (\sin 4x - \sin 2x)\,dx + \int_{c=\pi/6}^{\pi/4} (\sin 2x - \sin 4x)\,dx$$

Find c: $\sin 4x = \sin 2x$

$$2\sin 2x \cos 2x = \sin 2x$$

$$2\sin 2x \cos 2x - \sin 2x = 0$$

$$\sin 2x(2\cos 2x - 1) = 0$$

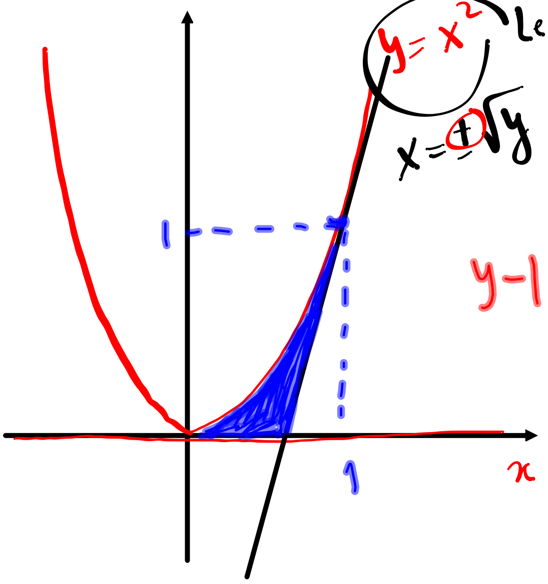
$\sin 2x = 0$ OR $2\cos 2x - 1 = 0$

$\Downarrow$ $\quad\cos 2x = \frac{1}{2}$

$x = 0$ $\quad 2x = \frac{\pi}{3}$

$$x = \boxed{\frac{\pi}{6} = c}$$

17. Determine the area of the region bounded by the x-axis, the curve $y = x^2$ and tangent line to this curve at the point $(1, 1)$.



Left fun. $y = x^2$

slope of tangent $m = 2x\Big|_{x=1} = 2$

$y - 1 = m(x-1) \Rightarrow y - 1 = 2(x-1)$

$y = 2x - 2 + 1$

R.F. $\boxed{y = 2x - 1}$ Tangent

$x = \frac{y+1}{2}$

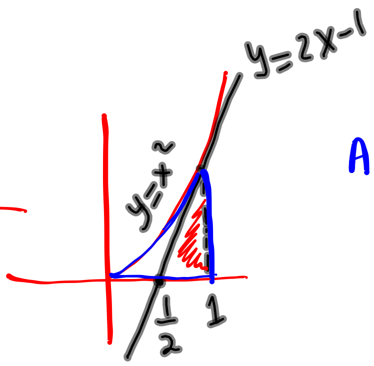
Way 1 $A = \int (UF) - (LF)\,dx \to 2$ integrals

Way 2 $A = \int_0^1 (RF)^{(y)} - (LF)^{(y)}\,dy = \int_0^1 \frac{y+1}{2} - \sqrt{y}\,dy =$

$= \frac{1}{2}\left(\frac{y^2}{2} + y\right) - \frac{2}{3} y^{3/2}\Big|_0^1 = \frac{1}{2}\left(\frac{1}{2} + 1\right) - \frac{2}{3} =$

$= \frac{1}{12}$

Way 3



$A = \int_0^1 x^2\,dx - \underbrace{\frac{1}{2} \cdot \frac{1}{2} \cdot 1}_{\text{TRIANGLE AREA}} = \frac{x^3}{3}\Big|_0^1 - \frac{1}{4} =$

$= \frac{1}{3} - \frac{1}{4} = \frac{1}{12}.$

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